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A REMARK ON EXTENSION OF ORDER PRESERVING OPERATOR INEQUALITY

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Abstract. We will give an extension of order preserving operator inequality of Furuta type.

1. Introduction

Each capital letter means a bounded linear operator on a Hilbert space. An operator T is said to be positive semidefinite (denoted by $0 \leqslant T$) if $0 \leqslant (Tx,x)$ for all vectors x. We denote by $0 < T$ if T is positive semidefinite and invertible.

THEOREM 1. [11], [10] *Let* $0 \leqslant p \leqslant 1$. *If* $0 \leqslant B \leqslant A$ *holds, then* $B^p \leqslant A^p$.

For $1 < p$, $0 \leqslant B \leqslant A$ does not always ensure $B^p \leqslant A^p$. The following result has been obtained from this point of view.

THEOREM 2. [2] *Let* $0 \leqslant p$, $1 \leqslant q$ *and* $0 \leqslant r$ *with* $p+r \leqslant (1+r)q$. *If* $0 \leqslant B \leqslant A$ *holds, then*

$$\left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1}{q}} \leqslant A^{\frac{p+r}{q}}.$$

It is known that the following result interpolates Theorem 2 and an inequality equivalent to the main result of Ando-Hiai log majorization [1] by the parameter $0 \leqslant t \leqslant 1$.

THEOREM 3. [3] *Let* $1 \leqslant p$, $1 \leqslant s$, $0 \leqslant t \leqslant 1$ *and* $t \leqslant r$. *If* $0 \leqslant B \leqslant A$ *with* $0 < A$, *then the following inequality holds:*

$$\left\{A^{\frac{r}{2}} \left(A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}}\right)^s A^{\frac{r}{2}}\right\}^{\frac{1-t+r}{(p-t)s+r}} \leqslant A^{1-t+r}.$$

An elementary one-page proof is in the Monograph [4], which the first author began his research of this direction by reading its Japanese edition.

Furuta [6] gave a further extension of Theorem 3, which is applied to development of the theory of operator functions and log majorization (e.g. [7], [8], [9]).

DEFINITION. [6]. Let n be a natural number. Put

$$\varphi[2n;r,t] = (\cdots(((p_1 - t)p_2 + t)p_3 - t)p_4 + \cdots - t)p_{2n} + r.$$

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THEOREM 4. [6] *Let* $1 \leqslant p_j\,(j=1,\cdots,2n)$, $0 \leqslant t \leqslant 1$ *and* $t \leqslant r$. *If* $0 \leqslant B \leqslant A$ *with* $0 < A$, *then the following inequality holds:*

$$\left\{A^{\frac{r}{2}}\left(A^{-\frac{t}{2}}\cdots\left(A^{\frac{t}{2}}\left(A^{-\frac{t}{2}}B^{p_1}A^{-\frac{t}{2}}\right)^{p_2}A^{\frac{t}{2}}\right)^{p_3}\cdots A^{-\frac{t}{2}}\right)^{p_{2n}}A^{\frac{r}{2}}\right\}^{\frac{1-t+r}{\varphi[2n;r,t]}} \leqslant A^{1-t+r}.$$

The following result by Uchiyama [12] is 3 operators version of Theorem 3.

THEOREM 5. [12] *Let* $1 \leqslant p_1, p_2$, $0 \leqslant t_1 \leqslant 1$ *and* $t_1 \leqslant t_2$. *If* $0 \leqslant B \leqslant A_1 \leqslant A_2$ *with* $0 < A_1$, *then the following inequality holds:*

$$\left\{A_2^{\frac{t_2}{2}}\left(A_1^{-\frac{t_1}{2}}B^{p_1}A_1^{-\frac{t_1}{2}}\right)^{p_2}A_2^{\frac{t_2}{2}}\right\}^{\frac{1-t_1+t_2}{(p_1-t_1)p_2+t_2}} \leqslant A_2^{1-t_1+t_2}.$$

Another simplified proof of Theorem 5 is also given in [5] by using Theorem 3. Yang and Wang [13] gave a unifying extension of Theorem 4 and Theorem 5.

DEFINITION. [13]. Let n be a natural number. Put

$$\eth[2n] = \{\cdots((((p_1-t_1)p_2+t_1)p_3-t_2)p_4+t_2)p_5-\cdots-t_n\}p_{2n}+t_n.$$

THEOREM 6. [13] *Let* $1 \leqslant p_j\,(j=1,\cdots,2n)$, $0 \leqslant t_k \leqslant 1\,(k=1,\cdots,n)$ *and* $t_n \leqslant r$. *If* $0 \leqslant B \leqslant A_1 \leqslant A_2 \leqslant \cdots \leqslant A_{2n-1} \leqslant A_{2n}$ *with* $0 < A_1$, *then the following inequality holds:*

$$\left\{A_{2n}^{\frac{r}{2}}\left(A_{2n-1}^{-\frac{t_n}{2}}\left(A_{2n-2}^{\frac{t_{n-1}}{2}}\cdots A_4^{\frac{t_2}{2}}\left[A_3^{-\frac{t_2}{2}}\left\{A_2^{\frac{t_1}{2}}\left(A_1^{-\frac{t_1}{2}}B^{p_1}A_1^{-\frac{t_1}{2}}\right)^{p_2}A_2^{\frac{t_1}{2}}\right\}^{p_3}A_3^{-\frac{t_2}{2}}\right]^{p_4}A_4^{\frac{t_2}{2}}\cdots\right.\right.\right.$$

$$\left.\left.\left.A_{2n-2}^{\frac{t_{n-1}}{2}}\right)^{p_{2n-1}}A_{2n-1}^{-\frac{t_n}{2}}\right)^{p_{2n}}A_{2n}^{\frac{r}{2}}\right\}^{\frac{1-t_n+r}{\eth[2n]-t_n+r}} \leqslant A_{2n}^{1-t_n+r}.$$

2. Results

We will give the following result which is an extension of Theorem 6.

DEFINITION. Let n be a natural number. We set

$$\alpha(2n) = 1-t_1+t_2-\cdots-t_{2n-1}+t_{2n}$$
$$\psi(2n) = \{\cdots(((p_1-t_1)p_2+t_2)p_3-t_3)p_4+\cdots-t_{2n-1}\}p_{2n}+t_{2n}.$$

THEOREM 7. *Let n be a natural number. Let* $1 \leqslant p_j\,(j=1,\cdots,2n)$, $0 \leqslant t_{2k-1} \leqslant 1$ *and* $t_{2k-1} \leqslant t_{2k}\,(k=1,\cdots,n)$. *If* $0 \leqslant B \leqslant A_1 \leqslant A_2$ *and*

$$A_{2k-2}^{\alpha(2k-2)} \leqslant A_{2k-1}^{\alpha(2k-2)} \leqslant A_{2k}^{\alpha(2k-2)} \quad (k=2,\cdots n) \tag{$*$}$$

with $0 < A_1$, *then the following inequality holds:*

$$\left\{A_{2n}^{\frac{t_{2n}}{2}}\left(A_{2n-1}^{-\frac{t_{2n-1}}{2}}\cdots\left(A_2^{\frac{t_2}{2}}\left(A_1^{-\frac{t_1}{2}}B^{p_1}A_1^{-\frac{t_1}{2}}\right)^{p_2}A_2^{\frac{t_2}{2}}\right)^{p_3}\cdots A_{2n-1}^{-\frac{t_{2n-1}}{2}}\right)^{p_{2n}}A_{2n}^{\frac{t_{2n}}{2}}\right\}^{\frac{\alpha(2n)}{\psi(2n)}}$$
$$\leqslant A_{2n}^{\alpha(2n)}. \tag{1}$$

LEMMA 8. *Under the assumption of Theorem 7,*

$$1 \leqslant \alpha(2n) \leqslant \psi(2n).$$

Proof. $\psi(2) = (p_1 - t_1)p_2 + t_2 \geqslant (1-t_1)p_2 + t_2 \geqslant 1 - t_1 + t_2 = \alpha(2) \geqslant 1$. Suppose $1 \leqslant \alpha(2k) \leqslant \psi(2k)$ for some k such that $1 \leqslant k \leqslant n-1$. Then we have

$$\begin{aligned}\psi(2k+2) &= (\psi(2k)p_{2k+1} - t_{2k+1})p_{2k+2} + t_{2k+2} \\ &\geqslant (\alpha(2k)p_{2k+1} - t_{2k+1})\,p_{2k+2} + t_{2k+2} \\ &\geqslant \alpha(2k) - t_{2k+1} + t_{2k+2} = \alpha(2k+2) \geqslant 1. \quad \square\end{aligned}$$

Proof of Theorem 7. For the case of $n=1$, it is exactly Theorem 5. Let $1 \leqslant p_{2n+1}, p_{2n+2}$, $0 \leqslant t_{2n+1} \leqslant 1$, $t_{2n+1} \leqslant t_{2n+2}$ and

$$A_{2n}^{\alpha(2n)} \leqslant A_{2n+1}^{\alpha(2n)} \leqslant A_{2n+2}^{\alpha(2n)}.$$

Suppose that the inequality (1) holds. We denote the left hand side of (1) by B_1. Put

$$p = \frac{\psi(2n)}{\alpha(2n)}p_{2n+1}, \quad t = \frac{t_{2n+1}}{\alpha(2n)}, \quad r = \frac{t_{2n+2}}{\alpha(2n)}, \quad s = p_{2n+2}.$$

Then it is easy to check that $1 \leqslant p$, $1 \leqslant s$, $0 \leqslant t \leqslant 1$ and $t \leqslant r$. We have

$$B_1^p = \left(A_{2n}^{\frac{t_{2n}}{2}}\left(A_{2n-1}^{-\frac{t_{2n-1}}{2}} \cdots \left(A_2^{\frac{t_2}{2}}\left(A_1^{-\frac{t_1}{2}} B^{p_1} A_1^{-\frac{t_1}{2}}\right)^{p_2} A_2^{\frac{t_2}{2}}\right)^{p_3} \cdots A_{2n-1}^{-\frac{t_{2n-1}}{2}}\right)^{p_{2n}} A_{2n}^{\frac{t_{2n}}{2}}\right)^{p_{2n+1}}.$$

Since

$$B_1 \leqslant A_{2n}^{\alpha(2n)} \leqslant A_{2n+1}^{\alpha(2n)} \leqslant A_{2n+2}^{\alpha(2n)},$$

applying Theorem 5, we have

$$\left\{\left(A_{2n+2}^{\alpha(2n)}\right)^{\frac{r}{2}}\left(\left(A_{2n+1}^{\alpha(2n)}\right)^{-\frac{t}{2}} B_1^p \left(A_{2n+1}^{\alpha(2n)}\right)^{-\frac{t}{2}}\right)^s \left(A_{2n+2}^{\alpha(2n)}\right)^{\frac{r}{2}}\right\}^{\frac{1-t+r}{(p-t)s+r}} \leqslant \left(A_{2n+2}^{\alpha(2n)}\right)^{1-t+r}. \tag{2}$$

Then we have

$$\alpha(2n)(1-t+r) = \alpha(2n) - t_{2n+1} + t_{2n+2} = \alpha(2n+2),$$

so the right hand side of (2) is $A_{2n+2}^{\alpha(2n+2)}$. It is obvious that

$$\left(A_{2n+2}^{\alpha(2n)}\right)^{\frac{r}{2}} = A_{2n+2}^{\frac{t_{2n+2}}{2}} \quad \text{and} \quad \left(A_{2n+1}^{\alpha(2n)}\right)^{-\frac{t}{2}} = A_{2n+1}^{-\frac{t_{2n+1}}{2}}.$$

Furthermore, it is easy to see that

$$\frac{1-t+r}{(p-t)s+r} = \frac{\alpha(2n+2)}{\psi(2n+2)}$$

holds, so we can conclude that

$$\left\{A_{2n+2}^{\frac{t_{2n+2}}{2}}\left(A_{2n+1}^{-\frac{t_{2n+1}}{2}}\cdots\left(A_2^{\frac{t_2}{2}}\left(A_1^{-\frac{t_1}{2}}B^{p_1}A_1^{-\frac{t_1}{2}}\right)^{p_2}A_2^{\frac{t_2}{2}}\right)^{p_3}\cdots A_{2n+1}^{-\frac{t_{2n+1}}{2}}\right)^{p_{2n+2}}A_{2n+2}^{\frac{t_{2n+2}}{2}}\right\}^{\frac{\alpha(2n+2)}{\psi(2n+2)}}$$
$$\leqslant A_{2n+2}^{\alpha(2n+2)}. \tag{3}$$

This completes the proof. □

DEFINITION. Let n be a natural number. We set

$$\beta(2n+1)=1+t_1-t_2+\cdots+t_{2n+1}$$
$$\gamma(2n+1)=\{\cdots((p_1+t_1)p_2-t_2)p_3+\cdots-t_{2n}\}p_{2n+1}+t_{2n+1}$$

THEOREM 9. *Let n be a natural number. Let $1\leqslant p_j\,(j=1,\cdots,2n+1)$, $0\leqslant t_1$, $0\leqslant t_{2k}\leqslant 1$ and $t_{2k}\leqslant t_{2k+1}\,(k=1,\cdots,n)$. If $0\leqslant B\leqslant A_1$ and*

$$A_{2k-1}^{\beta(2k-1)}\leqslant A_{2k}^{\beta(2k-1)}\leqslant A_{2k+1}^{\beta(2k-1)}\ \ (k=1,\cdots n)$$

with $0<A_1$, then the following inequality holds:

$$\left\{A_{2n+1}^{\frac{t_{2n+1}}{2}}\left(A_{2n}^{-\frac{t_{2n}}{2}}\cdots\left(A_2^{-\frac{t_2}{2}}\left(A_1^{\frac{t_1}{2}}B^{p_1}A_1^{\frac{t_1}{2}}\right)^{p_2}A_2^{-\frac{t_2}{2}}\right)^{p_3}\cdots A_{2n}^{-\frac{t_{2n}}{2}}\right)^{p_{2n+1}}A_{2n+1}^{\frac{t_{2n+1}}{2}}\right\}^{\frac{\beta(2n+1)}{\gamma(2n+1)}}$$
$$\leqslant A_{2n+1}^{\beta(2n+1)}.$$

Proof. Put $t_1=0$ and $p_1=1$ in the inequality (3). Then we have

$$\left\{A_{2n+2}^{\frac{t_{2n+2}}{2}}\left(A_{2n+1}^{-\frac{t_{2n+1}}{2}}\cdots\left(A_2^{\frac{t_2}{2}}B^{p_2}A_2^{\frac{t_2}{2}}\right)^{p_3}\cdots A_{2n+1}^{-\frac{t_{2n+1}}{2}}\right)^{p_{2n+2}}A_{2n+2}^{\frac{t_{2n+2}}{2}}\right\}^{\frac{\alpha(2n+2)}{\psi(2n+2)}}\leqslant A_{2n+2}^{\alpha(2n+2)}.$$

For $j=2,\cdots,2n+2$, rewrite t_j, p_j and A_j by t_{j-1}, p_{j-1} and A_{j-1}, respectively. □

COROLLARY 10. *Let n be a natural number and l be an even natural number. Let $1\leqslant p_j\,(j=1,\cdots,2n+l)$, $0\leqslant t_1,\cdots,t_n,t_{n+1},t_{n+3},\cdots,t_{n+l-1}\leqslant 1$ and $t_{n+1}\leqslant t_{n+2},\cdots,t_{n+l-1}\leqslant t_{n+l}$. $0\leqslant B\leqslant A_1\leqslant A_2\leqslant\cdots\leqslant A_{2n+2}$, with $0<A_1$, then the following inequality holds:*

$$\left\{A_{2n+2}^{\frac{t_{n+l}}{2}}\left(A_{2n+2}^{-\frac{t_{n+l-1}}{2}}\cdots\left(A_{2n+2}^{\frac{t_{n+2}}{2}}\left(A_{2n+1}^{-\frac{t_{n+1}}{2}}\left(A_{2n}^{\frac{t_n}{2}}\left(A_{2n-1}^{-\frac{t_n}{2}}\cdots\left(A_2^{\frac{t_1}{2}}\left(A_1^{-\frac{t_1}{2}}B^{p_1}A_1^{-\frac{t_1}{2}}\right)^{p_2}A_2^{\frac{t_1}{2}}\right)^{p_3}\right.\right.\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left.\left.\cdots\ A_{2n-1}^{-\frac{t_n}{2}}\right)^{p_{2n}}A_{2n}^{\frac{t_n}{2}}\right)^{p_{2n+1}}A_{2n+1}^{-\frac{t_{n+1}}{2}}\right)^{p_{2n+2}}A_{2n+2}^{\frac{t_{n+2}}{2}}\right)^{p_{2n+3}}\cdots A_{2n+2}^{-\frac{t_{n+l-1}}{2}}\right)^{p_{2n+l}}A_{2n+2}^{\frac{t_{n+l}}{2}}\right\}^{\frac{\alpha'}{\psi'}}$$
$$\leqslant A_{2n+2}^{\alpha'},$$

where

$$\begin{aligned}\alpha' &= 1 - t_{n+1} + t_{n+2} - \cdots - t_{n+l-1} + t_{n+l}\\ \psi' &= (\cdots(((((p_1 - t_1)p_2 + t_1)\, p_3 - \cdots - t_n)\, p_{2n} + t_n)\, p_{2n+1} - t_{n+1})\, p_{2n+2} + \cdots\\ &\qquad -t_{n+l-1})\, p_{2n+l} + t_{n+l}.\end{aligned}$$

Proof. By putting $A_{2n+2} = A_{2n+3} = \cdots = A_{2n+l}$ and replacing $t_1, t_2, \cdots, t_{2n-1}$, $t_{2n}, t_{2n+1}, \cdots, t_{2n+l}$ in Theorem 7 to $t_1, t_1, \cdots, t_n, t_n$, $t_{n+1}, \cdots, t_{n+l}$ in Corollary 10, respectively, we can apply Theorem 7 succesively. □

COROLLARY 11. *Let n be a natural number. Let $1 \leqslant p_j\, (j = 1, \cdots, 2n)$, $0 \leqslant t_{2k-1} \leqslant 1$ and $t_{2k-1} \leqslant t_{2k}\, (k = 1, \cdots, n)$. If $0 \leqslant B \leqslant A$ with $0 < A$, then the following inequality holds:*

$$\left\{A^{\frac{t_{2n}}{2}}\left(A^{-\frac{t_{2n-1}}{2}}\cdots\left(A^{\frac{t_2}{2}}\left(A^{-\frac{t_1}{2}}B^{p_1}A^{-\frac{t_1}{2}}\right)^{p_2}A^{\frac{t_2}{2}}\right)^{p_3}\cdots A^{-\frac{t_{2n-1}}{2}}\right)^{p_{2n}}A^{\frac{t_{2n}}{2}}\right\}^{\frac{\alpha(2n)}{\psi(2n)}}$$
$$\leqslant A^{\alpha(2n)}.$$

Proof. Put $A = A_1 = A_2 = \cdots = A_{2n}$ in Theorem 7. □

COROLLARY 12. *Let n be a natural number. Let $1 \leqslant p_j\, (j = 1, \cdots, 2n+1)$, $0 \leqslant t_1$, $0 \leqslant t_{2k} \leqslant 1$ and $t_{2k} \leqslant t_{2k+1}\, (k = 1, \cdots, n)$. If $0 \leqslant B \leqslant A$ with $0 < A$, then the following inequality holds:*

$$\left\{A^{\frac{t_{2n+1}}{2}}\left(A^{-\frac{t_{2n}}{2}}\cdots\left(A^{-\frac{t_2}{2}}\left(A^{\frac{t_1}{2}}B^{p_1}A^{\frac{t_1}{2}}\right)^{p_2}A^{-\frac{t_2}{2}}\right)^{p_3}\cdots A^{-\frac{t_{2n}}{2}}\right)^{p_{2n+1}}A^{\frac{t_{2n+1}}{2}}\right\}^{\frac{\beta(2n+1)}{\gamma(2n+1)}}$$
$$\leqslant A^{\beta(2n+1)}.$$

Proof. Put $A = A_1 = A_2 = \cdots = A_{2n+1}$ in Theorem 9. □

REMARK. If we omit the condition $(*)$ in Theorem 7 and assume only $0 \leqslant B \leqslant A_1 \leqslant A_2 \leqslant \cdots \leqslant A_{2n}$ with $0 < A_1$, we cannot always obtain the inequality (1). We may take operators $I \leqslant C_1 \leqslant C_2$ such that $C_1^2 \nleqslant C_2^2$, where I denotes the identity operator. Put $n = 2$, $p_1 = \cdots = p_4 = 1$, $t_1 = t_3 = 1$, $t_2 = t_4 = 2$ and $B = A_1 = I$, $A_2 = C_1$, $A_3 = A_4 = C_2$. In this case, $\alpha(4) = \psi(4) = 3$. If the inequality (1) holds, we would have

$$C_2 C_2^{-\frac{1}{2}} C_1^2 C_2^{-\frac{1}{2}} C_2 \leqslant C_2^3,$$

which leads to $C_1^2 \leqslant C_2^2$, a contradiction.

REMARK. Theorem 7 becomes Theorem 6, when $t_{2k-1} = t_{2k}\, (k = 1, \cdots, n-1)$, $t_{2n} = r$ and $0 \leqslant B \leqslant A_1 \leqslant A_2 \leqslant \cdots \leqslant A_{2n}$, and rewrite t_{2k-1} to $t_k\, (k = 1, \cdots, n)$.

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