Journal of
Mathematical
Inequalities
Volume 6, Number 1 (2012), 125–139

ON SOME PROPERTIES OF JENSEN–MERCER'S FUNCTIONAL

MARIO KRNIĆ, NEDA LOVRIČEVIĆ AND JOSIP PEČARIĆ

(*Communicated by I. Franjić*)

Abstract. Motivated by results of S.S. Dragomir, J.E. Pečarić and L.E. Persson, related to superadditivity and monotonicity of discrete Jensen's functional, in this paper we consider Jensen-Mercer's functional, for which we state and prove analogous results. In particular, these results are obtained for Jensen-Mercer's functional under Jensen-Steffensen's conditions. Integral versions are also given.

1. Introduction

As A. McD. Mercer proved a Jensen-type inequality in his paper [6], it was named the Jensen-Mercer inequality, after both mathematicians involved. Namely, for a convex function $f:[a,b]\to\mathbb{R}$, $[a,b]\subseteq\mathbb{R}$, $\mathbf{x}=(x_1,\dots,x_n)\in[a,b]^n$, and $\mathbf{p}=(p_1,\dots,p_n)$ a non-negative $n-$tuple, such that $P_n:=\sum_{i=1}^n p_i>0$, Mercer proved that the following inequality holds:

$$f\left(a+b-\frac{1}{P_n}\sum_{i=1}^n p_ix_i\right)\leqslant f(a)+f(b)-\frac{1}{P_n}\sum_{i=1}^n p_if(x_i). \tag{1}$$

In the sequel, the set of all non-negative $n-$ tuples $\mathbf{p}=(p_1,\dots,p_n)$, such that $P_n:=\sum_{i=1}^n p_i>0$ will be denoted with $\mathscr{P}_n^0$.

The difference between the right-hand and the left-hand side of inequality (1) defines discrete Jensen-Mercer's functional

$$\mathscr{M}(f,\mathbf{x},\mathbf{p}):=P_n[f(a)+f(b)]-\sum_{i=1}^n p_if(x_i)-P_nf\left(a+b-\frac{1}{P_n}\sum_{i=1}^n p_ix_i\right). \tag{2}$$

For a fixed function f and $n-$ tuple $\mathbf{x}$, $\mathscr{M}(f,\mathbf{x},\cdot)$ can be considered as a function on $\mathscr{P}_n^0$, which is a convex subset in $\mathbb{R}^n$. Furthermore, because of (1), $\mathscr{M}(f,\mathbf{x},\mathbf{p})\geqslant 0$, for all $\mathbf{p}\in\mathscr{P}_n^0$.

Mathematics subject classification (2010): Primary 26D10, 26A51.

Keywords and phrases: Inequalities, convex functions, Jensen's functional, Jensen-Mercer's inequality, Jensen-Mercer's functional, Jensen-Steffensen's inequality, superadditivity, monotonicity, Jensen-Steffensen's integral inequality.

Paper JMI-06-13

It was proved in [1] that (1) remains valid even when the condition on non-negativity of $\mathbf{p}$ is relaxed, that is, when $\mathbf{p}$ satisfies the conditions for Jensen-Steffensen's inequality. More precisely, (1) holds if $\mathbf{x} = (x_1, \dots, x_n) \in [a,b]^n$ is a monotonic $n-$ tuple and $\mathbf{p} = (p_1, \dots, p_n)$ is a real $n-$ tuple, such that for $P_k := \sum_{i=1}^{k} p_i,\ k = 1, \dots, n,$ we have

$$0 \leqslant P_k \leqslant P_n, \quad k = 1, \dots, n-1, \quad P_n > 0, \tag{3}$$

with f being a convex function as before. Here is also $\sum_{i=1}^{n} p_i x_i \in [a,b]$, (see [1]). In the sequel, the set of all $n-$ tuples $\mathbf{p} = (p_1, \dots, p_n)$, satisfying Jensen-Steffensen's conditions (3) will be denoted with $\mathscr{P}_n$. In this setting, the associated functional (2) is called (discrete) Jensen-Mercer's functional under Jensen-Steffensen's conditions.

Now that we introduced the basic notions to be dealt with in the sequel, let us explain the motivation for writing this paper. In paper [5], S.S. Dragomir and al. introduced and investigated discrete Jensen's functional

$$J_n(f, \mathbf{x}, \mathbf{p}) = \sum_{i=1}^{n} p_i f(x_i) - P_n f\left(\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i\right). \tag{4}$$

They proved that $J_n(f, \mathbf{x}, \cdot)$ is superadditive on $\mathscr{P}_n^0$, that is, if $\mathbf{p}, \mathbf{q} \in \mathscr{P}_n^0$, then

$$J_n(f, \mathbf{x}, \mathbf{p} + \mathbf{q}) \geqslant J_n(f, \mathbf{x}, \mathbf{p}) + J_n(f, \mathbf{x}, \mathbf{q}), \tag{5}$$

and is also increasing on $\mathscr{P}_n^0$, that is,

$$\text{if} \quad \mathbf{p} \geqslant \mathbf{q}, \quad \text{then} \quad J_n(f, \mathbf{x}, \mathbf{p}) \geqslant J_n(f, \mathbf{x}, \mathbf{q}) \geqslant 0. \tag{6}$$

(Here $\mathbf{p} \geqslant \mathbf{q}$ means $p_i \geqslant q_i,\ i = 1, \dots, n$.) However, monotonicity property (6) had been obtained by J.E. Pečarić, (see [7, p.717]), even before Dragomir unified both properties.

In our paper we are going to prove that superadditivity and monotonicity property hold in the case of Jensen-Mercer's functional, too, in all its variants. Namely, in Section 2 we deal with discrete Jensen-Mercer's functional and in Section 3 with discrete Jensen-Mercer's functional under Jensen-Steffensen's conditions. In the last two sections we give integral versions of the results from the first two sections. In the beginning of the first two sections we list some of the known results needed for further considerations. These are given in the paper of J. Barić and A. Matković (see [2]). The discrete notation given in Introduction is valid in the sequel.

2. Properties of discrete Jensen-Mercer's functional

The main results of this section are accompanied with a few consequent results, one of which is related to a result given in [2]. For that purpose we cite it here.

THEOREM A. *Let $\mathbf{p} = (p_1, \dots, p_n)$ and $\mathbf{q} = (q_1, \dots, q_n)$ be two $n-$ tuples from $\mathscr{P}_n^0$. Let m and M be any real constants such that*

$$m \geqslant 0, \quad p_i - m q_i \geqslant 0, \quad M q_i - p_i \geqslant 0, \quad i = 1, \dots, n.$$

If $f : [a,b] \to \mathbb{R}$ *is a convex function and* $\mathbf{x} = (x_1,\dots,x_n)$ *is any* $n-$ *tuple from* $[a,b]^n$, *then*

$$M\mathscr{M}(f,\mathbf{x},\mathbf{q}) \geqslant \mathscr{M}(f,\mathbf{x},\mathbf{p}) \geqslant m\mathscr{M}(f,\mathbf{x},\mathbf{q}).$$

In the following theorem we are concerned with proving the superadditivity property of the functional (2).

THEOREM 1. *Let* $\mathbf{p} = (p_1,\dots,p_n)$ *and* $\mathbf{q} = (q_1,\dots,q_n)$ *be two* $n-$ *tuples from* $\mathscr{P}_n^0$. *If* $f : [a,b] \to \mathbb{R}$, $[a,b] \subseteq \mathbb{R}$, *is a convex function and if* $\mathbf{x} = (x_1,\dots,x_n)$ *is an* $n-$ *tuple in* $[a,b]^n$, *then* $\mathscr{M}(f,\mathbf{x},\cdot)$ *defined by* (2) *is superadditive on* $\mathscr{P}_n^0$, *i.e.*

$$\mathscr{M}(f,\mathbf{x},\mathbf{p}+\mathbf{q}) \geqslant \mathscr{M}(f,\mathbf{x},\mathbf{p}) + \mathscr{M}(f,\mathbf{x},\mathbf{q}) \geqslant 0. \tag{7}$$

Proof. Starting from the definition we have

$$\begin{aligned}
\mathscr{M}(f,\mathbf{x},\mathbf{p}+\mathbf{q}) &= (P_n+Q_n)[f(a)+f(b)] - \sum_{i=1}^{n}(p_i+q_i)f(x_i) \\
&\quad -(P_n+Q_n)f\left(a+b-\frac{\sum_{i=1}^{n}(p_i+q_i)x_i}{P_n+Q_n}\right) \\
&= P_n[f(a)+f(b)] + Q_n[f(a)+f(b)] - \sum_{i=1}^{n} p_i f(x_i) - \sum_{i=1}^{n} q_i f(x_i) \\
&\quad -(P_n+Q_n)f\left(a+b-\frac{\sum_{i=1}^{n}(p_i+q_i)x_i}{P_n+Q_n}\right),
\end{aligned} \tag{8}$$

while, after arranging, convexity of f and Jensen's inequality yield

$$\begin{aligned}
&f\left(a+b-\frac{\sum_{i=1}^{n}(p_i+q_i)x_i}{P_n+Q_n}\right) = f\left(\frac{\sum_{i=1}^{n}(p_i+q_i)(a+b-x_i)}{P_n+Q_n}\right) \\
&= f\left(\frac{P_n}{P_n+Q_n}\frac{\sum_{i=1}^{n} p_i(a+b-x_i)}{P_n} + \frac{Q_n}{P_n+Q_n}\frac{\sum_{i=1}^{n} q_i(a+b-x_i)}{Q_n}\right) \\
&\leqslant \frac{P_n}{P_n+Q_n} f\left(a+b-\frac{\sum_{i=1}^{n} p_i x_i}{P_n}\right) + \frac{Q_n}{P_n+Q_n} f\left(a+b-\frac{\sum_{i=1}^{n} q_i x_i}{Q_n}\right).
\end{aligned} \tag{9}$$

Finally, combining relation (8) and inequality (9) we get

$$\begin{aligned}
\mathscr{M}(f,\mathbf{x},\mathbf{p}+\mathbf{q}) &\geqslant P_n[f(a)+f(b)] + Q_n[f(a)+f(b)] - \sum_{i=1}^{n} p_i f(x_i) - \sum_{i=1}^{n} q_i f(x_i) \\
&\quad -P_n f\left(a+b-\frac{\sum_{i=1}^{n} p_i x_i}{P_n}\right) - Q_n f\left(a+b-\frac{\sum_{i=1}^{n} q_i x_i}{Q_n}\right) \\
&= \mathscr{M}(f,\mathbf{x},\mathbf{p}) + \mathscr{M}(f,\mathbf{x},\mathbf{q}).
\end{aligned}$$

Because of (1) we have that $\mathscr{M}(f,\mathbf{x},\mathbf{p}) \geqslant 0$ and $\mathscr{M}(f,\mathbf{x},\mathbf{q}) \geqslant 0$, so the proposed right-hand side inequality in (7) holds. □

The functional (2) satisfies the monotonicity property, as is shown in the sequel.

THEOREM 2. *Let* $\mathbf{p} = (p_1, \ldots, p_n)$ *and* $\mathbf{q} = (q_1, \ldots, q_n)$ *be two* $n-$ *tuples from* $\mathscr{P}_n^0$, *such that* $\mathbf{p} \geqslant \mathbf{q}$, *(i.e.* $p_i \geqslant q_i$, $i = 1, \ldots, n$*). If* $f : [a,b] \to \mathbb{R}$, $[a,b] \subseteq \mathbb{R}$, *is a convex function and if* $\mathbf{x} = (x_1, \ldots, x_n)$ *is an* $n-$ *tuple in* $[a,b]^n$, *then for functional* $\mathscr{M}(f, \mathbf{x}, \cdot)$ *defined by* (2) *inequality*

$$\mathscr{M}(f, \mathbf{x}, \mathbf{p}) \geqslant \mathscr{M}(f, \mathbf{x}, \mathbf{q}) \tag{10}$$

holds on $\mathscr{P}_n^0$.

Proof. The monotonicity property follows directly from superadditivity. Since $\mathbf{p} \geqslant \mathbf{q}$, $\mathbf{p}$ can be represented as the sum of two $n-$ tuples: $\mathbf{p} - \mathbf{q}$ and $\mathbf{q}$. Applying (7) we have

$$\mathscr{M}(f, \mathbf{x}, \mathbf{p}) = \mathscr{M}(f, \mathbf{x}, \mathbf{p} - \mathbf{q} + \mathbf{q}) \geqslant \mathscr{M}(f, \mathbf{x}, \mathbf{p} - \mathbf{q}) + \mathscr{M}(f, \mathbf{x}, \mathbf{q}).$$

Finally, since by (1) $\mathscr{M}(f, \mathbf{x}, \mathbf{p} - \mathbf{q}) \geqslant 0$, we have that $\mathscr{M}(f, \mathbf{x}, \mathbf{p}) \geqslant \mathscr{M}(f, \mathbf{x}, \mathbf{q})$, which proves the theorem. □

REMARK 1. We can easily obtain the result from Theorem A from [2] by means of Theorem 2. Let $\mathbf{p}, \mathbf{q} \in \mathscr{P}_n^0$ and let m and M be real constants such that $\mathbf{p} - m\mathbf{q}$ and $M\mathbf{q} - \mathbf{p}$ are in $\mathscr{P}_n^0$. If $f : [a,b] \to \mathbb{R}$, $[a,b] \subseteq \mathbb{R}$, is a convex function and if $\mathbf{x} = (x_1, \ldots, x_n)$ is an $n-$ tuple in $[a,b]$, then by Theorem 2

$$\mathscr{M}(f, \mathbf{x}, \mathbf{p}) \geqslant \mathscr{M}(f, \mathbf{x}, \mathbf{p} - m\mathbf{q}) + \mathscr{M}(f, \mathbf{x}, m\mathbf{q}) \geqslant m\mathscr{M}(f, \mathbf{x}, \mathbf{q}).$$

Similarly we get

$$\mathscr{M}(f, \mathbf{x}, \mathbf{p}) \leqslant M\mathscr{M}(f, \mathbf{x}, \mathbf{q}),$$

that is

$$M\mathscr{M}(f, \mathbf{x}, \mathbf{q}) \geqslant \mathscr{M}(f, \mathbf{x}, \mathbf{p}) \geqslant m\mathscr{M}(f, \mathbf{x}, \mathbf{q}).$$

Applying Theorem 2, we are able to give the result on bounding the functional (2) by a non-weighted functional.

COROLLARY 1. *Let* $\mathbf{p}$, $\mathbf{x}$, f *and functional* $\mathscr{M}$ *be as in Theorem* 2*. Then*

$$\max_{1 \leqslant i \leqslant n} \{p_i\} \mathscr{M}_{\mathscr{N}}(f, \mathbf{x}) \geqslant \mathscr{M}(f, \mathbf{x}, \mathbf{p}) \geqslant \min_{1 \leqslant i \leqslant n} \{p_i\} \mathscr{M}_{\mathscr{N}}(f, \mathbf{x}),$$

where $\mathscr{M}_{\mathscr{N}}(f, \mathbf{x}) = n[f(a) + f(b)] - \sum_{i=1}^n f(x_i) - nf\left(a + b - \frac{1}{n}\sum_{i=1}^n x_i\right)$.

Proof. Let $\mathbf{p}_{\min} \in \mathscr{P}_n^0$ be a constant $n-$ tuple, i.e. $\mathbf{p}_{\min} = \left(\min_{1 \leqslant i \leqslant n} \{p_i\}, \ldots, \min_{1 \leqslant i \leqslant n} \{p_i\}\right)$. Then for any $\mathbf{p} \in \mathscr{P}_n^0$ we have $\mathbf{p} \geqslant \mathbf{p}_{\min}$. So applying Theorem 2 we have

$$\mathscr{M}(f, \mathbf{x}, \mathbf{p}) \geqslant \mathscr{M}(f, \mathbf{x}, \mathbf{p}_{\min}). \tag{11}$$

On the other hand,

$$\mathscr{M}(f,\mathbf{x},\mathbf{p}_{\min}) = \min_{1\leqslant i\leqslant n}\{p_i\}\left\{n[f(a)+f(b)] - \sum_{i=1}^{n} f(x_i) - nf\left(a+b-\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right\},$$

i.e. $\mathscr{M}(f,\mathbf{x},\mathbf{p}_{\min}) = \min_{1\leqslant i\leqslant n}\{p_i\}\mathscr{M}_{\mathscr{N}}(f,\mathbf{x})$, which is the required non-weighted bound. The left-hand side inequality is obtained similarly, by exchanging the roles of min and max. □

3. Properties of discrete Jensen-Mercer's functional under Jensen-Steffensen's conditions

The main results of this section are accompanied with a few consequent results, related to some results given in [2]. For that purpose we cite them here. These results in [2] are obtained for normalized Jensen-Mercer's functional.

THEOREM B. *Let* $\mathbf{p} = (p_1,\ldots,p_n)$ *and* $\mathbf{q} = (q_1,\ldots,q_n)$ *be two* $n-$ *tuples satisfying*

$$0 \leqslant P_k, Q_k \leqslant 1, \quad k = 1,\ldots,n-1, \quad P_n = Q_n = 1. \tag{12}$$

Let m *and* M *be real constants such that*

$$m \geqslant 0, \quad P_k - mQ_k \geqslant 0, \quad (1-P_k) - m(1-Q_k) \geqslant 0, \quad k = 1,\ldots,n-1 \tag{13}$$

and

$$MQ_k - P_k \geqslant 0, \quad M(1-Q_k) - (1-P_k) \geqslant 0, \quad k = 1,\ldots,n-1. \tag{14}$$

If $f : [a,b] \to \mathbb{R}$, $[a,b] \subseteq \mathbb{R}$, *is a convex function and if* $\mathbf{x} = (x_1,\ldots,x_n)$ *is a monotonic* $n-$ *tuple in* $[a,b]^n$, *then*

$$M\mathscr{M}(f,\mathbf{x},\mathbf{q}) \geqslant \mathscr{M}(f,\mathbf{x},\mathbf{p}) \geqslant m\mathscr{M}(f,\mathbf{x},\mathbf{q}). \tag{15}$$

The following corollary of Theorem B was also given in [2] and will be of interest in the sequel. It considers the uniform distribution $\mathbf{u} = (\frac{1}{n},\ldots,\frac{1}{n})$ and the corresponding nonweighted functional

$$\mathscr{M}(f,\mathbf{x}) := \mathscr{M}(f,\mathbf{x},\mathbf{u}) = f(a)+f(b) - \frac{1}{n}\sum_{i=1}^{n} f(x_i) - f\left(a+b-\frac{1}{n}\sum_{i=1}^{n} x_i\right).$$

COROLLARY A. *Let* $\mathbf{p} = (p_1,\ldots,p_n)$ *be an* $n-$ *tuple satisfying*

$$0 \leqslant P_k \leqslant 1, \quad k = 1,\ldots,n-1, \quad P_n = 1.$$

For $k \in \{1,\ldots,n\}$ *denote* $P_k := \sum_{i=1}^{k} p_i$ *and define*

$$\tilde{m}_0 := n\cdot\min\left\{\frac{P_k}{k}, \frac{1-P_k}{n-k} : k = 1,\ldots,n-1\right\},$$

$$\tilde{M}_0 := n \cdot \max\left\{ \frac{P_k}{k}, \frac{1-P_k}{n-k} : k = 1, \ldots, n-1 \right\}.$$

If $f : [a,b] \to \mathbb{R}$ is a convex function and if $\mathbf{x} = (x_1, \ldots, x_n) \in [a,b]^n$ is any monotonic $n-$ tuple, then

$$\tilde{M}_0 \mathscr{M}(f, \mathbf{x}) \geqslant \mathscr{M}(f, \mathbf{x}, \mathbf{p}) \geqslant \tilde{m}_0 \mathscr{M}(f, \mathbf{x}). \tag{16}$$

Just like in the previous section, we are first going to discuss the superadditivity property of the functional (2), this time - under Jensen-Steffensen's conditions.

THEOREM 3. *Let $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$ be two $n-$ tuples from $\mathscr{P}_n$. If $f : [a,b] \to \mathbb{R}$, $[a,b] \subseteq \mathbb{R}$, is a convex function and if $\mathbf{x} = (x_1, \ldots, x_n)$ is a monotonic $n-$ tuple in $[a,b]^n$, then $\mathscr{M}(f, \mathbf{x}, \cdot)$ defined by* (2) *is superadditive on $\mathscr{P}_n$, i.e.*

$$\mathscr{M}(f, \mathbf{x}, \mathbf{p} + \mathbf{q}) \geqslant \mathscr{M}(f, \mathbf{x}, \mathbf{p}) + \mathscr{M}(f, \mathbf{x}, \mathbf{q}) \geqslant 0. \tag{17}$$

Proof. The proof follows the same lines as in Theorem 1, but for the right-hand side inequality in (17) we use Jensen-Mercer's inequality under Jensen-Steffensen's conditions. □

In order to adjust monotonicity property (10) to functional (2) under Jensen-Steffensen's conditions, we are going to impose some extra conditions on $n-$ tuples $\mathbf{p}$ and $\mathbf{q}$ from $\mathscr{P}_n$, as follows.

THEOREM 4. *Let $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$ be two $n-$ tuples from $\mathscr{P}_n$. Let $P_k \geqslant Q_k$, $P_n - P_k \geqslant Q_n - Q_k$, $k = 1, \ldots, n-1$, and $P_n > Q_n$, where $P_k = \sum_{i=1}^k p_i$ and $Q_k = \sum_{i=1}^k q_i$. If $f : [a,b] \to \mathbb{R}$, $[a,b] \subseteq \mathbb{R}$, is a convex function and if $\mathbf{x} = (x_1, \ldots, x_n)$ is a monotonic $n-$ tuple in $[a,b]^n$, then for functional $\mathscr{M}(f, \mathbf{x}, \cdot)$ defined by* (2) *inequality*

$$\mathscr{M}(f, \mathbf{x}, \mathbf{p}) \geqslant \mathscr{M}(f, \mathbf{x}, \mathbf{q}) \tag{18}$$

holds on $\mathscr{P}_n$.

Proof. Write $\mathscr{M}(f, \mathbf{x}, \mathbf{p}) = \mathscr{M}(f, \mathbf{x}, \mathbf{p} - \mathbf{q} + \mathbf{q})$. Now, if we could apply superadditivity property (17) to $\mathbf{p} - \mathbf{q}$ and $\mathbf{q}$, monotonicity property would also be proved. And that would be the case if the $n-$ tuple $\mathbf{p} - \mathbf{q} = (p_1 - q_1, \ldots, p_n - q_n)$ belonged to $\mathscr{P}_n$. Hence the following conditions need to be satisfied: $0 \leqslant P_k - Q_k \leqslant P_n - Q_n$, $k = 1, \ldots, n-1$, and $P_n - Q_n > 0$, which yields: $0 \leqslant P_k - Q_k$, $P_k - Q_k \leqslant P_n - Q_n$, $k = 1, \ldots, n-1$, and $P_n - Q_n > 0$. Now, taking into account that by (1) $\mathscr{M}(f, \mathbf{x}, \mathbf{p} - \mathbf{q}) \geqslant 0$, we have

$$\mathscr{M}(f, \mathbf{x}, \mathbf{p}) = \mathscr{M}(f, \mathbf{x}, \mathbf{p} - \mathbf{q} + \mathbf{q}) \geqslant \mathscr{M}(f, \mathbf{x}, \mathbf{p} - \mathbf{q}) + \mathscr{M}(f, \mathbf{x}, \mathbf{q}) \geqslant \mathscr{M}(f, \mathbf{x}, \mathbf{q}).$$

□

REMARK 2. We can easily obtain the result of Theorem B by means of Theorem 4. Let $\mathbf{p},\mathbf{q}\in\mathscr{P}_n$ and let m and M be real constants such that $\mathbf{p}-m\mathbf{q}$ and $M\mathbf{q}-\mathbf{p}$ are in $\mathscr{P}_n$. If $f:[a,b]\to\mathbb{R}$ is a convex function and if $\mathbf{x}=(x_1,\dots,x_n)\in[a,b]^n$ is any monotonic $n-$ tuple, then by Theorem 4 is

$$\mathscr{M}(f,\mathbf{x},\mathbf{p})\geqslant\mathscr{M}(f,\mathbf{x},\mathbf{p}-m\mathbf{q})+\mathscr{M}(f,\mathbf{x},m\mathbf{q})\geqslant m\mathscr{M}(f,\mathbf{x},\mathbf{q}).$$

Similarly we get

$$\mathscr{M}(f,\mathbf{x},\mathbf{p})\leqslant M\mathscr{M}(f,\mathbf{x},\mathbf{q}),$$

that is

$$M\mathscr{M}(f,\mathbf{x},\mathbf{q})\geqslant\mathscr{M}(f,\mathbf{x},\mathbf{p})\geqslant m\mathscr{M}(f,\mathbf{x},\mathbf{q}). \tag{19}$$

Since $\mathbf{p}-m\mathbf{q}\in\mathscr{P}_n$ implies $P_k\geqslant mQ_k$ and $(P_n-P_k)\geqslant m(Q_n-Q_k)$, and since $M\mathbf{q}-\mathbf{p}\in\mathscr{P}_n$ implies $P_k\leqslant MQ_k$ and $(P_n-P_k)\leqslant M(Q_n-Q_k)$, $k=1,\dots,n-1$, which are the assumptions of Theorem B (only in a non-normalized form), by obtaining (19), we proved Theorem B.

Applying Theorem 4, we are able to give the result on bounding the functional (2) by a nonweighted functional. But, almost the same result, only in a slightly specialized form, is given in Corollary A, cited from [2]. Our proof would then be the alternative one, obtained via Theorem 4. Hence the detailed analysis is given in the form of a remark.

REMARK 3. In order not to derange our former consideration, we write Corollary A in a slightly different form, namely, for $P_n>0$:

Let $\mathbf{p}=(p_1,\dots,p_n)$ be an $n-$ tuple from $\mathscr{P}_n$. Define

$$m=\min_{1\leqslant k\leqslant n-1}\left\{\frac{P_k}{k},\frac{P_n-P_k}{n-k}\right\},\quad M=\max_{1\leqslant k\leqslant n-1}\left\{\frac{P_k}{k},\frac{P_n-P_k}{n-k}\right\},$$

where $P_k=\sum_{i=1}^k p_i$ and $P_n=\sum_{i=1}^n p_i$. If $f:[a,b]\to\mathbb{R}$ is a convex function and if $\mathbf{x}=(x_1,\dots,x_n)\in[a,b]^n$ is any monotonic $n-$ tuple, then

$$M\mathscr{M}_{\mathscr{N}}(f,\mathbf{x})\geqslant\mathscr{M}(f,\mathbf{x},\mathbf{p})\geqslant m\mathscr{M}_{\mathscr{N}}(f,\mathbf{x}),$$

where $\mathscr{M}_{\mathscr{N}}(f,\mathbf{x})=n[f(a)+f(b)]-\sum_{i=1}^n f(x_i)-nf\left(a+b-\frac{1}{n}\sum_{i=1}^n x_i\right).$

Alternative proof of Corollary A. Let $\mathbf{q}_{\min}\in\mathscr{P}_n^0$ be a constant $n-$ tuple, i.e. $q_{\min}=(\alpha,\alpha,\dots,\alpha)$, where $\alpha>0$, for $Q_n:=\sum_{i=1}^n q_i>0$ must be satisfied. Provided $P_k\geqslant Q_k=k\alpha$, $P_n-P_k\geqslant Q_n-Q_k=(n-k)\alpha$, $k=1,\dots,n-1$, and $P_n>Q_n=n\alpha$, Theorem 4 can be applied. Further, these imply corresponding conditions concerning α:

(i) $\alpha \leqslant \frac{P_k}{k}, \quad k = 1, \dots, n-1,$

(ii) $\alpha \leqslant \frac{P_n - P_k}{n-k}, \quad k = 1, \dots, n-1,$

(iii) $\alpha < \frac{P_n}{n}.$

In order to prove the right-hand side inequality, let us first denote

$$m = \min_{1 \leqslant k \leqslant n-1} \left\{ \frac{P_k}{k}, \frac{P_n - P_k}{n-k} \right\}.$$

Obviously, m satisfies conditions (i) and (ii), and is a candidate for the choice of α. However, (iii) needs some extra considerations. Fix $k \in \{1, \dots, n\}$. Then (i) and (ii) imply $n\alpha \leqslant P_n$, i.e. $\alpha \leqslant \frac{P_n}{n}$. Now we distinguish two cases:

1° $\alpha < \dfrac{P_n}{n}$. Condition (iii) is instantly satisfied and Theorem 4 yields

$$\mathscr{M}(f, \mathbf{x}, \mathbf{p}) \geqslant \mathscr{M}(f, \mathbf{x}, \mathbf{q}_{\min}).$$

2° $\alpha = \dfrac{P_n}{n}$, i.e. $P_n = n\alpha$. From (ii) we get $n\alpha - P_k \geqslant n\alpha - k\alpha$, i.e. $P_k \leqslant k\alpha$. But from (i) is also $P_k \geqslant k\alpha$, hence $P_k = k\alpha$, $k = 1, \dots, n-1$. Since in that case $\mathbf{p} = (\alpha, \alpha, \dots, \alpha) = \mathbf{q}_{\min}$, inequality

$$\mathscr{M}(f, \mathbf{x}, \mathbf{p}) \geqslant \mathscr{M}(f, \mathbf{x}, \mathbf{q}_{\min})$$

holds again.

So m is a good choice for α. Now, respecting notation from the corollary statement we get

$$\begin{aligned} \mathscr{M}(f, \mathbf{x}, \mathbf{q}_{\min}) &= m \left(n[f(a) + f(b)] - \sum_{i=1}^{n} f(x_i) - nf\left(a + b - \frac{1}{n} \sum_{i=1}^{n} x_i \right) \right) \\ &= m\mathscr{M}_{\mathscr{N}}(f, \mathbf{x}). \end{aligned}$$

Lower bound provided by the non-weighted functional is then

$$\mathscr{M}(f, \mathbf{x}, \mathbf{p}) \geqslant m\mathscr{M}_{\mathscr{N}}(f, \mathbf{x}).$$

Upper bound is obtained similarly, by exchanging the roles of $\mathbf{p}$ and $\mathbf{q}$, and with

$$M = \max_{1 \leqslant k \leqslant n-1} \left\{ \frac{P_k}{k}, \frac{P_n - P_k}{n-k} \right\}.$$

$\square$

4. Properties of integral Jensen-Mercer's functional

In [3] Cheung, Matković and Pečarić have proved that if $(\Omega, \mathscr{A}, \mu)$ is a probability space and $x : \Omega \to [a,b]$, $(-\infty < a < b < \infty,)$ is a measurable function, then for any continuous convex function $f : [a,b] \to \mathbb{R}$ the following inequality holds:

$$f\left(a+b-\int_{\Omega} x \,\mathrm{d}\mu\right) \leqslant f(a)+f(b)-\int_{\Omega} f(x)\,\mathrm{d}\mu. \tag{20}$$

It can analogously be proved that for a measure space $(\Omega, \mathscr{A}, \mu)$ with $0 < \mu(\Omega) < \infty$ the integral version of Jensen-Mercer's inequality

$$f\left(a+b-\frac{1}{\mu(\Omega)}\int_{\Omega} x \,\mathrm{d}\mu\right) \leqslant f(a)+f(b)-\frac{1}{\mu(\Omega)}\int_{\Omega} f(x)\,\mathrm{d}\mu \tag{21}$$

holds. In a special case, when $\Omega = [\alpha, \beta]$, where $-\infty < \alpha < \beta < \infty$ and $\lambda : [\alpha,\beta] \to \mathbb{R}$ is any nondecreasing function such that $\lambda(\beta) \neq \lambda(\alpha)$ inequality (21) becomes

$$\begin{aligned} &f\left(a+b-\tfrac{1}{\lambda(\beta)-\lambda(\alpha)}\int_{\alpha}^{\beta} x(t)\,\mathrm{d}\lambda(t)\right)\\ &\qquad \leqslant f(a)+f(b)-\tfrac{1}{\lambda(\beta)-\lambda(\alpha)}\int_{\alpha}^{\beta} f(x(t))\,\mathrm{d}\lambda(t). \end{aligned} \tag{22}$$

Inequality (22) can also be rewritten as:

$$\begin{aligned} &f\left(\tfrac{1}{\lambda(\beta)-\lambda(\alpha)}\int_{\alpha}^{\beta} (a+b-x(t))\,\mathrm{d}\lambda(t)\right)\\ &\qquad \leqslant \tfrac{1}{\lambda(\beta)-\lambda(\alpha)}\int_{\alpha}^{\beta} (f(a)+f(b)-f(x(t)))\,\mathrm{d}\lambda(t). \end{aligned} \tag{23}$$

Let us abbreviate the former notation by setting $\lambda(\beta)-\lambda(\alpha) := \lambda_{\alpha}^{\beta}$. Now we consider the corresponding integral Jensen-Mercer's functional

$$\begin{aligned} \mathscr{M}(f,x,\lambda) \;:=\; &\lambda_{\alpha}^{\beta}[f(a)+f(b)]-\int_{\alpha}^{\beta} f(x(t))\,\mathrm{d}\lambda(t)\\ &-\lambda_{\alpha}^{\beta} f\left(a+b-\frac{1}{\lambda_{\alpha}^{\beta}}\int_{\alpha}^{\beta} x(t)\,\mathrm{d}\lambda(t)\right). \end{aligned} \tag{24}$$

Because of (22) we always have $\mathscr{M}(f,x,\lambda) \geqslant 0$. For the sake of the considerations in the next section, let us denote with $\Lambda_{[\alpha,\beta]}$, $-\infty < \alpha < \beta < \infty$, the class of all functions $\lambda : [\alpha,\beta] \to \mathbb{R}$ which are either continuous or of bounded variation and satisfy the conditions

$$\lambda(\alpha) \leqslant \lambda(t) \leqslant \lambda(\beta) \quad \text{for all} \quad t \in [\alpha,\beta], \quad \lambda(\beta)-\lambda(\alpha) > 0. \tag{25}$$

As for now, we notice that any nondecreasing function $\lambda : [\alpha,\beta] \to \mathbb{R}$ with $\lambda(\beta) - \lambda(\alpha) > 0$ belongs to $\Lambda_{[\alpha,\beta]}$ and are concerned with such functions, while dealing with the integral versions of the results from Section 2. We start with the integral analogue of Theorem 1.

THEOREM 5. *Let λ and μ be two nondecreasing functions from $\Lambda_{[\alpha,\beta]}$ and let $x : [\alpha,\beta] \to [a,b]$, $-\infty < a < b < \infty$, be a continuous function. If $f : [a,b] \to \mathbb{R}$ is a continuous convex function, then functional $\mathscr{M}(f,x,\cdot)$ defined by* (24) *is superadditive on $\Lambda_{[\alpha,\beta]}$, i.e.*

$$\mathscr{M}(f,x,\lambda+\mu) \geqslant \mathscr{M}(f,x,\lambda) + \mathscr{M}(f,x,\mu) \geqslant 0. \tag{26}$$

Proof. Since λ and μ are nondecreasing functions, so is $\lambda+\mu$, so $\mathscr{M}(f,x,\lambda+\mu)$ is well defined. From the definition we have

$$\begin{aligned} \mathscr{M}(f,x,\lambda+\mu) = {} & \left(\lambda_\alpha^\beta + \mu_\alpha^\beta\right)[f(a)+f(b)] - \int_\alpha^\beta f(x(t))\,\mathrm{d}(\lambda+\mu)(t) \\ & - \left(\lambda_\alpha^\beta + \mu_\alpha^\beta\right) \cdot f\left(a+b-\frac{1}{\lambda_\alpha^\beta + \mu_\alpha^\beta}\int_\alpha^\beta x(t)\,\mathrm{d}(\lambda+\mu)(t)\right), \end{aligned} \tag{27}$$

while convexity of f and (integral) Jensen's inequality yield

$$\begin{aligned} & f\left(a+b-\frac{1}{\lambda_\alpha^\beta + \mu_\alpha^\beta}\int_\alpha^\beta x(t)\,\mathrm{d}(\lambda+\mu)(t)\right) = \\ & = f\left(\frac{\lambda_\alpha^\beta}{\lambda_\alpha^\beta + \mu_\alpha^\beta}\cdot\frac{\int_\alpha^\beta (a+b-x(t))\,\mathrm{d}\lambda(t)}{\lambda_\alpha^\beta} + \frac{\mu_\alpha^\beta}{\lambda_\alpha^\beta + \mu_\alpha^\beta}\cdot\frac{\int_\alpha^\beta (a+b-x(t))\,\mathrm{d}\mu(t)}{\mu_\alpha^\beta}\right) \\ & \leqslant \frac{\lambda_\alpha^\beta}{\lambda_\alpha^\beta + \mu_\alpha^\beta}\cdot f\left(\frac{\int_\alpha^\beta (a+b-x(t))\,\mathrm{d}\lambda(t)}{\lambda_\alpha^\beta}\right) + \frac{\mu_\alpha^\beta}{\lambda_\alpha^\beta + \mu_\alpha^\beta}\cdot f\left(\frac{\int_\alpha^\beta (a+b-x(t))\,\mathrm{d}\mu(t)}{\mu_\alpha^\beta}\right). \end{aligned}$$

Finally, combining the last inequality with (27) we get

$$\begin{aligned} & \mathscr{M}(f,x,\lambda+\mu) \geqslant \\ & \geqslant \lambda_\alpha^\beta[f(a)+f(b)] - \int_\alpha^\beta f(x(t))\,\mathrm{d}\lambda(t) - \lambda_\alpha^\beta \cdot f\left(a+b-\frac{\int_\alpha^\beta x(t)\,\mathrm{d}\lambda(t)}{\lambda_\alpha^\beta}\right) \\ & \quad + \mu_\alpha^\beta[f(a)+f(b)] - \int_\alpha^\beta f(x(t))\,\mathrm{d}\mu(t) - \mu_\alpha^\beta \cdot f\left(a+b-\frac{\int_\alpha^\beta x(t)\,\mathrm{d}\mu(t)}{\mu_\alpha^\beta}\right) \\ & = \mathscr{M}(f,x,\lambda) + \mathscr{M}(f,x,\mu). \end{aligned}$$

Because of the inequality (22) we have that $\mathscr{M}(f,x,\lambda) \geqslant 0$ and $\mathscr{M}(f,x,\mu) \geqslant 0$, so the proposed right-hand side inequality in (26) holds. □

We proceed with the integral analogue of Theorem 2.

THEOREM 6. *Let* λ *and* μ *be functions from* $\Lambda_{[\alpha,\beta]}$, *let* $x:[\alpha,\beta]\to[a,b]$, $-\infty<a<b<\infty$, *be a continuous function and* $f:[a,b]\to\mathbb{R}$ *be a continuous convex function. If* μ *and* $\rho:=\lambda-\mu$ *are nondecreasing functions, then for functional* $\mathscr{M}(f,x,\cdot)$ *defined by* (24) *inequality*

$$\mathscr{M}(f,x,\lambda)\geqslant\mathscr{M}(f,x,\mu) \tag{28}$$

holds on $\Lambda_{[\alpha,\beta]}$.

Proof. Since μ and $\rho:=\lambda-\mu$ are nondecreasing functions, so is the function λ. Hence $\mathscr{M}(f,x,\lambda)$, $\mathscr{M}(f,x,\mu)$ and $\mathscr{M}(f,x,\lambda-\mu)$ are all well defined. If we write $\mathscr{M}(f,x,\lambda)=\mathscr{M}(f,x,\lambda-\mu+\mu)$, we can apply (26) in the following way:

$$\mathscr{M}(f,x,\lambda)=\mathscr{M}(f,x,\lambda-\mu+\mu)\geqslant\mathscr{M}(f,x,\lambda-\mu)+\mathscr{M}(f,x,\mu).$$

Since by (22) $\mathscr{M}(f,x,\lambda-\mu)\geqslant 0$, we have that $\mathscr{M}(f,x,\lambda)\geqslant\mathscr{M}(f,x,\mu)$, which ends the proof. □

REMARK 4. Theorem A, cited from [2] in Section 2, has its integral version, also given in [2]. Hence the latter one can easily be obtained by means of Theorem 6. (Follows the same lines as in the discrete case, in Remark 1.)

Just like in the discrete case (Section 2), we are able to give the result on bounding the functional (24) by a nonweighted functional. Almost the same result, only in a slightly specialized form, was given in [2]. What we obtain here is its alternative proof, via Theorem 6. Hence we give our consideration in the form of a remark.

REMARK 5. According to our former considerations, we slightly alter the notation from [2], so that the result reads:

> *Let* λ *be a nondecreasing function from* $\Lambda_{[\alpha,\beta]}$. *Let* $x:[\alpha,\beta]\to[a,b]$, $-\infty<a<b<\infty$, *be a continuous function and let* $f:[a,b]\to\mathbb{R}$ *be a continuous and convex function. If* m *and* M *are defined by*
>
> $$m:=\inf_{\alpha<t<\beta}\left\{\inf\left\{\frac{\lambda(t)-\lambda(s)}{t-s},\alpha\leqslant s\leqslant\beta,s\neq t\right\}\right\},$$
>
> $$M:=\sup_{\alpha<t<\beta}\left\{\sup\left\{\frac{\lambda(t)-\lambda(s)}{t-s},\alpha\leqslant s\leqslant\beta,s\neq t\right\}\right\},$$
>
> *then*
>
> $$M\mathscr{M}(f,x)\geqslant\mathscr{M}(f,x,\lambda)\geqslant m\mathscr{M}(f,x), \tag{29}$$
>
> *where*
>
> $$\mathscr{M}(f,x):=(\beta-\alpha)[f(a)+f(b)]-\int_\alpha^\beta f(x(t))\,\mathrm{d}t-(\beta-\alpha)f\left(a+b-\tfrac{1}{\beta-\alpha}\int_\alpha^\beta x(t)\,\mathrm{d}t\right).$$

Proof. Let us prove the right-hand side inequality in (29). According to the definition of m, $m \leqslant \dfrac{\lambda(t)-\lambda(s)}{t-s}$. It follows that $\lambda(t)-mt-(\lambda(s)-ms) \geqslant 0$. Let μ be a function from $\Lambda_{[\alpha,\beta]}$, such that $\mu(t)=mt$, $\mu(s)=ms$. Function μ is nondecreasing. Function $\rho := \lambda - \mu$ is also nondecreasing, since $\rho(t) \geqslant \rho(s)$. Hence by Theorem 6 we have that

$$\mathscr{M}(f,x,\lambda) = \mathscr{M}(f,x,\lambda-\mu+\mu) \geqslant \mathscr{M}(f,x,\mu). \tag{30}$$

On the other hand, we have

$$\begin{aligned}\mathscr{M}(f,x,\mu) &= (m\beta - m\alpha)[f(a)+f(b)] - \int_\alpha^\beta f(x(t))\,\mathrm{d}(mt) \\ &\quad -(m\beta-m\alpha)f\left(a+b-\frac{1}{m\beta-m\alpha}\int_\alpha^\beta x(t)\,\mathrm{d}(mt)\right) \\ &= m\mathscr{M}(f,x). \end{aligned}\tag{31}$$

Now, combining (30) and (31) we have the right-hand side inequality in (29) proved. The left-hand side inequality is obtained similarly, by exchanging the roles of λ and μ. □

5. Properties of integral Jensen-Mercer's functional under Jensen-Steffensen's conditions

One of the integral analogues of Jensen-Steffensen's inequality was given by R. P. Boas. For a continuous and monotonic function $x:[\alpha,\beta] \to \langle a,b\rangle$, $-\infty < \alpha < \beta < \infty$ and $-\infty \leqslant a < b \leqslant \infty$, for a convex function $f:\langle a,b\rangle \to \mathbb{R}$ and for $\lambda:[\alpha,\beta]\to\mathbb{R}$, either continuous function or of bounded variation, satisfying

$$\lambda(\alpha) \leqslant \lambda(t) \leqslant \lambda(\beta) \quad \text{for all} \quad t\in[\alpha,\beta], \qquad \lambda(\beta)-\lambda(\alpha) > 0, \tag{32}$$

Boas proved that Jensen-Steffensen's inequality

$$f\left(\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta x(t)\,\mathrm{d}\lambda(t)\right) \leqslant \frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta f(x(t))\,\mathrm{d}\lambda(t).$$

holds.

Barić and Matković proved in [2] that Jensen-Mercer's inequality (22) also remains valid when the condition "λ is a nondecreasing function" is relaxed, as in the Boas' result. Namely, they proved that for a continuous and monotonic function $x:[\alpha,\beta]\to[a,b]$, $-\infty<\alpha<\beta<\infty$ and $-\infty<a<b<\infty$, for a continuous and convex function $f:[a,b]\to\mathbb{R}$ and for $\lambda:[\alpha,\beta]\to\mathbb{R}$, either continuous function or of bounded variation, satisfying (32), inequality (22) holds. Only in this section, the use of $\Lambda_{[\alpha,\beta]}$, $-\infty<\alpha<\beta<\infty$, the class of all functions $\lambda:[\alpha,\beta]\to\mathbb{R}$ which are either continuous or of bounded variation and satisfy (32), is fully justified. (In Section 3 we dealt only with the nondecreasing functions from the defined class.) The first integral result that we give here is the integral analogue of Theorem 3.

THEOREM 7. *Let λ and μ be functions from $\Lambda_{[\alpha,\beta]}$, either both continuous or both of bounded variation. If $x:[\alpha,\beta]\to[a,b]$, $-\infty<a<b<\infty$, is a continuous and monotonic function and if $f:[a,b]\to\mathbb{R}$ is a continuous and convex function, then functional $\mathscr{M}(f,x,\cdot)$ defined by* (24) *is superadditive on $\Lambda_{[\alpha,\beta]}$, i.e.*

$$\mathscr{M}(f,x,\lambda+\mu)\geqslant\mathscr{M}(f,x,\lambda)+\mathscr{M}(f,x,\mu)\geqslant 0. \tag{33}$$

Proof. The proof follows the same lines as in Theorem 5, but for the right-hand side inequality in (33) we use integral Jensen-Mercer's inequality (22) under Jensen-Steffensen's conditions. □

The integral analogue of Theorem 4 is given in the form of the result that follows.

THEOREM 8. *Let λ and μ be functions from $\Lambda_{[\alpha,\beta]}$, either both continuous or both of bounded variation. Let*

$$\lambda(\alpha)-\mu(\alpha)\leqslant\lambda(t)-\mu(t)\leqslant\lambda(\beta)-\mu(\beta),\quad t\in[\alpha,\beta],$$

$$\lambda(\beta)-\mu(\beta)>\lambda(\alpha)-\mu(\alpha).$$

If $x:[\alpha,\beta]\to[a,b]$, $-\infty<a<b<\infty$, is a continuous and monotonic function and if $f:[a,b]\to\mathbb{R}$ is a continuous and convex function, then for functional $\mathscr{M}(f,x,\cdot)$ defined by (24) *inequality*

$$\mathscr{M}(f,x,\lambda)\geqslant\mathscr{M}(f,x,\mu) \tag{34}$$

holds on $\Lambda_{[\alpha,\beta]}$.

Proof. Write $\mathscr{M}(f,x,\lambda)=\mathscr{M}(f,x,\lambda-\mu+\mu)$. If we could apply superadditivity property (33) to $\lambda-\mu$ and μ, monotonicity property would also be proved. And that would be the case if $\lambda-\mu$, also continuous or of bounded variation, belonged to $\Lambda_{[\alpha,\beta]}$, which, according to the assumptions of the theorem, is the case. Now, since by (22) is $\mathscr{M}(f,x,\lambda-\mu)\geqslant 0$, we have

$$\mathscr{M}(f,x,\lambda)=\mathscr{M}(f,x,\lambda-\mu+\mu)\geqslant\mathscr{M}(f,x,\lambda-\mu)+\mathscr{M}(f,x,\mu)\geqslant\mathscr{M}(f,x,\mu),$$

which was to be proved. □

REMARK 6. Theorem B, cited from [2] in Section 3, has its integral version, also given in [2]. Hence the latter one can easily be obtained by means of Theorem 8. (Follows the same lines as in the discrete case, in Remark 2.)

In the sequel we lean on Remark 3, and in this setting that means - bounding of functional (24) by a nonweighted functional. As before, we only give an alternative proof of the integral version of Corollary A, cited from [2], so our result is given within another remark.

REMARK 7. With a slightly altered notation from that in [2], according to our former considerations, the result reads:

Let λ be a function from $\Lambda_{[\alpha,\beta]}$. Let $x:[\alpha,\beta]\to[a,b]$, $-\infty<a<b<\infty$, be a continuous and monotonic function and let $f:[a,b]\to\mathbb{R}$ be a continuous and convex function. If m and M are defined by

$$m:=\inf_{\alpha<t<\beta}\left\{\frac{\lambda(t)-\lambda(\alpha)}{t-\alpha},\frac{\lambda(\beta)-\lambda(t)}{\beta-t}\right\},$$

$$M:=\sup_{\alpha<t<\beta}\left\{\frac{\lambda(t)-\lambda(\alpha)}{t-\alpha},\frac{\lambda(\beta)-\lambda(t)}{\beta-t}\right\},$$

then

$$M\mathscr{M}(f,x)\geqslant\mathscr{M}(f,x,\lambda)\geqslant m\mathscr{M}(f,x),\tag{35}$$

where

$$\mathscr{M}(f,x):=(\beta-\alpha)[f(a)+f(b)]-\int_\alpha^\beta f(x(t))\,\mathrm{d}t-(\beta-\alpha)f\left(a+b-\tfrac{1}{\beta-\alpha}\int_\alpha^\beta x(t)\,\mathrm{d}t\right).$$

Proof. Let us prove the right-hand side inequality in (35). According to the definition of m we have that $m\leqslant\dfrac{\lambda(t)-\lambda(\alpha)}{t-\alpha}$, $m\leqslant\dfrac{\lambda(\beta)-\lambda(t)}{\beta-t}$. Hence the following inequalities hold: $\lambda(\alpha)-m\alpha\leqslant\lambda(t)-mt\leqslant\lambda(\beta)-m\beta$. Let μ be a function from $\Lambda_{[\alpha,\beta]}$, such that $\mu(t)=mt$, $\mu(\alpha)=m\alpha$ and $\mu(\beta)=m\beta$. Then we write $\lambda(\alpha)-\mu(\alpha)\leqslant\lambda(t)-\mu(t)\leqslant\lambda(\beta)-\mu(\beta)$. Hence by Theorem 8 we have that

$$\mathscr{M}(f,x,\lambda)=\mathscr{M}(f,x,\lambda-\mu+\mu)\geqslant\mathscr{M}(f,x,\mu).\tag{36}$$

On the other hand, we have that $\mathscr{M}(f,x,\mu)=m\mathscr{M}(f,x)$, as in (31), so the right-hand side inequality in (35) is proved. The left-hand side inequality is obtained similarly, by exchanging the roles of λ and μ. □

Acknowledgments

This research was supported by the Croatian Ministry of Science, Education, and Sports, under Research Grants 036–1170889–1054 (first author), 083–0000000–3227 (second author), and 117–1170889–0888 (third author).

REFERENCES

[1] S. ABRAMOVICH, M. KLARIČIĆ BAKULA, M. MATIĆ, J. E. PEČARIĆ, *A variant of Jensen-Steffensen's inequality and quasi-arithmetic means,* J. Math. Anal. Appl. **307** (2005), 370–386.

[2] J. BARIĆ, A. MATKOVIĆ, *Bounds for the normalized Jensen-Mercer functional,* J. Math. Inequal. **3**, 4 (2009), 529–541.

[3] W. S. CHEUNG, A. MATKOVIĆ, J. PEČARIĆ, *A variant of Jessen's inequality and generalized means,* J. Inequal. Pure Appl. Math. **7**, 1 (2006), Article 10.

[4] S. S. DRAGOMIR, *Bounds for the normalized Jensen's functional*, Bull. Austral. Math. Soc. **74**, 3 (2006), 471–478.
[5] S. S. DRAGOMIR, J. E. PEČARIĆ, L. E. PERSSON, *Properties of some functionals related to Jensen's inequality*, Acta Math. Hungar. **70**, 1–2 (1996), 129–143.
[6] A. MCD. MERCER, *A variant of Jensen's inequality,* J. Inequal. Pure Appl. Math. **4**, 4 (2003), Article 73.
[7] D. S. MITRINOVIĆ, J. E. PEČARIĆ, A. M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1993.
[8] J. E. PEČARIĆ, F. PROSCHAN, Y. L. TONG, *Convex functions, partial orderings, and statistical applications*, Academic Press, Inc, 1992.

(Received May 31, 2011)

Mario Krnić
Faculty of Electrical Engineering and Computing
Unska 3
10000 Zagreb
Croatia
e-mail: Mario.Krnic@fer.hr

Neda Lovričević
Faculty of Civil Engineering and Architecture
Matice hrvatske 15
21 000 Split
Croatia
e-mail: neda.lovricevic@gradst.hr

Josip Pečarić
Faculty of Textile Technology
University of Zagreb
Prilaz baruna Filipovića 28a
10000 Zagreb
Croatia
e-mail: pecaric@element.hr