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INEQUALITIES FOR THE JACOBIAN ELLIPTIC FUNCTIONS WITH COMPLEX MODULUS

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Abstract. Despite the fact that there is a huge amount on papers and books devoted to the theory of Jacobian elliptic functions, very little is known when the modulus k of these functions lies outside the unit interval $[0,1]$. In this note, we prove some simple inequalities for the absolute value of Jacobian elliptic functions with complex modulus.

1. Introduction and Main Result

Consider the Jacobian elliptic functions $\operatorname{sn}(z,k)$, $\operatorname{cn}(z,k)$, $\operatorname{dn}(z,k)$ with complex parameter (modulus) $k \in \mathbb{C}$. Starting with the works of Jacobi in the 1820's until now, there exists a huge amount on papers and books devoted to the theory of Jacobian elliptic functions, see, e.g., [3], [2], or [6]. Almost all of these contributions are restricted to the case when the modulus k is in the unit interval $[0,1]$. Exceptions are, e.g., the articles of Walker [4], [5]. Since the functions $\operatorname{sn}(z,k)$, $\operatorname{cn}(z,k)$, and $\operatorname{dn}(z,k)$ depend on k^2 rather than k, we shall use $m = k^2$ as parameter but use the same notation $\operatorname{sn}(z,m)$, $\operatorname{cn}(z,m)$, and $\operatorname{dn}(z,m)$.

There seems to exist no estimates for the absolute value of $\operatorname{sn}(z,m)$, $\operatorname{cn}(z,m)$, and $\operatorname{dn}(z,m)$ in terms of elementary functions. In this paper, we give such estimates.

THEOREM 1. *Let $m \in \mathbb{C}$, $|m| \leqslant 1$, and $z \in \mathbb{C}$, $|z| \leqslant R < \frac{\pi}{2}$. Then the inequalities*

$$|\operatorname{sn}(z,m)| \leqslant \frac{\operatorname{sn}(|z|,m_1)}{\operatorname{cn}(|z|,m_1)} \leqslant \tan|z|, \tag{1}$$

$$|\operatorname{cn}(z,m)| \leqslant \frac{1}{\operatorname{cn}(|z|,m_1)} \leqslant \frac{1}{\cos|z|}, \tag{2}$$

$$|\operatorname{dn}(z,m)| \leqslant \frac{\operatorname{dn}(|z|,m_1)}{\operatorname{cn}(|z|,m_1)} \leqslant \frac{1}{\cos|z|}, \tag{3}$$

hold, where $m_1 := 1-|m| \in [0,1]$. For all inequalities we have equality if $z = 0$ or if $z = \mathrm{i}y$, $-R \leqslant y \leqslant R$, and $m = 1$. Further, in the first inequalities of (1)–(3)*, equality is attained if $z = \mathrm{i}y$, $-R \leqslant y \leqslant R$, and $m = |m|$.*

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2. Proof

Proof of Theorem 1. Starting point for the proof are the Taylor expansions of $\operatorname{sn}(z,m)$, $\operatorname{cn}(z,m)$, and $\operatorname{dn}(z,m)$, respectively, given by [4, Eq. (3.6)]

$$\operatorname{sn}(z,m) = \sum_{n=0}^{\infty} (-1)^n s_n(m) \frac{z^{2n+1}}{(2n+1)!}, \tag{4}$$

$$\operatorname{cn}(z,m) = \sum_{n=0}^{\infty} (-1)^n c_n(m) \frac{z^{2n}}{(2n)!}, \tag{5}$$

$$\operatorname{dn}(z,m) = \sum_{n=0}^{\infty} (-1)^n d_n(m) \frac{z^{2n}}{(2n)!}, \tag{6}$$

where $s_n(m)$, $c_n(m)$, and $d_n(m)$ are polynomials with positive integer coefficients. Thus, we have the inequalities

$$|s_n(m)| \leqslant s_n(|m|) \leqslant s_n(1), \tag{7}$$

$$|c_n(m)| \leqslant c_n(|m|) \leqslant c_n(1), \tag{8}$$

$$|d_n(m)| \leqslant d_n(|m|) \leqslant d_n(1). \tag{9}$$

Note that each of the power series (7)–(9) are absolutely convergent for $|m| \leqslant 1$ and $|z| < \frac{\pi}{2}$, see [4, Thm. 3.2]. Especially,

$$\operatorname{sn}(\mathrm{i}z,m) = \mathrm{i} \sum_{n=0}^{\infty} s_n(m) \frac{z^{2n+1}}{(2n+1)!}, \tag{10}$$

$$\operatorname{cn}(\mathrm{i}z,m) = \sum_{n=0}^{\infty} c_n(m) \frac{z^{2n}}{(2n)!}, \tag{11}$$

$$\operatorname{dn}(\mathrm{i}z,m) = \sum_{n=0}^{\infty} d_n(m) \frac{z^{2n}}{(2n)!}. \tag{12}$$

If we put $m = 1$ in formulae (10)–(12), we get (note that $c_n(1) = d_n(1)$)

$$\tanh(\mathrm{i}z) = \mathrm{i} \sum_{n=0}^{\infty} s_n(1) \frac{z^{2n+1}}{(2n+1)!}, \tag{13}$$

$$\frac{1}{\cosh(\mathrm{i}z)} = \sum_{n=0}^{\infty} c_n(1) \frac{z^{2n}}{(2n)!} = \sum_{n=0}^{\infty} d_n(1) \frac{z^{2n}}{(2n)!}. \tag{14}$$

Hence, the inequalities

$$
\begin{aligned}
|\operatorname{sn}(z,m)| &\leqslant \sum_{n=0}^{\infty} |s_n(m)| \frac{|z|^{2n+1}}{(2n+1)!} && \text{by (4)} \\
&\leqslant \sum_{n=0}^{\infty} s_n(|m|) \frac{|z|^{2n+1}}{(2n+1)!} =: (*_1) && \text{by (7)} \\
&= \frac{1}{\mathrm{i}} \operatorname{sn}(\mathrm{i}|z|, |m|) && \text{by (10)} \\
&= \frac{\operatorname{sn}(|z|, 1-|m|)}{\operatorname{cn}(|z|, 1-|m|)} && \text{by [2, Eq. (2.6.12)]}
\end{aligned}
$$

and

$$
\begin{aligned}
|\operatorname{cn}(z,m)| &\leqslant \sum_{n=0}^{\infty} |c_n(m)| \frac{|z|^{2n}}{(2n)!} && \text{by (5)} \\
&\leqslant \sum_{n=0}^{\infty} c_n(|m|) \frac{|z|^{2n}}{(2n)!} =: (*_2) && \text{by (8)} \\
&= \operatorname{cn}(\mathrm{i}|z|, |m|) && \text{by (11)} \\
&= \frac{1}{\operatorname{cn}(|z|, 1-|m|)} && \text{by [2, Eq. (2.6.12)]}
\end{aligned}
$$

and

$$
\begin{aligned}
|\operatorname{dn}(z,m)| &\leqslant \sum_{n=0}^{\infty} |d_n(m)| \frac{|z|^{2n}}{(2n)!} && \text{by (6)} \\
&\leqslant \sum_{n=0}^{\infty} d_n(|m|) \frac{|z|^{2n}}{(2n)!} =: (*_3) && \text{by (9)} \\
&= \operatorname{dn}(\mathrm{i}|z|, |m|) && \text{by (12)} \\
&= \frac{\operatorname{dn}(|z|, 1-|m|)}{\operatorname{cn}(|z|, 1-|m|)} && \text{by [2, Eq. (2.6.12)]}
\end{aligned}
$$

hold and we have proved the first inequalities of (1)–(3), respectively. Analogously,

$$
\begin{aligned}
(*_1) &\leqslant \sum_{n=0}^{\infty} s_n(1) \frac{|z|^{2n+1}}{(2n+1)!} && \text{by (8)} \\
&= \frac{1}{\mathrm{i}} \tanh(\mathrm{i}|z|) && \text{by (13)} \\
&= \tan|z|
\end{aligned}
$$

and

$$\begin{aligned}(*_2),(*_3) &\leqslant \sum_{n=0}^{\infty} c_n(1)\,\frac{|z|^{2n}}{(2n)!} \qquad \text{by (8)}\\ &= \frac{1}{\cosh(\mathrm{i}|z|)} \qquad \text{by (14)}\\ &= \frac{1}{\cos|z|}\end{aligned}$$

holds. □

REMARK 1. Another (much more complicated) possibility for proving the second inequalities of (1)–(3), respectively, is the following: For fixed $u \in [0, \frac{\pi}{2}]$, consider the functions

$$f_1(m_1) := \frac{\mathrm{sn}(u,m_1)}{\mathrm{cn}(u,m_1)},\ f_2(m_1) := \frac{1}{\mathrm{cn}(u,m_1)},\ f_3(m_1) := \frac{\mathrm{dn}(u,m_1)}{\mathrm{cn}(u,m_1)}$$

for $0 \leqslant m_1 \leqslant 1$. Then one has to prove that these functions are strictly monotone decreasing in m_1, i.e. $f_j'(m_1) < 0$, $j = 1,2,3$, using formulae (710.54), (710.57), and (710.60) of [1].

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