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A NEW INTERPRETATION OF CHEBYSHEV'S INEQUALITY FOR SEQUENCES OF REAL NUMBERS AND QUASI–ARITHMETIC MEANS

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Abstract. The main purpose of the paper is to give a new interpretation of Chebyshev's inequality for the sequences of real numbers from a standpoint of composition functions. As an application, an n-version of the concavity (or the convexity) of a quasi-arithmetic mean function is obtained under some conditions.

1. Introduction

The well-known Chebyshev's inequality for sequences of real numbers may be as follows:

$$(x_1 + \dots + x_n)(y_1 + \dots + y_n) \leqslant n(x_1y_1 + \dots + x_ny_n)$$

holds whenever both $\{x_1,\dots,x_n\}$ and $\{y_1,\dots,y_n\}$ are simultaneously monotone increasing or monotone decreasing. We give a generalization of this inequality from a standpoint of composition functions (see Theorem 1). However, this generalization can be regarded as a new interpretation of Chebyshev's inequality.

Throughout the paper, we denote by (Ω,Σ,μ), I and f a probability space, an interval of $\mathbb{R}$ and a real-valued Σ-measurable function on Ω with $f(\omega) \in I$ for almost all $\omega \in \Omega$, respectively. Let $C(I)$ be the real linear space of all continuous real-valued functions defined on I. Let $C^+_{sm}(I)$ (resp. $C^-_{sm}(I)$) be the set of all $\varphi \in C(I)$ which is strictly monotone increasing (resp. decreasing) on I. Put $C_{sm}(I) = C^+_{sm}(I) \cup C^-_{sm}(I)$. Then $C_{sm}(I)$ denotes the set of all strictly monotone continuous functions on I.

Let $C_{sm,f}(I)$ be the set of all $\varphi \in C_{sm}(I)$ with $\varphi \circ f \in L^1(\Omega,\Sigma,\mu)$. Let φ be an arbitrary function of $C_{sm,f}(I)$. Since $\varphi(I)$ is an interval of $\mathbb{R}$ and μ is a probability measure on Ω, it follows that

$$\int \varphi \circ f d\mu \in \varphi(I).$$

Then there exists a unique real number $M_\varphi(f) \in I$ such that $\int \varphi \circ f d\mu = \varphi(M_\varphi(f))$. Since φ is one-to-one, it follows that

$$M_\varphi(f) = \varphi^{-1}\left(\int \varphi \circ f d\mu\right).$$

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We call $M_\varphi(f)$ a φ–quasi-arithmetic mean of f with respect to μ (or simply, φ–mean of f). These means are somewhat different from C^1-means introducing by J. I. Fujii, et al. [1], but they include many known numerical means (cf. [3]).

As an application of Theorem 1, we show that a quasi-arithmetic mean function: $\varphi \to M_\varphi(f)$ has an n-version of the concavity on a suitable convex subset of $C_{sm}(I)$ under some conditions (see Theorem 2). Also this function has an n-version of the convexity on a suitable convex subset of $C_{sm}(I)$ under another conditions (see Theorem 3).

2. Lemmas

The first lemma describes a geometric property of convex function, but this will be standard, so we will omit the proof (cf. [3, Lemma 1]).

LEMMA 1. *Let φ be a real-valued function on I. Then φ is convex (resp. strictly convex) on I if and only if whenever c is in the interior of I a function $\lambda_{c,\varphi}$ defined by*

$$\lambda_{c,\varphi}(x) = \frac{\varphi(x)-\varphi(c)}{x-c} \quad (x \in I \setminus \{c\})$$

is monotone increasing (resp. strictly monotone increasing) on $I \setminus \{c\}$.

The following result is our key lemma in which one can feel an atmosphere of Chebyshev's inequality for sequences of real numbers. This follows from the proof of [3, Lemma 5] but we will give a proof for the sake of completeness.

LEMMA 2. *Let φ and ψ be two functions on I such that $\psi - \varphi$ is monotone increasing on I and ψ is convex on I. Then*

$$((1-t)\varphi + t\psi)((1-t)x+ty) \leqslant (1-t)\varphi(x) + t\psi(y) \tag{1}$$

holds for all $t \in (0,1)$ and $x,y \in I$ with $x \leqslant y$.

The equality holds in (1) if and only if $x = y$, otherwise, $(\psi-\varphi)(x) = (\psi-\varphi)(z)$ and $\lambda_{z,\psi}(x) = \lambda_{z,\psi}(y)$, where $z = (1-t)x+ty$.

Proof. Suppose that $\psi - \varphi$ is monotone increasing on I and ψ is convex on I. Let $x,y \in I$ with $x \leqslant y$ and $t \in (0,1)$. If $x = y$, then (1) is clearly true. Indeed, the equality holds in that case. Then we suppose that $x < y$ and put $z = (1-t)x+ty$. Since $x < z < y$ and $\psi - \varphi$ is monotone increasing on I, it follows that

$$\psi(z) - \psi(x) - \varphi(z) + \varphi(x) \geqslant 0.$$

Also since ψ is convex on I, it follows from Lemma 1 that $\lambda_{z,\psi}$ is monotone increasing on $I\setminus\{z\}$. Therefore we have

$$\begin{aligned}
&(1-t)\varphi(x)+t\psi(y)-((1-t)\varphi+t\psi)(z)\\
=&t(\psi(y)-\psi(z))-(1-t)(\varphi(z)-\varphi(x))\\
\geqslant&t(\psi(y)-\psi(z))-(1-t)(\psi(z)-\psi(x))\\
=&t(1-t)(y-x)\left(\frac{\psi(y)-\psi(z)}{(1-t)(y-x)}-\frac{\psi(z)-\psi(x)}{t(y-x)}\right)\\
=&t(1-t)(y-x)\left(\frac{\psi(y)-\psi(z)}{y-z}-\frac{\psi(x)-\psi(z)}{x-z}\right)\\
=&t(1-t)(y-x)(\lambda_{z,\psi}(y)-\lambda_{z,\psi}(x))\\
\geqslant&0,
\end{aligned}$$

and so (1) holds. Moreover, we see from the above equations that the equality holds in (1) if and only if

$$t(\psi(y)-\psi(z))-(1-t)(\varphi(z)-\varphi(x))=t(\psi(y)-\psi(z))-(1-t)(\psi(z)-\psi(x))$$

and

$$t(1-t)(y-x)(\lambda_{z,\psi}(y)-\lambda_{z,\psi}(x))=0.$$

Since $t(1-t)(x-y)\neq 0$, it follows that the equality holds in (1) if and only if $(\psi-\varphi)(x)=(\psi-\varphi)(z)$ and $\lambda_{z,\psi}(x)=\lambda_{z,\psi}(y)$. □

REMARK 1. If we replace "increasing" and "convex" by "decreasing" and "concave", respectively in Lemma 2, then we obtain the reverse inequality of (1)

3. Main result

The following result is an n-version of Lemma 2 but it is also a generalization of a weighted Chebyshev's inequality for sequences of real numbers (see Corollary 1).

THEOREM 1. *Let I and J be two intervals of $\mathbb{R}$. Let $n\geqslant 2$ and $w_1,\ldots,w_n>0$ with $\sum_{i=1}^n w_i=1$. Suppose that $\varphi_1,\ldots,\varphi_n$ are real-valued functions on I such that $\sum_{i=1}^k w_i(\varphi_{k+1}-\varphi_i)$ is monotone increasing on I and φ_{k+1} is convex on I for each $k=1,\ldots,n-1$, and that $\psi_1,\ldots,\psi_n$ are functions from J to I such that $\sum_{i=1}^k w_i(\psi_{k+1}-\psi_i)\geqslant 0$ on J for each $k=1,\ldots,n-1$. Then*

$$\sum_{i=1}^n w_i\varphi_i\circ\sum_{i=1}^n w_i\psi_i\leqslant\sum_{i=1}^n w_i(\varphi_i\circ\psi_i) \tag{2}$$

holds on J.

If $\psi_1=\ldots=\psi_n$, then the equality holds in (2). Conversely if the equality holds in (2), then $\psi_1=\ldots=\psi_n$, under the assumption that either $\sum_{i=1}^k w_i(\varphi_{k+1}-\varphi_i)$ is strictly monotone increasing on I or φ_{k+1} is strictly convex on I for each $k=1,\ldots,n-1$.

Proof. Let $x \in J$ be arbitrary. Put $x_k = \psi_k(x)$ and $W(k) = \sum_{i=1}^k w_i$ for each $k = 1,\ldots,n$. By hypothesis, we see easily that $\sum_{i=1}^k \frac{w_i}{W(k)} x_i \leqslant x_{k+1}$ and $\varphi_{k+1} - \sum_{i=1}^k \frac{w_i}{W(k)} \varphi_i$ is monotone increasing on I for each $k = 1,\ldots,n-1$. Since φ_{k+1} is convex on I, it follows from Lemma 2 that

$$\begin{aligned}&\frac{W(k)}{W(k+1)} \sum_{i=1}^k \frac{w_i}{W(k)} \varphi_i \left(\sum_{i=1}^k \frac{w_i}{W(k)} x_i \right) + \frac{w_{k+1}}{W(k+1)} \varphi_{k+1}(x_{k+1}) \\ &\geqslant \sum_{i=1}^{k+1} \frac{w_i}{W(k+1)} \varphi_i \left(\sum_{i=1}^{k+1} \frac{w_i}{W(k+1)} x_i \right)\end{aligned}$$

for each $k = 1,\ldots,n-1$. These inequalities yield easily the following inequalities:

$$\sum_{i=1}^k w_i \varphi_i \left(\sum_{i=1}^k \frac{w_i}{W(k)} x_i \right) + w_{k+1} \varphi_{k+1}(x_{k+1}) \geqslant \sum_{i=1}^{k+1} w_i \varphi_i \left(\sum_{i=1}^{k+1} \frac{w_i}{W(k+1)} x_i \right)$$

for each $k = 1,\ldots,n-1$. Therefore we have

$$\begin{aligned}\sum_{i=1}^n w_i \varphi_i(x_i) &\geqslant \sum_{i=1}^2 w_i \varphi_i \left(\sum_{i=1}^2 \frac{w_i}{W(2)} x_i \right) + \sum_{i=3}^n w_i \varphi_i(x_i) \\ &\geqslant \sum_{i=1}^3 w_i \varphi_i \left(\sum_{i=1}^3 \frac{w_i}{W(3)} x_i \right) + \sum_{i=4}^n w_i \varphi_i(x_i) \\ &\ \ \vdots \\ &\geqslant \sum_{i=1}^n w_i \varphi_i \left(\sum_{i=1}^n \frac{w_i}{W(n)} x_i \right) \\ &= \sum_{i=1}^n w_i \varphi_i \left(\sum_{i=1}^n w_i x_i \right)\end{aligned}$$

and then

$$\sum_{i=1}^n w_i (\varphi_i \circ \psi_i)(x) \geqslant \left(\sum_{i=1}^n w_i \varphi_i \circ \sum_{i=1}^n w_i \psi_i \right)(x).$$

Since x is arbitrary, we obtain the desired inequality (2).

Now if $\psi_1 = \ldots = \psi_n$ holds, we can easily verify that the equality holds in (2).

Conversely suppose that the equality holds in (2) and that either $\sum_{i=1}^k w_i(\varphi_{k+1} - \varphi_i)$ is strictly monotone increasing on I or φ_{k+1} is strictly convex on I for each $k = 1,\ldots,n-1$. Then the first assumtion implies that

$$\sum_{i=1}^k w_i \varphi_i \left(\sum_{i=1}^k \frac{w_i}{W(k)} x_i \right) + w_{k+1} \varphi_{k+1}(x_{k+1}) = \sum_{i=1}^{k+1} w_i \varphi_i \left(\sum_{i=1}^{k+1} \frac{w_i}{W(k+1)} x_i \right),$$

that is,

$$\frac{W(k)}{W(k+1)}\sum_{i=1}^{k}\frac{w_i}{W(k)}\varphi_i\left(\sum_{i=1}^{k}\frac{w_i}{W(k)}x_i\right)+\frac{w_{k+1}}{W(k+1)}\varphi_{k+1}(x_{k+1})$$
$$=\sum_{i=1}^{k+1}\frac{w_i}{W(k+1)}\varphi_i\left(\sum_{i=1}^{k+1}\frac{w_i}{W(k+1)}x_i\right)$$

must hold for each $k=1,...,n-1$ by the above argument. By Lemma 2, we have that for any $k=1,...,n-1$, $\sum_{i=1}^{k}\frac{w_i}{W(k)}x_i=x_{k+1}$, otherwise,

$$\left(\varphi_{k+1}-\sum_{i=1}^{k}\frac{w_i}{W(k)}\varphi_i\right)(y)=\left(\varphi_{k+1}-\sum_{i=1}^{k}\frac{w_i}{W(k)}\varphi_i\right)(z) \tag{3}$$

and

$$\lambda_{z,\varphi_{k+1}}(x_{k+1})=\lambda_{z,\varphi_{k+1}}(y) \tag{4}$$

where $y=\sum_{i=1}^{k}\frac{w_i}{W(k)}x_i$ and $z=\sum_{i=1}^{k+1}\frac{w_i}{W(k+1)}x_i$. However since $y<z<x_{k+1}$, it follows that either (3) or (4) do not occur for each $k=1,...,n-1$ by the second assumption (see the strict case of Lemma 1) and then $\sum_{i=1}^{k}\frac{w_i}{W(k)}x_i=x_{k+1}$ must hold for any $k=1,...,n-1$. This implies easily that $x_1=...=x_n$, that is, $\psi_1(x)=...=\psi_n(x)$. Since x is arbitrary, it follows that $\psi_1=...=\psi_n$. □

REMARK 2. (i) If we replace "increasing" and "convex" by "decreasing" and "concave", respectively in Theorem 1, then we obtain the reverse inequality of (2)

(ii) If all $\varphi_{i+1}-\varphi_i$ $(1\leqslant i\leqslant n-1)$ are monotone increasing on I, then all $\sum_{i=1}^{k}w_i(\varphi_{k+1}-\varphi_i)$ $(1\leqslant k\leqslant n-1)$ also are monotone increasing on I. However the converse does not hold (cf. Proof of Corollary 1 and Proposition 1).

(iii) If $\psi_1\leqslant...\leqslant\psi_n$ on J, then $\sum_{i=1}^{k}w_i(\psi_{k+1}-\psi_i)\geqslant 0$ for all $k=1,...,n-1$. However the converse does not hold (cf. Proof of Corollary 1 and Proposition 1).

The following inequality may be a weighted Chebyshev's inequality for sequences of real numbers.

COROLLARY 1. *If both sequences* $\{x_1,...,x_n\}$ *and* $\{y_1,...,y_n\}$ *in* $\mathbb{R}$ *are simultaneously monotone increasing or monotone decreasing, then*

$$\left(\sum_{i=1}^{n}w_ix_i\right)\left(\sum_{i=1}^{n}w_iy_i\right)\leqslant\sum_{i=1}^{n}w_ix_iy_i$$

for all $w_1,...,w_n>0$ *with* $\sum_{i=1}^{n}w_i=1$.

Proof. Suppose that both sequences $\{x_1,...,x_n\}$ and $\{y_1,...,y_n\}$ are simultaneously monotone increasing. Put $\varphi_i(x)=x_ix$ and $\psi_i(x)=y_i$ for each $i=1,2,...,n$ and $x\in\mathbb{R}$. Then we have $\psi_1\leqslant...\leqslant\psi_n$ on $\mathbb{R}$ and all $\varphi_{i+1}-\varphi_i$ are monotone increasing on $\mathbb{R}$ and all φ_i are convex on $\mathbb{R}$. Therefore the desired result follows immediately from Theorem 1 and Remark 2.

For the decreasing case, we also obtain the same result replacing x_i and y_i by $-x_i$ and $-y_i$, respectively in the above argument. □

REMARK 3. In case of $w_1 = ... = w_n = \frac{1}{n}$, it is well-known that this inequality follows from the rearrangement inequality. Also there are other proofs ([2, p. 108] and so on).

4. Applications

For each $\varphi \in C_{sm}(I)$, $t \in [0,1]$ and $x, y \in I$, put

$$x\nabla_{t,\varphi}\, y = \varphi^{-1}((1-t)\varphi(x) + t\varphi(y)).$$

This can be regarded as a φ–mean of $\{x,y\}$ with respect to a probability measure which represents a weighted arithmetic mean $(1-t)x + ty$.

For each $\varphi \in C_{sm}(I)$, denote by ∇_φ a three variable real-valued function:

$$(t,x,y) \to x\nabla_{t,\varphi}\, y$$

on $(0,1) \times \{(x,y) \in I^2 : x \neq y\}$. For each $\varphi, \psi \in C_{sm}(I)$, we write $\nabla_\varphi \leqslant \nabla_\psi$ (resp. $\nabla_\varphi < \nabla_\psi$) if

$$x\nabla_{t,\varphi}\, y \leqslant x\nabla_{t,\psi}\, y \quad (\text{resp. } x\nabla_{t,\varphi}\, y < x\nabla_{t,\psi}\, y)$$

for all $t \in (0,1)$ and $x, y \in I$ with $x \neq y$.

REMARK 4. The continuity of φ implies that $\nabla_\varphi \leqslant \nabla_\psi$ (resp. $\nabla_\varphi < \nabla_\psi$) if and only if

$$x\nabla_{\frac{1}{2},\varphi}\, y \leqslant x\nabla_{\frac{1}{2},\psi}\, y \quad (\text{resp. } x\nabla_{\frac{1}{2},\varphi}\, y < x\nabla_{\frac{1}{2},\psi}\, y)$$

for all $x, y \in I$ with $x \neq y$.

The following lemma is just [3, Theorem 1] which asserts that a φ–mean function: $\nabla_\varphi \to M_\varphi(f)$ is well-defined and order-preserving, and simultaneously gives a new interpretation of Jensen's inequality.

LEMMA 3. *Suppose that f is non-constant on I and $\varphi, \psi \in C_{sm,f}(I)$. Then*

(i) If $\nabla_\varphi \leqslant \nabla_\psi$ holds, then $M_\varphi(f) \leqslant M_\psi(f)$.

(ii) If $\nabla_\varphi < \nabla_\psi$ holds, then $M_\varphi(f) < M_\psi(f)$.

The following result is an n-version of [3, Theorem 3–(i)] and asserts that a quasi-arithmetic mean function: $\varphi \to M_\varphi(f)$ has an n-version of the concavity on a suitable convex subset of $C_{sm}(I)$.

THEOREM 2. *Let $n \geqslant 2$ and $w_1, ..., w_n > 0$ with $\sum_{i=1}^n w_i = 1$. Suppose that f is non-constant on I and that $\varphi_1, ..., \varphi_n \in C_{sm,f}(I)$ are such that* *(i)* $\nabla_{\varphi_1} \leqslant \nabla_{\varphi_2} \leqslant \ldots \leqslant \nabla_{\varphi_n}$,

(ii) all $\varphi_1, \sum_{i=1}^k w_i(\varphi_{k+1} - \varphi_i)$ $(k = 1, \ldots, n-1)$ *are monotone increasing on* I,

(iii) all $\varphi_2, \ldots, \varphi_n$ *are convex on* I.

Then $\sum_{i=1}^n w_i \varphi_i$ *is strictly monotone increasing on* I *and*

$$\sum_{i=1}^n w_i M_{\varphi_i}(f) \leqslant M_{\sum_{i=1}^n w_i \varphi_i}(f) \tag{5}$$

holds.

If $M_{\varphi_1}(f) = \ldots = M_{\varphi_n}(f)$, *then the equality holds in* (5). *Conversely if the equality holds in* (5), *then* $M_{\varphi_1}(f) = \ldots = M_{\varphi_n}(f)$, *under the assumption that either* $\sum_{i=1}^k w_i(\varphi_{k+1} - \varphi_i)$ *is strictly monotone increasing on* I *or* φ_{k+1} *is strictly convex on* I *for any* $k = 1, \ldots, n-1$.

Proof. Put $x_1 = M_{\varphi_1}(f), \ldots, x_n = M_{\varphi_n}(f)$. Then all $x_1, \ldots, x_n$ are in I and it follows that $x_1 \leqslant \ldots \leqslant x_n$ from the condition (i) and Lemma 3. Moreover put $\psi_1(x) = x_1, \ldots, \psi_n(x) = x_n$ for each $x \in I$. Since $\psi_1 \leqslant \ldots \leqslant \psi_n$ on I and then $\sum_{i=1}^k w_i(\psi_{k+1} - \psi_i) \geqslant 0$ on I for each $k = 1, \ldots, n-1$, it follows from the conditions (ii), (iii) and Theorem 1 that

$$\left(\sum_{i=1}^n w_i \varphi_i\right)\left(\sum_{i=1}^n w_i x_i\right) \leqslant \sum_{i=1}^n w_i \varphi_i(x_i). \tag{6}$$

Since φ_1 is strictly monotone increasing on I and $\varphi_2 - \varphi_1$ is monotone increasing on I, it follows that φ_2 also is strictly monotone increasing on I. Then $\frac{w_1\varphi_1 + w_2\varphi_2}{w_1 + w_2}$ is strictly monotone increasing on I. Moreover since $w_1(\varphi_3 - \varphi_1) + w_2(\varphi_3 - \varphi_2)$ is monotone increasing on I, it follows that $\varphi_3 - \frac{w_1\varphi_1 + w_2\varphi_2}{w_1 + w_2}$ is monotone increasing on I and hence φ_3 also is strictly monotone increasing on I. Similarly, we see that all $\varphi_1, \ldots, \varphi_n$ are strictly monotone increasing on I. Thus $\sum_{i=1}^n w_i \varphi_i$ also is strictly monotone increasing on I. Put $u = M_{\sum_{i=1}^n w_i \varphi_i}(f)$ and then $u \in I$. Since

$$\left(\sum_{i=1}^n w_i \varphi_i\right)(u) = \int (\sum_{i=1}^n w_i \varphi_i) \circ f \, d\mu = \sum_{i=1}^n w_i \int \varphi_i \circ f \, d\mu = \sum_{i=1}^n w_i \varphi_i(x_i),$$

it follows from (6) that

$$\left(\sum_{i=1}^n w_i \varphi_i\right)\left(\sum_{i=1}^n w_i x_i\right) \leqslant \left(\sum_{i=1}^n w_i \varphi_i\right)(u). \tag{7}$$

Since $\sum_{i=1}^n w_i \varphi_i$ is strictly monotone increasing on I, it follows from (7) that $\sum_{i=1}^n w_i x_i \leqslant u$, that is, $\sum_{i=1}^n w_i M_{\varphi_i}(f) \leqslant M_{\sum_{i=1}^n w_i \varphi_i}(f)$.

Note that the equality holds in (5) if and only if the equality holds in (7) and hence (6). Therefore if $M_{\varphi_1}(f) = \ldots = M_{\varphi_n}(f)$ and hence $x_1 = \ldots = x_n$, then the equality holds in (6) and hence (5).

Conversely suppose that the equality holds in (5) and that either $\sum_{i=1}^{k} w_i(\varphi_{k+1}-\varphi_i)$ is strictly monotone increasing on I or φ_{k+1} is strictly convex on I for any $k=1,\dots,n-1$. Then the equality holds in (6) and hence it follows from Theorem 1 that $\psi_1=\dots=\psi_n$ on I, that is, $M_{\varphi_1}(f)=\dots=M_{\varphi_n}(f)$. □

The following result is an n-version of [3, Theorem 3–(ii)] and asserts that a quasi-arithmetic mean function: $\varphi\to M_\varphi(f)$ has an n-version of the convexity on a suitable convex subset of $C_{sm}(I)$.

THEOREM 3. *Let $n\geqslant 2$ and $w_1,\dots,w_n>0$ with $\sum_{i=1}^{n} w_i=1$. Suppose that f is non-constant on I and that $\varphi_1,\dots,\varphi_n\in C_{sm,f}(I)$ are such that*

(i) $\nabla_{\varphi_1}\leqslant\nabla_{\varphi_2}\leqslant\dots\leqslant\nabla_{\varphi_n}$,

(ii) all $\sum_{i=1}^{k} w_i(\varphi_{k+1}-\varphi_i)\,(k=1,\dots,n-1)$ are monotone increasing on I,

(iii) all $\varphi_2,\dots,\varphi_n$ are monotone decreasing and convex on I.

Then $\sum_{i=1}^{n} w_i\varphi_i$ is strictly monotone decreasing on I and

$$\sum_{i=1}^{n} w_i M_{\varphi_i}(f)\geqslant M_{\sum_{i=1}^{n} w_i\varphi_i}(f) \tag{8}$$

holds.

If $M_{\varphi_1}(f)=\dots=M_{\varphi_n}(f)$, then the equality holds in (8). Conversely if the equality holds in (8), then $M_{\varphi_1}(f)=\dots=M_{\varphi_n}(f)$, under the assumption that either $\sum_{i=1}^{k} w_i(\varphi_{k+1}-\varphi_i)$ is strictly monotone increasing on I or φ_{k+1} is strictly convex on I for any $k=1,\dots,n-1$.

Proof. Put $x_1=M_{\varphi_1}(f),\dots,x_n=M_{\varphi_n}(f)$. Then all $x_1,x_2,\dots,x_n$ are in I and it follows that $x_1\leqslant x_2\leqslant\dots\leqslant x_n$ from the condition (i) and Lemma 3. Moreover put $\psi_1(x)=x_1,\dots,\psi_n(x)=x_n$ for each $x\in I$. Since $\psi_1\leqslant\dots\leqslant\psi_n$ on I, it follows from the conditions (ii), (iii) and Theorem 1 that (6) holds. Since φ_2 is strictly monotone decreasing on I and $\varphi_1-\varphi_2$ is monotone decreasing on I, it follows that φ_1 also is strictly monotone decreasing on I. Then all $\varphi_1,\dots,\varphi_n$ are strictly monotone decreasing on I by the condition (iii). Thus $\sum_{i=1}^{n} w_i\varphi_i$ also is strictly monotone decreasing on I. Put $u=M_{\sum_{i=1}^{n} w_i\varphi_i}(f)$ and then $u\in I$. Therefore, as can be observed in the proof of Theorem 2, it follows that (6) implies (7). Since $\sum_{i=1}^{n} w_i\varphi_i$ is strictly monotone decreasing on I, it follows from (7) that $\sum_{i=1}^{n} w_i x_i\geqslant u$, that is, (8).

For the equality condition, it is just the same with the proof of Theorem 2. □

5. Examples

Let $\alpha\in\mathbb{R}$, $n\geqslant 3$, $\{a_1,\dots,a_{n-1}\}\subset\mathbb{R}$ and $w_1,\dots,w_n>0$ with $\sum_{i=1}^{n} w_i=1$. Let $\{x_1,x_2,\dots,x_n\}$ be a sequence defined by

$$x_1=\alpha,\, x_{k+1}=\frac{1}{W(k)}\sum_{i=1}^{k} w_i x_i+a_k\ (k=1,\dots,n-1),$$

where $W(k)=\sum_{i=1}^{k} w_i$. Let $1 \leqslant k \leqslant n-1$ and put $S_k=\sum_{i=1}^{k} w_i x_i$. Then $x_{k+1}=\frac{S_{k+1}-S_k}{w_{k+1}}$ and hence we have

$$\frac{S_{k+1}-S_k}{w_{k+1}}=\frac{S_k}{W(k)}+a_k.$$

This implies easily that

$$\frac{S_{k+1}}{W(k+1)}-\frac{S_k}{W(k)}=\frac{w_{k+1}a_k}{W(k+1)}.$$

By addition, we obtain

$$\frac{S_{k+1}}{W(k+1)}-\frac{S_1}{W(1)}=\sum_{i=1}^{k}\frac{w_{i+1}a_i}{W(i+1)}.$$

Since $\frac{S_1}{W(1)}=\frac{w_1x_1}{w_1}=x_1=\alpha$, it follows that

$$S_{k+1}=W(k+1)\alpha+W(k+1)\sum_{i=1}^{k}\frac{w_{i+1}a_i}{W(i+1)}.$$

Then we have

$$\begin{aligned}
x_k&=\frac{S_k-S_{k-1}}{w_k}\\
&=\frac{W(k)\alpha+W(k)\sum_{i=1}^{k-1}\frac{w_{i+1}a_i}{W(i+1)}-W(k-1)\alpha-W(k-1)\sum_{i=1}^{k-2}\frac{w_{i+1}a_i}{W(i+1)}}{w_k}\\
&=\frac{w_k\alpha+w_k\sum_{i=1}^{k-2}\frac{w_{i+1}a_i}{W(i+1)}+W(k)\frac{w_ka_{k-1}}{W(k)}}{w_k}\\
&=\alpha+\sum_{i=1}^{k-2}\frac{w_{i+1}a_i}{W(i+1)}+a_{k-1}
\end{aligned}$$

for each $3 \leqslant k \leqslant n$. Of course, $x_1=\alpha, x_2=\alpha+a_1$. Moreover, we have

$$\begin{aligned}
x_{k+1}-x_k&=\alpha+\sum_{i=1}^{k-1}\frac{w_{i+1}a_i}{W(i+1)}+a_k-\alpha-\sum_{i=1}^{k-2}\frac{w_{i+1}a_i}{W(i+1)}-a_{k-1}\\
&=a_k-\frac{w_1+\ldots+w_{k-1}}{w_1+\ldots+w_k}a_{k-1}.
\end{aligned}$$

for each $3 \leqslant k \leqslant n-1$. However since

$$x_3-x_2=\alpha+\frac{w_2}{w_1+w_2}a_1+a_2-\alpha-a_1=a_2-\frac{w_1}{w_1+w_2}a_1,$$

it follows that

$$x_{k+1}-x_k=a_k-\frac{w_1+\ldots+w_{k-1}}{w_1+\ldots+w_k}a_{k-1}$$

for each $2 \leqslant k \leqslant n-1$.

Then the above observation implies easily the following

PROPOSITION 1. *Let* $\alpha \in \mathbb{R}$, $n \geqslant 3$ *and* $w_1,\dots,w_n > 0$ *with* $\sum_{i=1}^n w_i = 1$. *Let* $\{a_1,\dots,a_{n-1}\}$ *be such that*

$$0 < a_k < \frac{w_1+\dots+w_{k-1}}{w_1+\dots+w_k} a_{k-1}$$

for each $2 \leqslant k \leqslant n-1$. *Then a sequence* $\{x_1,\dots,x_n\}$ *defined by*

$$x_1 = \alpha, x_2 = \alpha + a_1, x_k = \alpha + a_{k-1} + \sum_{i=1}^{k-2} \frac{w_{i+1} a_i}{w_1+\dots+w_{i+1}} \ (3 \leqslant k \leqslant n)$$

satisfies the following properties:

(i) $x_{k+1} > \frac{1}{W(k)} \sum_{i=1}^k w_i x_i \ (k = 1,\dots,n-1)$.

(ii) If $x_2 > x_3 > \dots > x_n > x_1$.

REMARK 5. (i) Let $n \geqslant 3$, $w_1,\dots,w_n > 0$ with $\sum_{i=1}^n w_i = 1$ and $\{x_1,\dots,x_n\}$ a strictly monotone increasing sequence of real numbers. Then by Theorem 1 and Proposition 1, we can found many sequences $\{y_1,\dots,y_n\}$ such that

$$\left(\sum_{i=1}^n w_i x_i\right)\left(\sum_{i=1}^n w_i y_i\right) < \sum_{i=1}^n w_i x_i y_i$$

and

$$y_2 > y_3 > \dots > y_n > y_1.$$

(ii) In Proposition 1, let $\alpha = 0$, $w_1 = \dots = w_n = \frac{1}{n}$ and $a_k = \frac{1}{k^2}$ for each $k = 1,\dots,n-1$. In this time, we can easily see that

$$0 < a_k < \frac{w_1+\dots+w_{k-1}}{w_1+\dots+w_k} a_{k-1} \ (2 \leqslant k \leqslant n-1)$$

and

$$x_1 = 0, x_2 = 1, x_k = \frac{1}{(k-1)^2} + \sum_{i=1}^{k-2} \frac{1}{i^2(i+1)} \ (3 \leqslant k \leqslant n).$$

However we also see that

$$x_k = \frac{1}{(k-1)^2} + \frac{1}{k-1} - 1 + \sum_{i=1}^{k-2} \frac{1}{i^2} \ (3 \leqslant k \leqslant n).$$

Therefore we have

$$x_\infty \equiv \lim_{k\to\infty} x_k = \sum_{i=1}^{\infty} \frac{1}{i^2} - 1 = \frac{\pi^2}{6} - 1 \ (Euler, 1735)$$

and hence

$$x_2 = 1 > x_3 = \frac{3}{4} > x_4 > \dots > x_\infty = \frac{\pi^2}{6} - 1 > x_1 = 0.$$

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