

SCHUR-CONVEXITY OF THE WEIGHTED ČEBYŠEV FUNCTIONAL II

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(Communicated by A. Vukelić)

Abstract. In this paper the weighted Čebyšev functional $T(p; f, g; a, b)$ is regarded as a function of two variables

$$T(p; f, g; x, y) = \frac{\int_x^y p(t)f(t)g(t)dt}{\int_x^y p(t)dt} - \left(\frac{\int_x^y p(t)f(t)dt}{\int_x^y p(t)dt} \right) \left(\frac{\int_x^y p(t)g(t)dt}{\int_x^y p(t)dt} \right), \quad (x, y) \in [a, b] \times [a, b]$$

where f, g and $p > 0$ are Lebesgue integrable functions. For a function

$$K(p; f, g; x, y) = \left(\int_x^y p(t)dt \right)^2 T(p; f, g; x, y) \quad (x, y) \in [a, b] \times [a, b]$$

the property of Schur-convexity, Schur-geometric convexity, Schur-harmonic convexity and $(1, 1)$ -convexity is proved.

1. Introduction

Let f, g and $p > 0$ be Lebesgue integrable functions on the interval $I = [a, b] \subseteq \mathbb{R}$. In this paper the weighted Čebyšev functional $T(p; f, g; a, b)$ is regarded as a function of two variables

$$T(p; f, g; x, y) = \frac{\int_x^y p(t)f(t)g(t)dt}{\int_x^y p(t)dt} - \left(\frac{\int_x^y p(t)f(t)dt}{\int_x^y p(t)dt} \right) \left(\frac{\int_x^y p(t)g(t)dt}{\int_x^y p(t)dt} \right), \quad (x, y) \in I^2.$$

In [4] we proved Schur-convexity of a function $T(1; f, g; x, y)$ with $(x, y) \in I^2$.

THEOREM A 1. *Let f and g be Lebesgue integrable functions on $I = [a, b] \subseteq \mathbb{R}$. If they are monotone in the same sense (in the opposite sense) then $T(x, y) := T(1; f, g; x, y)$, $(x, y) \in I^2$ is Schur-convex (Schur-concave) on I .*

Using the following notations:

$$\overline{P}(x, y) := \int_x^y p(t)dt, \\ \overline{f_p}(x, y) := \frac{1}{\int_x^y p(t)dt} \int_x^y p(t)f(t)dt \quad \text{and} \quad \overline{g_p}(x, y) := \frac{1}{\int_x^y p(t)dt} \int_x^y p(t)g(t)dt$$

we obtained next result for the weighted Čebyšev functional (see [5]):

Mathematics subject classification (2010): Primary 26D15, Secondary 26D99.

Keywords and phrases: Convex functions, Schur-convex function, Čebyšev functional.

THEOREM A 2. *Let f and g be Lebesgue integrable functions on $I = [a, b] \subseteq \mathbb{R}$ and let p be a positive continuous weight on I such that pf and pg are also Lebesgue integrable functions on I . Then $T(x, y) := T(p; f, g; x, y)$ is Schur-convex (Schur-concave) on I^2 if and only if the inequality*

$$T(x, y) \leq \frac{p(x)(\overline{f_p}(x, y) - f(x))(\overline{g_p}(x, y) - g(x)) + p(y)(\overline{f_p}(x, y) - f(y))(\overline{g_p}(x, y) - g(y))}{p(x) + p(y)}. \quad (1)$$

holds (reverses) for all x, y in I .

For a function $K : I^2 \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$K(p; f, g; x, y) = \left(\int_x^y p(t) dt \right)^2 \cdot T(p; f, g; x, y), \quad (x, y) \in I^2 \quad (2)$$

the author in [17] proved the following statement:

THEOREM A 3. *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be Lebesgue integrable functions, and let $p : [a, b] \rightarrow \mathbb{R}_+$ be a Lebesgue integrable function. A function $K(x, y) := K(p; f, g; x, y)$ defined as (2) is increasing (decreasing) with y on $I = [a, b]$ and decreasing (increasing) with x on I if f and g are monotone in the same sense (in the opposite sense).*

In this paper we prove the property of Schur-convexity, Schur-geometric convexity and Schur-harmonic convexity of a function $K(x, y)$ with (x, y) , depending of monotonicity and simultan ordering of the functions f and g . We also show $(1, 1)$ -convexity of a function K .

2. Definitions and properties

The concepts of majorizations and Schur-convex functions involve convex functions and measure of the diversity of the components of an n -tuple in \mathbb{R}^n . Most of the basic results are given in Marshall and Olkin's book [8]. In the recently references [1], [2], [3], [10], [11], [13], [15], [16], we can find the definitions and applications of the Schur-convex, Schur-geometrically convex and Schur-harmonic convex functions.

In this section we will recall usefull definitions, lemmas and theorems:

DEFINITION 1. Let \mathbf{x}, \mathbf{y} be in $E \subseteq \mathbb{R}^n$ and let $x_{[i]}, y_{[i]}$ denote the i th largest component in \mathbf{x} and \mathbf{y} . We say \mathbf{y} majorizes \mathbf{x} , denote $\mathbf{x} \prec \mathbf{y}$ if

$$\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}, \quad k = 1, \dots, n-1,$$

$$\sum_{i=1}^n x_{[i]} = \sum_{i=1}^n y_{[i]}.$$

DEFINITION 2. Let \mathbf{x}, \mathbf{y} be in $E \subseteq \mathbb{R}^n$. A function $F : E \rightarrow \mathbb{R}$ is called a Schur-convex function on E if

$$F(x_1, x_2, \dots, x_n) \leq F(y_1, y_2, \dots, y_n)$$

for each \mathbf{x} and \mathbf{y} in E such that $\mathbf{x} \prec \mathbf{y}$.

A function F is Schur-concave if and only if $-F$ is a Schur-convex function.

DEFINITION 3. Let \mathbf{x}, \mathbf{y} be in $E \subseteq \mathbb{R}_+^n$. A function $F : E \rightarrow [0, \infty)$ is called a Schur-geometrically convex function on E if

$$F(x_1, x_2, \dots, x_n) \leq F(y_1, y_2, \dots, y_n)$$

for each two positive \mathbf{x} and \mathbf{y} in E such that $(\ln x_1, \ln x_2, \dots, \ln x_n) \prec (\ln y_1, \ln y_2, \dots, \ln y_n)$ i.e \mathbf{y} logarithm majorizes \mathbf{x} .

A function F is Schur-geometrically concave if and only if $-F$ is a Schur-geometrically convex function.

DEFINITION 4. Let \mathbf{x}, \mathbf{y} be in $E \subseteq \mathbb{R}_+^n$. A function $F : E \rightarrow [0, \infty)$ is called a Schur-harmonic convex function on E if

$$F(x_1, x_2, \dots, x_n) \leq F(y_1, y_2, \dots, y_n)$$

for each two positive \mathbf{x} and \mathbf{y} on E such that it holds $(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n}) \prec (\frac{1}{y_1}, \frac{1}{y_2}, \dots, \frac{1}{y_n})$.

A function F is Schur-harmonic concave if and only if $-F$ is a Schur-harmonic convex function.

The next lemmas gives us the characterisations of Schur-convexity, Schur-geometrically convexity and Schur-harmonic convexity (see [8, p. 57], [11, p. 333], [14], [15, p. 108], [16]):

LEMMA A 1. Let $E \subseteq \mathbb{R}^n$ be a symmetric convex set with a nonempty interior. Let $F : E \rightarrow \mathbb{R}$ be a continuous function on E and differentiable on the interior of E . Then F is Schur-convex (Schur-concave) if and only if it is symmetric and

$$\left(\frac{\partial F}{\partial x_2} - \frac{\partial F}{\partial x_1} \right) (x_2 - x_1) \geq 0 \ (\leq 0) \quad (3)$$

holds for all \mathbf{x} in the interior of E , $x_1 \neq x_2$.

LEMMA A 2. Let $E \subseteq \mathbb{R}_+^n$ be a symmetric logarithm convex set with a nonempty interior, i.e. $\ln E = \{\ln \mathbf{x} = (\ln x_1, \dots, \ln x_n) : \mathbf{x} \in E\}$ is a convex set. Let $F : E \rightarrow [0, \infty)$ be a continuous function on E and differentiable on the interior of E . Then F is Schur-geometrically convex (Schur-geometrically concave) if it is symmetric and the inequality

$$\left(x_2 \frac{\partial F}{\partial x_2} - x_1 \frac{\partial F}{\partial x_1} \right) (\ln x_2 - \ln x_1) \geq 0 \ (\leq 0) \quad (4)$$

holds for all \mathbf{x} in the interior of E , $x_1 \neq x_2$.

LEMMA A 3. Let $E \subseteq \mathbb{R}_+^n$ be a symmetric harmonic convex set with a nonempty interior, i.e. $\mathbf{1}/E = \{\mathbf{1}/\mathbf{x} = (\frac{1}{x_1}, \dots, \frac{1}{x_n}) : \mathbf{x} \in E\}$ is a convex set. Let $F : E \rightarrow [0, \infty)$ be a continuous function on E and differentiable on the interior of E . Then F is Schur-harmonic convex (Schur-harmonic concave) if it is symmetric and

$$\left(x_2^2 \frac{\partial F}{\partial x_2} - x_1^2 \frac{\partial F}{\partial x_1}\right)(x_2 - x_1) \geq 0 \ (\leq 0) \quad (5)$$

holds for all \mathbf{x} in the interior of E , $x_1 \neq x_2$.

DEFINITION 5. The functions f and $g : I^n \rightarrow \mathbb{R}$ are similarly ordered if

$$(f(x_1, x_2, \dots, x_n) - f(y_1, y_2, \dots, y_n)) \cdot (g(x_1, x_2, \dots, x_n) - g(y_1, y_2, \dots, y_n)) \geq 0,$$

for each two n -tuples \mathbf{x} and \mathbf{y} on I^n .

Function f and g are oppositely ordered if f and $-g$ are similarly ordered.

We recall the well-known Čebyšev inequality for monotone functions (see [9, p. 239], [11, p. 197]) and for similar ordered functions (see [7, p. 168], [9, p. 252]):

THEOREM A 4. Let f and g be Lebesgue integrable on an interval $I = [a, b] \subseteq \mathbb{R}$ and let p be a positive continuous weight on I such that pf , pg and pg are also Lebesgue integrable functions on I . If f and g are monotone in the same sense (in the opposite sense) then it holds

$$T(p; f, g; a, b) \geq 0 \ (\leq 0).$$

THEOREM A 5. Let f and g be Lebesgue integrable functions on $I = [a, b] \subseteq \mathbb{R}$ and let p be a positive continuous weight on I such that pf , pg and pg are also Lebesgue integrable functions on I . If f and g are similarly (oppositely) ordered then it holds

$$T(p; f, g; a, b) \geq 0 \ (\leq 0).$$

Popoviciu in ([12, p. 60]) used the (n, m) -divided difference of the function in the definition of the (n, m) -convexity (concavity) (see also [11, p. 18]):

DEFINITION 6. A function $F : I^2 \rightarrow \mathbb{R}$ is (n, m) -convex (concave) if for all distinct points $x_0, x_1, \dots, x_n \in I$ and $y_0, y_1, \dots, y_m \in I$ yields

$$\left[\begin{array}{c} x_0, x_1, \dots, x_n \\ y_0, y_1, \dots, y_m \end{array} \right] F = \sum_{i=0}^n \sum_{j=0}^m \frac{F(x_i, x_j)}{\omega'(x_i) \cdot w'(y_j)} \geq 0 \ (\leq 0),$$

where $\omega(x) = \prod_{i=0}^n (x - x_i)$, $w(y) = \prod_{j=0}^m (y - y_j)$.

The next lemma give us the necessary and sufficient conditions for verifying the (n, m) -convexity (concavity):

LEMMA A 4. *If the partial derivative $\frac{\partial^{(n+m)}F}{\partial x^n \partial y^m}$ exists then $F : I^2 \rightarrow \mathbb{R}$ is (n, m) -convex (concave) if and only if*

$$\frac{\partial^{(n+m)}F}{\partial x^n \partial y^m} \geq 0 \ (\leq 0).$$

3. Results

THEOREM 3.1. *Let f and g be Lebesgue integrable on interval $I = [a, b] \subseteq \mathbb{R}$. Let p be a positive continuous weight on I such that pf, pg and pf, pg are also Lebesgue integrable functions on I . If f and g are monotone in the same sense (in the opposite sense) then for a function $K(x, y) := K(p; f, g; x, y)$ defined by (2) holds*

- (i) $K(x, y) \geq 0$ (≤ 0), for $(x, y) \in I^2$;
- (ii) $K(x, y)$ is Schur-convex (Schur-concave) with (x, y) on $I^2 \subseteq \mathbb{R}^2$;
- (iii) $K(x, y)$ Schur-geometrical convex (Schur-geometrical concave) with (x, y) on $I^2 \subseteq \mathbb{R}_+^2$;
- (iv) $K(x, y)$ is Schur-harmonic convex (Schur-harmonic concave) with (x, y) on $I^2 \subseteq \mathbb{R}_+^2$;
- (v) $K(x, y) := K(p; f, g; x, y)$ is an $(1, 1)$ -concave (convex) function on $I^2 \subseteq \mathbb{R}^2$.

Proof. Let f and g be monotone in the same sense (in the opposite sense). Let p be a positive continuous weight on I such that pf, pg and pf, pg are also Lebesgue integrable functions on $I = [a, b]$.

We may assume that $x < y$ without loss of generality.

Now, we calculate $\frac{\partial K(x, y)}{\partial y}$, $\frac{\partial K(x, y)}{\partial x}$ and $\frac{\partial^2 K(x, y)}{\partial x \partial y}$:

$$\frac{\partial K(x, y)}{\partial y} = p(y) \int_x^y p(t)(f(t) - f(y))(g(t) - g(y))dt; \quad (6)$$

$$\frac{\partial K(x, y)}{\partial x} = -p(x) \int_x^y p(t)(f(t) - f(x))(g(t) - g(x))dt; \quad (7)$$

$$\frac{\partial^2 K(x, y)}{\partial x \partial y} = -p(x)p(y)(f(y) - f(x))(g(y) - g(x)) \quad (8)$$

(i) Applying Čebyšev inequality, Theorem A4 to the function $K(p; f, g; x, y) = [P(x, y)]^2 \cdot T(p; f, g; x, y)$ we obtain that holds $K(x, y) \geq 0$ (≤ 0).

(ii) To prove Schur-convexity of $K(p; x, y)$ (or Schur-concavity) we apply Lemma A1. It is sufficient to discuss the following inequality $\left(\frac{\partial K(x, y)}{\partial y} - \frac{\partial K(x, y)}{\partial x}\right)(y - x) \geq 0$ (≤ 0), for all $x, y \in [a, b]$, since the function $K(x, y)$ is evidently symmetric. According Theorem A3 we know that $K(x, y)$ is increasing (decreasing) with y on I and decreasing (increasing) with x on I . So, it follows Schur-convexity (Schur-concavity) of K as in the statement (ii).

(iii) The set $I^2 \subseteq \mathbb{R}_+^2$ is a symmetric logarithm convex set. By applying the condition in Lemma A2 to the function $K(x, y)$ we conclude that $\left(y \frac{\partial K(x, y)}{\partial y} - x \frac{\partial K(x, y)}{\partial x}\right)$.

$(\ln y - \ln x) \geq 0$ (≤ 0), $(x, y) \in I^2 \subseteq \mathbb{R}_+^2$, i.e. $K(x, y)$ is Schur-geometrically convex (Schur - geometrically concave) with (x, y) on $I^2 \subseteq \mathbb{R}_+^2$.

(iv) The set $I^2 \subseteq \mathbb{R}_+^2$ is a symmetric harmonic convex set. According Lemma A3 we conclude that $\left(y^2 \frac{\partial K(x, y)}{\partial y} - x^2 \frac{\partial K(x, y)}{\partial x}\right)(y - x) \geq 0$ (≤ 0), $(x, y) \in I^2 \subseteq \mathbb{R}_+^2$, i.e. $K(x, y)$ is Schur-harmonic convex (Schur-harmonic concave) with (x, y) on $I^2 \subseteq \mathbb{R}_+^2$.

(v) Since $\frac{\partial^2 K(x, y)}{\partial x \partial y} \leq 0$ (≥ 0), Lemma A4 implies that $K(x, y)$ is the (1, 1)-concave (convex) function on $I^2 \subseteq \mathbb{R}^2$. \square

THEOREM 3.2. *Let f and g be Lebesgue integrable functions on $I = [a, b] \subseteq \mathbb{R}_+$ and let p be a positive continuous weight on I such that pf, pg and pg are also Lebesgue integrable functions on $I = [a, b]$. If f and g are similarly (oppositely) ordered then for the function $K(x, y) := K(p; f, g; x, y)$ defined by (2) holds*

(i) $K(x, y)$ is increasing (decreasing) with y on I and decreasing (increasing) with x on I ;

(ii) $K(x, y) \geq 0$ (≤ 0), for $(x, y) \in I^2 \subseteq \mathbb{R}_+^2$;

(iii) $K(x, y)$ is Schur-convex (Schur-concave) with (x, y) on $I^2 \subseteq \mathbb{R}^2$;

(iv) $K(x, y)$ is Schur-geometrical convex (Schur-geometrical concave) with (x, y) on $I^2 \subseteq \mathbb{R}_+^2$;

(v) $K(x, y)$ is Schur-harmonic convex (Schur-harmonic concave) with (x, y) on $I^2 \subseteq \mathbb{R}_+^2$;

(vi) $K(x, y)$ is an (1, 1)-concave (convex) function on $I^2 \subseteq \mathbb{R}^2$.

Proof. Let f and g are similarly ordered (oppositely ordered) on $I = [a, b] \subseteq \mathbb{R}_+$. Let p be a positive continuous weight on I such that pf, pg and pg are also Lebesgue integrable functions on I .

(i) From (6) and (7) we have that $\frac{\partial K(x, y)}{\partial y} \geq 0$ (≤ 0) and $\frac{\partial K(x, y)}{\partial x} \leq 0$ (≥ 0) So, it holds statement (i).

(ii) $K(p; f, g; x, y) = [P(x, y)]^2 \cdot T(p; f, g; x, y)$ and Čebyšev inequality, Theorem A5 implies statement (ii).

(iii) The claim (i) implies that $\left(\frac{\partial K(x, y)}{\partial y} - \frac{\partial K(x, y)}{\partial x}\right)(y - x) \geq 0$ (≤ 0) on I^2 and according Lemma A3 it follows the property of Schur-convexity (Schur-concavity) of $K(x, y)$ on I^2 .

(iv) Similarly, by statement (i) we conclude that $\left(y \frac{\partial K(x, y)}{\partial y} - x \frac{\partial K(x, y)}{\partial x}\right) \cdot (\ln y - \ln x) \geq 0$ (≤ 0) on I^2 and according Lemma A2 we obtain the property of Schur-geometrically convexity (Schur-geometrically concavity) of K on I^2 .

(v) The claim (i) implies that $\left(y^2 \frac{\partial K(x, y)}{\partial y} - x^2 \frac{\partial K(x, y)}{\partial x}\right)(y - x) \geq 0$ (≤ 0) on I^2 and according Lemma A3 we obtain the property of Schur-harmonic convexity (Schur-harmonic concavity) of K on I^2 .

(vi) Applying (8) for similarly (opposit) ordered functions f and g we have $\frac{\partial^2 K(x, y)}{\partial x \partial y} \leq 0$ (≥ 0). Lemma A4 implies that $K(x, y)$ is an (1, 1)-concave (convex) function. \square

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(Received June 22, 2011)

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