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THE ROLE OF CONCAVITY IN APPLICATIONS OF AVERY TYPE FIXED POINT THEOREMS TO HIGHER ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this article we apply an extension of an Avery type fixed point theorem to a family of boundary value problems for higher order ordinary differential equations. The theorem employs concave and convex functionals defined on a cone in a Banach space. We begin by extending a known application to a right focal boundary value problem for a second order problem to a conjugate boundary value problem for a second order problem. We then extend inductively to a two point boundary value problem for a higher order equation. Concavity of differentiable functions plays a key role in the application to second order equations. A concept of generalized concavity plays the same key role in the application to the higher order equation.

1. Introduction

Richard Avery and co-authors [1, 2, 3, 4] have extended the Leggett-Williams fixed point theorem [6] in various ways; a recent extension [2] employs topological methods rather than index theory and as a result the recent extension does not require the functional boundaries to be invariant with respect to a functional wedge. It is shown [3] that this extension applies in a natural way to second order right focal boundary value problems. The concept of concavity implies an inequality that is fundamental to the technical arguments with respect to concave and convex functionals.

In this article we seek further applications of the extension of the fixed point theorem. Initially, we shall extend the applications from a two point right focal boundary value problem [3] to a two point conjugate boundary value problem. This is made possible by employing symmetry of solutions about the midpoint of the interval. Then we shall inductively construct a boundary value problem for a higher order ordinary differential equation from the conjugate problem for the second order problem. This work is clearly motivated by the work in [3]; we shall borrow the same notations.

In Section 2 we shall introduce the appropriate definitions and state the fixed point theorem. In Section 3, we shall apply the fixed point theorem to a conjugate boundary value problem for a second order problem. We shall briefly mention how concavity is employed to generate the appropriate inequalities. In Section 4, we shall define a two point boundary value problem for a kth order differential equation and apply the fixed point theorem. We shall also show how a generalized concept of concavity to higher order differential inequalities [5] generates the appropriate inequalities in Section 4.

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2. Preliminaries

DEFINITION 2.1. Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a *cone* if it satisfies the following two conditions:

(i) $x \in P, \lambda \geqslant 0$ implies $\lambda x \in P$;

(ii) $x \in P, -x \in P$ implies $x = 0$.

Every cone $P \subset E$ induces an ordering in E given by

$$x \leqslant y \text{ if and only if } y - x \in P.$$

DEFINITION 2.2. An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

DEFINITION 2.3. A map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E if $\alpha : P \to [0, \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geqslant t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0,1]$. Similarly we say the map β is a nonnegative continuous convex functional on a cone P of a real Banach space E if $\beta : P \to [0, \infty)$ is continuous and

$$\beta(tx + (1-t)y) \leqslant t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in P$ and $t \in [0,1]$.

Let α and ψ be non-negative continuous concave functionals on P and δ and β be non-negative continuous convex functionals on P; then, for non-negative real numbers a, b, c and d, we define the following sets:

$$A := A(\alpha, \beta, a, d) = \{x \in P : a \leqslant \alpha(x) \text{ and } \beta(x) \leqslant d\}, \tag{2.1}$$

$$B := B(\alpha, \delta, \beta, a, b, d) = \{x \in A : \delta(x) \leqslant b\}, \tag{2.2}$$

and

$$C := C(\alpha, \psi, \beta, a, c, d) = \{x \in A : c \leqslant \psi(x)\}. \tag{2.3}$$

We say that A is a *functional wedge with concave functional boundary* defined by the concave functional α and convex functional boundary defined by the convex functional β. We say that an operator $T : A \to P$ is *invariant with respect to the concave functional boundary*, if $a \leqslant \alpha(Tx)$ for all $x \in A$, and that T is *invariant with respect to the convex functional boundary*, if $\beta(Tx) \leqslant d$ for all $x \in A$. Note that A is a convex set. The following theorem, proved in [2], is an extension of the original Leggett-Williams fixed point theorem [6].

THEOREM 2.4. *Suppose P is a cone in a real Banach space E, α and ψ are non-negative continuous concave functionals on P, δ and β are non-negative continuous convex functionals on P, and for non-negative real numbers a, b, c and d the sets A, B and C are as defined in* (2.1), (2.2) *and* (2.3). *Furthermore, suppose that A is a bounded subset of P, that $T : A \to P$ is completely continuous and that the following conditions hold:*

(A1) $\{x \in A : c < \psi(x) \text{ and } \delta(x) < b\} \neq \emptyset$, $\{x \in P : \alpha(x) < a \text{ and } d < \beta(x)\} = \emptyset$;

(A2) $\alpha(Tx) \geqslant a$ *for all* $x \in B$;

(A3) $\alpha(Tx) \geqslant a$ *for all* $x \in A$ *with* $\delta(Tx) > b$;

(A4) $\beta(Tx) \leqslant d$ *for all* $x \in C$; *and,*

(A5) $\beta(Tx) \leqslant d$ *for all* $x \in A$ *with* $\psi(Tx) < c$.

Then T has a fixed point $x^* \in A$.

A fixed point of T will also be called a solution of T.

3. The second order problem

Let $f : \mathbb{R} \to \mathbb{R}^+$ is a continuous map and let $n > 0$ be fixed. We consider the two point conjugate boundary value problem,

$$-x''(t) = f(x(t)), \quad t \in [0,n], \tag{3.1}$$

$$x(0) = 0, \quad x(n) = 0. \tag{3.2}$$

The Green's function for the problem is known and has the form,

$$G(t,s) = \begin{cases} \frac{s(n-t)}{n} & : 0 \leqslant s < t \leqslant n, \\ \frac{t(n-s)}{n} & : 0 \leqslant t < s \leqslant n. \end{cases}$$

G satisfies the symmetry property

$$G(n-t, n-s) = G(t,s), \quad (t,s) \in [0,n] \times [0,n],$$

since

$$\begin{aligned} G(n-t,n-s) &= \begin{cases} \frac{(n-t)(n-(n-s))}{n} & : 0 \leqslant n-t < n-s \leqslant n, \\ \frac{(n-s)(n-(n-t))}{n} & : 0 \leqslant n-s < n-t \leqslant n, \end{cases} \\ &= \begin{cases} \frac{s(n-t)}{n} & : 0 \leqslant s < t \leqslant n, \\ \frac{t(n-s)}{n} & : 0 \leqslant t < s \leqslant n, \end{cases} = G(t,s). \end{aligned}$$

The Green's function plays the following role.

LEMMA 3.1. *x is a solution of the boundary value problem, (3.1), (3.2), if, and only if, $x \in C[0,n]$, and*

$$x(t) = \int_0^n G(t,s) f(x(s)) ds, \quad 0 \leqslant t \leqslant n.$$

Let $E = C[0,n]$, equiped with the usual supremum norm, denote the Banach space. Define the cone $P \subset E = C[0,n]$ by

$$P := \left\{ x \in E \ : x(n-t) = x(t),\ x \text{ is nonnegative and nondecreasing on } \left[0, \frac{n}{2}\right], \right.$$
$$\left. \text{and if } 0 \leqslant y \leqslant w \leqslant n, \text{ then } wx(y) \geqslant yx(w) \right\}.$$

Note that, due to the definition of P, we shall obtain symmetric solutions. We do not know how to apply Theorem 2.4 to a two-point conjugate problem without requiring the solutions to be symmetric.

Define $T : E \to E$ by

$$Tx(t) = \int_0^n G(t,s) f(x(s)) ds.$$

LEMMA 3.2. *For any $y, w \in [0, \frac{n}{2}]$ with $y \leqslant w$ we have*

$$\min_{s \in [0,n]} \frac{G(y,s)}{G(w,s)} \geqslant \frac{y}{w}. \tag{3.3}$$

Proof. For $y \leqslant w \leqslant s$,

$$\frac{G(y,s)}{G(w,s)} = \frac{y(\frac{n-s}{n})}{w(\frac{n-s}{n})} = \frac{y}{w}.$$

For $y \leqslant s \leqslant w$,

$$\frac{G(y,s)}{G(w,s)} = \frac{y(\frac{n-s}{n})}{s(\frac{n-w}{n})} = \frac{y(n-s)}{s(n-w)} \geqslant \frac{y(n-w)}{w(n-w)} = \frac{y}{w}.$$

For $s \leqslant y \leqslant w$,

$$\frac{G(y,s)}{G(w,s)} = \frac{s(\frac{n-y}{n})}{s(\frac{n-w}{n})} = \frac{n-y}{n-w} \geqslant 1 \geqslant \frac{y}{w}. \qquad \square$$

REMARK 3.3. Concavity of x on $[0, \frac{n}{2}]$ implies $wx(y) \geqslant yx(w)$ whenever $0 < y \leqslant w \leqslant \frac{n}{2}$. We shall show this implication in the higher order case in Section 4. So, note that Lemma 3.2 points out that G is concave. Moreover, it is the inequality $wx(y) \geqslant yx(w)$ that is used repeatedly in the proof Theorem 3.5.

LEMMA 3.4. *Assume* $f:\mathbb{R}\to\mathbb{R}^+$ *is a continuous map. Then*

$$T:P\to P.$$

Proof. To see that $Tx(n-t)=Tx(t)$,

$$\begin{aligned}
Tx(n-t) &= \int_0^n G(n-t,s)f(x(s))ds\\
&= -\int_n^0 G(n-t,n-\sigma)f(x(n-\sigma))d\sigma\\
&= \int_0^n G(n-t,n-\sigma)f(x(\sigma))d\sigma = \int_0^n G(t,\sigma)f(x(\sigma))d\sigma\\
&= Tx(t).
\end{aligned}$$

Clearly, $Tx(t)\geqslant 0$ on $[0,n]$ since $G(t,s)\geqslant 0$ on $[0,n]\times[0,n]$ and $f:\mathbb{R}\to\mathbb{R}^+$. To see that $Tx(t)$ is nondecreasing on $[0,\frac{n}{2}]$, we calculate $\frac{d}{dt}Tx(t)$.

$$\begin{aligned}
&\frac{d}{dt}\left[\int_0^t \frac{s(n-t)}{n}f(x(s))ds+\int_t^n \frac{t(n-s)}{n}f(x(s))\right]ds\\
&=\frac{t(n-t)}{n}f(x(t))+\int_0^t -\frac{s}{n}f(x(s))ds+\int_t^n \frac{(n-s)}{n}f(x(s))ds-\frac{t(n-t)}{n}f(x(t))\\
&=\int_0^t -\frac{s}{n}f(x(s))ds+\int_t^n \frac{(n-s)}{n}f(x(s))ds.
\end{aligned}$$

Set $s=n-\sigma$ and recall $t\in[0,\frac{n}{2}]$.

$$=\int_0^t -\frac{s}{n}f(x(s))ds+\int_t^n \frac{(n-s)}{n}f(x(s))ds$$

$$\begin{aligned}
&\int_0^t -\frac{s}{n}f(x(s))ds+\int_t^{\frac{n}{2}} \frac{(n-s)}{n}f(x(s))ds+\int_{\frac{n}{2}}^0 \frac{\sigma}{n}f(x(n-\sigma))(-d\sigma)\\
&=\int_0^t -\frac{s}{n}f(x(s))ds+\int_t^{\frac{n}{2}} \frac{(n-s)}{n}f(x(s))ds+\int_0^{\frac{n}{2}} \frac{s}{n}f(x(s))ds\\
&=\int_t^{\frac{n}{2}} f(x(s))ds\geqslant 0.
\end{aligned}$$

Now let $0<y\leqslant w\leqslant n$. Apply Lemma 3.2 to see that

$$Tx(y)=\int_0^n G(y,s)f(x(s))ds\geqslant \frac{y}{w}\int_0^n G(w,s)f(x(s))ds.$$

Thus, $wTx(y)\geqslant yTx(w)$ and Tx satisfies the concavity condition. □

For fixed $\nu,\tau,\mu\in[0,\frac{n}{2}]$ and $x\in P$, define the concave functionals α and ψ on P by

$$\alpha(x):=\min_{t\in[\tau,\frac{n}{2}]}x(t)=x(\tau),\ \ \psi(x):=\min_{t\in[\mu,\frac{n}{2}]}x(t)=x(\mu),$$

and the convex functionals δ and β on P by

$$\delta(x) := \max_{t\in[0,\nu]}(x(t)) = x(\nu), \;\; \beta(x) := \max_{t\in[0,\frac{n}{2}]} x(t) = x\left(\frac{n}{2}\right).$$

THEOREM 3.5. *Assume $\tau,\nu,\mu \in (0,\frac{n}{2}]$ are fixed with $\tau \leqslant \mu < \nu$, d and L are positive real numbers with $0 < L \leqslant \frac{2d\mu}{n}$ and $f:[0,\infty)\to[0,\infty)$ is a continuous function such that*

- (a) $f(w) \geqslant \frac{4d}{n(\nu-\tau)}$ *for* $w \in [\frac{2\tau d}{n}, \frac{2\nu d}{n}]$,
- (b) $f(w)$ *is decreasing for* $w \in [0,L]$ *with* $f(L) \geqslant f(w)$ *for* $w \in [L,d]$, *and*
- (c) $\int_0^t s\, f\left(\frac{Ls}{\mu}\right)\, ds \leqslant \frac{2d - f(L)(n^2-t^2)}{2}, \quad 0 \leqslant t \leqslant \mu.$

Then the operator T has at least one positive solution $x^* \in A(\alpha,\beta,\frac{2\tau d}{n},d)$.

Proof. Let $a = \frac{2\tau d}{n}$, $b = \frac{2\nu d}{n} = \frac{a\nu}{\tau}$, and $c = \frac{2\mu d}{n}$. Let $x \in A(\alpha,\beta,a,d)$. An immediate corollary of Lemma 3.4 is

$$T : A(\alpha,\beta,a,d) \to P.$$

By the Arzela-Ascoli Theorem it is a standard exercise to show that T is a completely continuous operator using the properties of G and f; by the definition of β, A is a bounded subset of the cone P. Also, if $x \in P$ and $\beta(x) > d$, then by the properties of the cone P (in particular, the concavity of x),

$$\alpha(x) = x(\tau) \geqslant \left(\frac{2\tau}{n}\right) x\left(\frac{n}{2}\right) = \frac{2\tau}{n}\beta(x) > \frac{2\tau d}{n} = a.$$

Thus,

$$\{x \in P \; : \; \alpha(x) < a \text{ and } d < \beta(x)\} = \emptyset.$$

For any $r \in \left(\frac{4d}{n(n-\mu)}, \frac{4d}{n(n-\nu)}\right)$ define x_r by

$$x_r(t) \equiv \int_0^n rG(t,s)ds = r\left(\int_0^t \frac{s(n-t)}{n}ds + \int_t^n \frac{t(n-s)}{n}ds\right) = \frac{rt(n-t)}{2}.$$

We claim $x_r \in A$.

$$\alpha(x_r) = x_r(\tau) = \frac{r\tau(n-\tau)}{2} > \frac{2d\tau(n-\tau)}{n(n-\mu)} \geqslant \frac{2d\tau}{n} = a,$$

$$\beta(x_r) = x_r\left(\frac{n}{2}\right) = \frac{r(\frac{n}{2})(n-\frac{n}{2})}{2} < \frac{dn}{2(n-\nu)} \leqslant d,$$

since $\frac{n}{2(n-\nu)} \leqslant 1 \leqslant \frac{n-\tau}{n-\mu}$. Thus, the claim is true. Moreover, x_r has the properties that

$$\psi(x_r) = x_r(\mu) = \frac{r\mu(n-\mu)}{2} > \left(\frac{4d}{n(n-\mu)}\right)\left(\frac{\mu(n-\mu)}{2}\right) = \frac{2d\mu}{n} = c$$

and

$$\delta(x_r) = x_r(\nu) = \frac{r\nu(n-\nu)}{2} < \left(\frac{4d}{n(n-\nu)}\right)\left(\frac{\nu(n-\nu)}{2}\right) = \frac{2d\nu}{n} = b.$$

In particular,

$$\{x \in A \ : \ c < \psi(x) \text{ and } \delta(x) < b\} \neq \emptyset.$$

We have shown that condition (A1) of Theorem 2.4 is satisfied.

We now verify that condition (A2) of Theorem 2.4, $\alpha(Tx) \geqslant a$ for all $x \in B$, is satisfied. Let $x \in B$. Apply condition (a) of Theorem 3.5, and

$$\begin{aligned}\alpha(Tx) &= \int_0^n G(\tau,s)\, f(x(s))\, ds \geqslant \left(\frac{4d}{n(\nu-\tau)}\right) \int_\tau^\nu G(\tau,s)\, ds \\ &= \left(\frac{2a}{\tau(\nu-\tau)}\right)\left(\tau(\nu-\tau)[1-\frac{(\nu+\tau)}{2n}]\right) \geqslant \frac{2a}{2} = a.\end{aligned}$$

We now verify that condition (A3) of Theorem 2.4, $\alpha(Tx) \geqslant a$, for all $x \in A$ with $\delta(Tx) > b$, is satisfied. Let $x \in A$ with $\delta(Tx) > b$. Apply Lemma 3.2 to obtain

$$\begin{aligned}\alpha(Tx) &= \int_0^n G(\tau,s)\, f(x(s))\, ds \geqslant \left(\frac{\tau}{\nu}\right) \int_0^n G(\nu,s)\, f(x(s))\, ds \\ &= \left(\frac{\tau}{\nu}\right) \delta(Tx) > \left(\frac{\tau}{\nu}\right) (\frac{2d\nu}{n}) = a.\end{aligned}$$

We now verify that condition (A4) of Theorem 2.4, $\beta(Tx) \leqslant d$, for all $x \in C$, is satisfied. Let $x \in C$. Since $c = \frac{2d\mu}{n}$, $0 < L \leqslant \frac{2d\mu}{n} = c$. The concavity of x implies, for $s \in [0,\mu]$,

$$x(s) \geqslant \frac{cs}{\mu} \geqslant \frac{Ls}{\mu}.$$

Let $t \leqslant \mu$ be such that $x(t) = L$. Apply properties (b) and (c) of Theorem 3.5, to obtain

$$\begin{aligned}\beta(Tx) &= \int_0^n G(\frac{n}{2},s)\, f(x(s))\, ds = \int_0^{\frac{n}{2}} s\, f(x(s))\, ds \\ &= \int_0^t s\, f(x(s))\, ds + \int_t^{\frac{n}{2}} s\, f(x(s))\, ds \\ &\leqslant \int_0^t s\, f\left(\frac{Ls}{\mu}\right)\, ds + f(L)\int_t^{\frac{n}{2}} s\, ds \\ &\leqslant \frac{2d - f(L)((\frac{n}{2})^2 - t^2)}{2} + \frac{f(L)((\frac{n}{2})^2 - t^2)}{2} = d.\end{aligned}$$

We close the proof by verifying that condition (A5), $\beta(Tx) \leqslant d$, for all $x \in A$ with $\psi(Tx) < c$ is satisfied. Let $x \in A$ with $\psi(Tx) < c$. Apply Lemma 3.2 to obtain

$$\beta(Tx) = \int_0^n G(\frac{n}{2},s)\, f(x(s))\, ds \leqslant \left(\frac{n}{2\mu}\right) \int_0^n G(\mu,s)\, f(x(s))\, ds$$
$$= \left(\frac{n}{2\mu}\right) Tx(\mu) = \left(\frac{n}{2\mu}\right) \psi(Tx) \leqslant \left(\frac{n}{2\mu}\right) c = d.$$

Therefore, the hypotheses of Theorem 2.4 have been satisfied; thus the operator T has at least one positive solution $x^* \in A(\alpha,\beta,a,d)$. □

4. The higher order problem

In this section, we apply Theorem 2.4 to a two point boundary value problem for a kth order ordinary differential equation. Throughout this section, we shall assume that $k \geqslant 3$. The concept of concavity is extended to kth order differential inequalities as developed in [5]. We consider the kth order problem

$$-x^{(k)}(t) = f(x(t)), \quad t \in [0,n], \tag{4.1}$$

$$x(0) = x'(0) = ... = x^{(k-2)}(0) = 0, \quad x^{(k-2)}(n) = 0, \tag{4.2}$$

where $f : \mathbb{R} \to \mathbb{R}^+$ is a continuous map.

The Green's function for this problem has the form,

$$G(t,s) = \begin{cases} \frac{t^{k-1}(n-s)}{(k-1)!n} - \frac{(t-s)^{k-1}}{(k-1)!} & : 0 \leqslant s < t \leqslant n, \\ \frac{t^{k-1}(n-s)}{(k-1)!n} & : 0 \leqslant t < s \leqslant n. \end{cases}$$

Again, let $E = C[0,n]$, equiped with the usual supremum norm, denote the Banach space. Define the cone $P \subset E = C[0,n]$ by

$$P := \{x \in E : x \text{ is nonnegative and nondecreasing on } [0,n],$$
$$\text{and if } 0 < y \leqslant w \leqslant n, \text{ then } w^{k-1}x(y) \geqslant y^{k-1}x(w)\}.$$

Define $T : E \to E$ by

$$Tx = \int_0^n G(t,s) f(x(s)) ds.$$

For fixed $\nu, \tau, \mu \in [0,n]$ and $x \in P$, define the concave functionals α and ψ on P as before by

$$\alpha(x) := \min_{t \in [\tau,n]} x(t) = x(\tau), \;\; \psi(x) := \min_{t \in [\mu,n]} x(t) = x(\mu),$$

and the convex functionals δ and β on P as before by

$$\delta(x) := \max_{t\in[0,\nu]}(x(t)) = x(\nu), \;\; \beta(x) := \max_{t\in[0,n]} x(t) = x(n).$$

LEMMA 4.1. *For any $y,w \in [0,n]$ with $0 < y \leqslant w$,*

$$\min_{s\in[0,n]} \frac{G(y,s)}{G(w,s)} \geqslant \frac{y^{k-1}}{w^{k-1}}. \tag{4.3}$$

Proof. For $0 < y \leqslant w \leqslant s$,

$$\frac{G(y,s)}{G(w,s)} = \frac{y^{k-1}(\frac{n-s}{(k-1)!n})}{w^{k-1}(\frac{n-s}{(k-1)!n})} = \frac{y^{k-1}}{w^{k-1}}.$$

For $0 < y \leqslant s \leqslant w$,

$$\frac{G(y,s)}{G(w,s)} = \frac{y^{k-1}\frac{n-s}{(k-1)!n}}{w^{k-1}\frac{n-w}{(k-1)!n} - \frac{(w-s)^{k-1}}{(k-1)!}} \geqslant \frac{\frac{y^{k-1}(n-w)}{(k-1)!n}}{\frac{w^{k-1}(n-w)}{(k-1)!n}} = \frac{y^{k-1}}{w^{k-1}}.$$

For $0 < s \leqslant y \leqslant w$,

$$\frac{(y-s)^{k-1}}{(k-1)!} \leqslant \frac{y^{k-1}(w-s)^{k-1}}{(k-1)!w^{k-1}}.$$

Thus,

$$\begin{aligned} G(y,s) &= \frac{y^{k-1}(n-s)}{(k-1)!n} - \frac{(y-s)^{k-1}}{(k-1)!} \geqslant \frac{y^{k-1}(n-s)}{(k-1)!n} - \frac{y^{k-1}(w-s)^{k-1}}{(k-1)!w^{k-1}} \\ &= \frac{y^{k-1}}{w^{k-1}}\Big(\frac{w^{k-1}(n-s)}{(k-1)!n} - \frac{(w-s)^{k-1}}{(k-1)!}\Big) = \frac{y^{k-1}}{w^{k-1}}G(w,s). \qquad \square \end{aligned}$$

REMARK 4.2. Assume x satisfies the differential inequality $-x^{(k)}(t) \geqslant 0$, $0 < t < n$, and x satisfies the boundary conditions (4.2). Then $0 < y \leqslant w \leqslant n$, implies

$$w^{k-1}x(y) \leqslant y^{k-1}x(w). \tag{4.4}$$

This is the extended concept of concavity that will be employed in the proof of the following theorem. Note that Lemma 4.1 merely states the extended concavity of G. To see that (4.4) is true, let $0 < y < w \leqslant n$, and let $p(t)$ denote the solution of the boundary value problem,

$$-p^{(k)}(t) = 0, \quad 0 < t < n,$$

$$p(0) = p'(0) = ... = p^{(k-2)}(0) = 0, \quad p(w) = x(w).$$

Then $p(t) = \frac{x(w)}{w^{k-1}} t^{k-1}$. Moreover, $z = x - p$ satisfies a boundary value problem for a differential inequality of the form

$$-z^{(k)}(t) \geqslant 0, \quad z(0) = z'(0) = ... = z^{(k-2)}(0) = 0, \quad z(w) = 0.$$

The Green's function for this conjugate boundary value problem is positive and so $x(t) \geqslant p(t)$, $0 < t < w$; evaluate this inequality at $t = y$ to obtain (4.4).

THEOREM 4.3. *Assume* $\tau, \nu, \mu \in (0, \frac{n}{2}]$ *are fixed with* $\tau \leqslant \mu < \nu$, d *and* L *are positive real numbers with* $0 < L \leqslant \frac{d\mu^{k-1}}{n^{k-1}}$ *and* $f : [0,\infty) \to [0,\infty)$ *is a continuous function such that*

(a) $f(w) \geqslant \frac{2(k-1)!d}{n^{k-1}(\nu-\tau)}$ *for* $w \in [\frac{\tau^{k-1}d}{n^{k-1}}, \frac{\nu^{k-1}d}{n^{k-1}}]$,

(b) $f(w)$ *is decreasing for* $w \in [0,L]$ *with* $f(L) \geqslant f(w)$ *for* $w \in [L,d]$, *and*

(c) $\int_0^t \frac{(n-s)(n^{k-2}-(n-s)^{k-2})}{(k-1)!} f(\frac{Ls^{k-1}}{\mu^{k-1}}) ds \leqslant \frac{2k!d - f(L)(kn^{k-2}(n-t)^2 - 2(n-t)^k)}{2k!}, \quad 0 \leqslant t \leqslant \mu.$

Then the operator T *has at least one positive solution* $x^* \in A(\alpha, \beta, \frac{\tau^{k-1}d}{n^{k-1}}, d)$.

Proof. Let $a = \frac{\tau^{k-1}d}{n^{k-1}}$, $b = \frac{\nu^{k-1}d}{n^{k-1}}$, and $c = \frac{\mu^{k-1}d}{n^{k-1}}$. Let $x \in A(\alpha, \beta, a, d)$. As before,

$$T : A(\alpha, \beta, a, d) \to P,$$

and T is a completely continuous operator. Also, if $x \in P$ and $\beta(x) > d$, then by the properties of the cone P, in particular, the kth order concavity of x,

$$\alpha(x) = x(\tau) \geqslant \left(\frac{\tau^{k-1}}{n^{k-1}}\right) x(n) = \frac{\tau^{k-1}}{n^{k-1}} \beta(x) > \frac{\tau^{k-1}d}{n^{k-1}} = a.$$

Thus,

$$\{x \in P : \alpha(x) < a \text{ and } d < \beta(x)\} = \emptyset.$$

For $r \in (\frac{2k!d}{n^{k-1}(kn-2\mu)}, \frac{2k!d}{n^{k-1}(kn-2\nu)})$ define x_r by

$$x_r(t) = r \int_0^n G(t,s) ds = \frac{rt^{k-1}(kn-2t)}{2k!}.$$

We claim $x_r \in A$.

$$\alpha(x_r) = x_r(\tau) = \frac{r\tau^{k-1}(kn-2\tau)}{2k!} > \frac{\tau^{k-1}d(kn-2\tau)}{n^{k-1}(kn-2\mu)} \geqslant \frac{\tau^{k-1}}{n^{k-1}} d = a.$$

$$\beta(x_r) = x_r(n) = \frac{rn^{k-1}(kn-2n)}{2k!} < \frac{d(kn-2n)}{kn-2\nu} \leqslant d.$$

Thus, $x_r \in A$. Moreover,

$$\psi(x_r) = x_r(\mu) = \frac{r\mu^{k-1}(kn-2\mu)}{2k!} > \frac{2k!d}{n^{k-1}(kn-2\mu)}\frac{\mu^{k-1}(kn-2\mu)}{2k!} = \frac{d\mu^{k-1}}{n^{k-1}} = c$$

and

$$\delta(x_r) = x_r(\nu) = \frac{r\nu^{k-1}(kn-2\nu)}{2k!} < \frac{2k!d}{n^{k-1}(kn-2\nu)}\frac{\nu^{k-1}(kn-2\nu)}{2k!} = \frac{d\nu^{k-1}}{n^{k-1}} = b.$$

We now verify that condition (A2), $\alpha(Tx) \geqslant a$ for all $x \in B$, is satisfied. Let $x \in B$. Apply condition (a) of Theorem 3.5, and

$$\begin{aligned}\alpha(Tx) &= \int_0^n G(\tau,s)\, f(x(s))\, ds \geqslant \left(\frac{2(k-1)!d}{n^{k-1}(\nu-\tau)}\right)\int_\tau^\nu G(\tau,s)\, ds \\ &= \left(\frac{2(k-1)!a}{\tau^{k-1}(\nu-\tau)}\right)\left(\frac{\tau^{k-1}(\nu-\tau)}{(k-1)!}\left[1-\frac{(\nu+\tau)}{2n}\right]\right) \geqslant \frac{2a}{2} = a.\end{aligned}$$

We now verify that condition (A3), $\alpha(Tx) \geqslant a$, for all $x \in A$ with $\delta(Tx) > b$, is satisfied. Let $x \in A$ with $\delta(Tx) > b$. Apply Theorem 3.2 to obtain

$$\begin{aligned}\alpha(Tx) &= \int_0^n G(\tau,s)\, f(x(s))\, ds \geqslant \left(\frac{\tau^{k-1}}{\nu^{k-1}}\right)\int_0^n G(\nu,s)\, f(x(s))\, ds \\ &= \left(\frac{\tau^{k-1}}{\nu^{k-1}}\right)\delta(Tx) > \left(\frac{\tau^{k-1}}{\nu^{k-1}}\right)\left(\frac{d\nu^{k-1}}{n^{k-1}}\right) = a.\end{aligned}$$

We now verify that condition (A4), $\beta(Tx) \leqslant d$, for all $x \in C$, is satisfied. Let $x \in C$. Since $c = \frac{d\mu^{k-1}}{n^{k-1}}$, $0 < L \leqslant \frac{d\mu^{k-1}}{n^{k-1}} = c$. The concavity of x implies, for $s \in [0, \mu^{k-1}]$,

$$x(s) \geqslant \frac{cs^{k-1}}{\mu^{k-1}} \geqslant \frac{Ls^{k-1}}{\mu^{k-1}}.$$

Let $t \leqslant \mu$ be such that $x(t) = L$. Apply properties (b) and (c) of Theorem 4.3, to obtain

$$\begin{aligned}\beta(Tx) &= \int_0^n G(n,s)\, f(x(s))\, ds \\ &= \int_0^n \left(\frac{n^{k-1}(n-s)}{(k-1)!n} - \frac{(n-s)^{k-1}}{(k-1)!}\right) f(x(s))\, ds \\ &= \int_0^t \left(\frac{n^{k-1}(n-s)}{(k-1)!n} - \frac{(n-s)^{k-1}}{(k-1)!}\right) f(x(s))\, ds \\ &\quad + \int_t^n \left(\frac{n^{k-1}(n-s)}{(k-1)!n} - \frac{(n-s)^{k-1}}{(k-1)!}\right) f(x(s))\, ds \\ &\leqslant \int_0^t \left(\frac{n^{k-1}(n-s)}{(k-1)!n} - \frac{(n-s)^{k-1}}{(k-1)!}\right) f\left(\frac{Ls^{k-1}}{\mu^{k-1}}\right) ds\end{aligned}$$

$$+f(L)\int_t^n \Big(\frac{n^{k-1}(n-s)}{(k-1)!n} - \frac{(n-s)^{k-1}}{(k-1)!}\Big)\, ds$$
$$\leqslant \frac{2k!d - f(L)(kn^{k-2}(n-t)^2 - 2(n-t)^k)}{2k!}$$
$$+\frac{f(L)(kn^{k-2}(n-t)^2 - 2(n-t)^k)}{2k!} = d.$$

We close the proof by verifying that condition (A5), $\beta(Tx) \leqslant d$, for all $x \in A$ with $\psi(Tx) < c$. Let $x \in A$ with $\psi(Tx) < c$. Thus by Lemma 4.1,

$$\beta(Tx) = \int_0^n G(n,s)\, f(x(s))\, ds \leqslant \left(\frac{n^{k-1}}{\mu^{k-1}}\right)\int_0^n G(\mu,s)\, f(x(s))\, ds$$
$$= \left(\frac{n^{k-1}}{\mu^{k-1}}\right) Tx(\mu) = \left(\frac{n^{k-1}}{\mu^{k-1}}\right)\psi(Tx) \leqslant \left(\frac{n^{k-1}}{\mu^{k-1}}\right) c = d.$$

Therefore, the hypotheses of Theorem 2.4 have been satisfied; thus the operator T has at least one positive solution $x^* \in A(\alpha,\beta,a,d)$. □

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