Journal of
Mathematical
Inequalities
Volume 6, Number 1 (2012), 141–147

SCHUR–CONVEXITY OF THE WEIGHTED ČEBYŠEV FUNCTIONAL II

V. ČULJAK

(*Communicated by A. Vukelić*)

Abstract. In this paper the weighted Čebyšev functional $T(p;f,g;a,b)$ is regarded as a function of two variables

$$T(p;f,g;x,y)=\frac{\int_x^y p(t)f(t)g(t)dt}{\int_x^y p(t)dt}-\left(\frac{\int_x^y p(t)f(t)dt}{\int_x^y p(t)dt}\right)\left(\frac{\int_x^y p(t)g(t)dt}{\int_x^y p(t)dt}\right),\ (x,y)\in[a,b]\times[a,b]$$

where f,g and $p>0$ are Lebesgue integrable functions. For a function

$$K(p;f,g;x,y)=\left(\int_x^y p(t)dt\right)^2 T(p;f,g;x,y)\ (x,y)\in[a,b]\times[a,b]$$

the property of Schur-covexity, Schur-geometric convexity, Schur-harmonic convexity and $(1,1)$-convexity is proved.

1. Introduction

Let f,g and $p>0$ be Lebesgue integrable functions on the interval $I=[a,b]\subseteq\mathbb{R}$. In this paper the weighted Čebyšev functional $T(p;f,g;a,b)$ is regarded as a function of two variables

$$T(p;f,g;x,y)=\frac{\int_x^y p(t)f(t)g(t)dt}{\int_x^y p(t)dt}-\left(\frac{\int_x^y p(t)f(t)dt}{\int_x^y p(t)dt}\right)\left(\frac{\int_x^y p(t)g(t)dt}{\int_x^y p(t)dt}\right),\ (x,y)\in I^2.$$

In [4] we proved Schur-convexity of a function $T(1;f,g;x,y)$ with $(x,y)\in I^2$.

THEOREM A 1. *Let f and g be Lebesgue integrable functions on $I=[a,b]\subseteq\mathbb{R}$. If they are monotone in the same sense (in the opposite sense) then $T(x,y):=T(1;f,g;x,y)$, $(x,y)\in I^2$ is Schur-convex (Schur-concave) on I.*

Using the following notations:
$P(x,y):=\int_x^y p(t)dt$,
$\overline{f_p}(x,y):=\frac{1}{\int_x^y p(t)dt}\int_x^y p(t)f(t)dt$ and $\overline{g}_p(x,y):=\frac{1}{\int_x^y p(t)dt}\int_x^y p(t)g(t)dt$
we obtained next result for the weighted Čebyšev functional (see [5]):

Mathematics subject classification (2010): Primary 26D15, Secondary 26D99.
Keywords and phrases: Convex functions, Schur-convex function, Čebyšev functional.

Paper JMI-06-14

THEOREM A 2. *Let f and g be Lebesgue integrable functions on $I=[a,b]\subseteq\mathbb{R}$ and let p be a positive continuous weight on I such that pf and pg are also Lebesgue integrable functions on I. Then $T(x,y):=T(p;f,g;x,y)$ is Schur-convex (Schur-concave) on I^2 if and only if the inequality*

$$T(x,y)\leqslant\frac{p(x)(\overline{f_p}(x,y)-f(x))(\overline{g_p}(x,y)-g(x))+p(y)(\overline{f_p}(x,y)-f(y))(\overline{g_p}(x,y)-g(y))}{p(x)+p(y)}. \tag{1}$$

holds (reverses) for all x,y in I.

For a function $K:I^2\subseteq\mathbb{R}^2\to\mathbb{R}$ defined by

$$K(p;f,g;x,y)=\left(\int_x^y p(t)dt\right)^2\cdot T(p;f,g;x,y),\ (x,y)\in I^2 \tag{2}$$

the author in [17] proved the following statement:

THEOREM A 3. *Let $f,g:[a,b]\to\mathbb{R}$ be Lebesgue integrable functions, and let $p:[a,b]\to\mathbb{R}_+$ be a Lebesgue integrable function. A function $K(x,y):=K(p;f,g;x,y)$ defined as (2) is increasing (decreasing) with y on $I=[a,b]$ and decreasing (increasing) with x on I if f and g are monotone in the same sense (in the opposite sense).*

In this paper we prove the property of Schur-convexity, Schur-geometric convexity and Schur-harmonic convexity of a function $K(x,y)$ with (x,y), depending of monotonicity and simultan ordering of the functions f and g. We also show $(1,1)$-convexity of a function K.

2. Definitions and properties

The concepts of majorizations and Schur-convex functions involve convex functions and measure of the diversity of the components of an n-tuple in $\mathbb{R}^n$. Most of the basic results are given in Marshall and Olkin's book [8]. In the recently references [1], [2], [3], [10], [11], [13], [15], [16], we can find the definitions and applications of the Schur-convex, Schur-geometrically convex and Schur-harmonic convex functions.

In this section we will recall usefull definitions, lemmas and theorems:

DEFINITION 1. Let $\mathbf{x},\mathbf{y}$ be in $E\subseteq\mathbb{R}^n$ and let $x_{[i]}$, $y_{[i]}$ denote the i th largest component in $\mathbf{x}$ and $\mathbf{y}$. We say $\mathbf{y}$ majorizes $\mathbf{x}$, denote $\mathbf{x}\prec\mathbf{y}$ if

$$\sum_{i=1}^{k}x_{[i]}\leqslant\sum_{i=1}^{k}y_{[i]},\ k=1,...,n-1,$$
$$\sum_{i=1}^{n}x_{[i]}=\sum_{i=1}^{n}y_{[i]}.$$

DEFINITION 2. Let $\mathbf{x},\mathbf{y}$ be in $E \subseteq \mathbb{R}^n$. A function $F : E \to \mathbb{R}$ is called a Schur-convex function on E if

$$F(x_1,x_2,..,x_n) \leqslant F(y_1,y_2,..,y_n)$$

for each $\mathbf{x}$ and $\mathbf{y}$ in E such that $\mathbf{x} \prec \mathbf{y}$.

A function F is Schur-concave if and only if $-F$ is a Schur-convex function.

DEFINITION 3. Let $\mathbf{x},\mathbf{y}$ be in $E \subseteq \mathbb{R}^n_+$. A function $F : E \to [0,\infty)$ is called a Schur-geometrically convex function on E if

$$F(x_1,x_2,..,x_n) \leqslant F(y_1,y_2,..,y_n)$$

for each two positive $\mathbf{x}$ and $\mathbf{y}$ in E such that $(\ln x_1, \ln x_2,.., \ln x_n) \prec (\ln y_1, \ln y_2,.., \ln y_n)$ i.e $\mathbf{y}$ logarithm majorizes $\mathbf{x}$.

A function F is Schur-geometrically concave if and only if $-F$ is a Schur-geometrically convex function.

DEFINITION 4. Let $\mathbf{x},\mathbf{y}$ be in $E \subseteq \mathbb{R}^n_+$. A function $F : E \to [0,\infty)$ is called a Schur-harmonic convex function on E if

$$F(x_1,x_2,..,x_n) \leqslant F(y_1,y_2,..,y_n)$$

for each two positive $\mathbf{x}$ and $\mathbf{y}$ on E such that it holds $(\frac{1}{x_1}, \frac{1}{x_2},.., \frac{1}{x_n}) \prec (\frac{1}{y_1}, \frac{1}{y_2},.., \frac{1}{y_n})$.

A function F is Schur-harmonic concave if and only if $-F$ is a Schur-harmonic convex function.

The next lemmas gives us the characterisations of Schur-convexity, Schur-geometrically convexity and Schur-harmonic convexity (see [8, p. 57], [11, p. 333], [14], [15, p. 108], [16]):

LEMMA A 1. *Let $E \subseteq \mathbb{R}^n$ be a symmetric convex set with a nonempty interior. Let $F : E \to \mathbb{R}$ be a continuous function on E and differentiable on the interior of E. Then F is Schur-convex (Schur-concave) if and only if it is symmetric and*

$$\left(\frac{\partial F}{\partial x_2} - \frac{\partial F}{\partial x_1}\right)(x_2 - x_1) \geqslant 0 \ (\leqslant 0) \tag{3}$$

holds for all $\mathbf{x}$ in the interior of E, $x_1 \neq x_2$.

LEMMA A 2. *Let $E \subseteq \mathbb{R}^n_+$ be a symmetric logarithm convex set with a nonempty interior, i.e. $\ln E = \{\ln \mathbf{x} = (\ln x_1,..., \ln x_n) : \mathbf{x} \in E\}$ is a convex set. Let $F : E \to [0,\infty)$ be a continuous function on E and differentiable on the interior of E. Then F is Schur-geometrically convex (Schur-geometrically concave) if it is symmetric and the inequality*

$$\left(x_2\frac{\partial F}{\partial x_2} - x_1\frac{\partial F}{\partial x_1}\right)(\ln x_2 - \ln x_1) \geqslant 0 \ (\leqslant 0) \tag{4}$$

holds for all $\mathbf{x}$ in the interior of E, $x_1 \neq x_2$.

LEMMA A 3. *Let $E \subseteq \mathbb{R}^n_+$ be a symmetric harmonic convex set with a nonempty interior, i.e. $\mathbf{1}/E = \{1/\mathbf{x} = (\frac{1}{\mathbf{x}_1}, \ldots, \frac{1}{\mathbf{x}_n}) : \mathbf{x} \in E\}$ is a convex set. Let $F : E \to [0,\infty)$ be a continuous function on E and differentiable on the interior of E. Then F is Schur-harmonic convex (Schur-harmonic concave) if it is symmetric and*

$$\left(x_2^2 \frac{\partial F}{\partial x_2} - x_1^2 \frac{\partial F}{\partial x_1}\right)(x_2 - x_1) \geqslant 0 \ (\leqslant 0) \tag{5}$$

holds for all $\mathbf{x}$ in the interior of E, $x_1 \neq x_2$.

DEFINITION 5. The functions f and $g : I^n \to \mathbb{R}$ are similarly ordered if

$$(f(x_1,x_2,..,x_n) - f(y_1,y_{2,..},y_n)) \cdot (g(x_1,x_2,.,x_n) - g(y_1,y_{2,..},y_n)) \geqslant 0,$$

for each two n-tuples $\mathbf{x}$ and $\mathbf{y}$ on I^n.

Function f and g are oppositely ordered if f and $-g$ are similarly ordered.

We recall the well-known Čebyšev inequality for monotone functions (see [9, p. 239], [11, p. 197]) and for similar ordered functions (see [7, p. 168], [9, p. 252]):

THEOREM A 4. *Let f and g be Lebesgue integrable on an interval $I = [a,b] \subseteq \mathbb{R}$ and let p be a positive continuous weight on I such that pfg, pf and pg are also Lebesgue integrable functions on I. If f and g are monotone in the same sense (in the opposite sense) then it holds*

$$T(p;f,g;a,b) \geqslant 0 \ (\leqslant 0).$$

THEOREM A 5. *Let f and g be Lebesgue integrable functions on $I = [a,b] \subseteq \mathbb{R}$ and let p be a positive continuous weight on I such that pfg, pf and pg are also Lebesgue integrable functions on I. If f and g are similarly (oppositely) ordered then it holds*

$$T(p;f,g;a,b) \geqslant 0 \ (\leqslant 0).$$

Popoviciu in ([12, p. 60]) used the (n,m) - divided difference of the function in the definition of the $(n,m)-$ convexity (concavity) (see also [11, p. 18]):

DEFINITION 6. A function $F : I^2 \to \mathbb{R}$ is (n,m)-convex (concave) if for all distinct points $x_0, x_1, .., x_n \in I$ and $y_0, y_1, .., y_m \in I$ yilds

$$\begin{bmatrix} x_0, & x_1, & ., & ., & x_n \\ y_0, & y_1, & ., & ., & y_m \end{bmatrix} F = \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{F(x_i, x_j)}{\omega'(x_i) \cdot w'(y_j)} \geqslant 0 \ (\leqslant 0),$$

where $\omega(x) = \prod_{i=0}^{n} (x - x_i)$, $w(y) = \prod_{j=0}^{m} (y - y_j)$.

The next lemma give us the necessary and sufficient conditions for verifying the $(n,m)-$ convexity (concavity):

LEMMA A 4. *If the partial derivative* $\frac{\partial^{(n+m)}F}{\partial x^n \partial y^m}$ *exists then* $F : I^2 \to \mathbb{R}$ *is* (n,m)*-convex (concave) if and only if*

$$\frac{\partial^{(n+m)}F}{\partial x^n \partial y^m} \geqslant 0 \ (\leqslant 0).$$

3. Results

THEOREM 3.1. *Let* f *and* g *be Lebesgue integrable on interval* $I = [a,b] \subseteq \mathbb{R}$*. Let* p *be a positive continuous weight on* I *such that* pfg*,* pf *and* pg *are also Lebesgue integrable functions on* I*. If* f *and* g *are monotone in the same sense (in the opposite sense) then for a function* $K(x,y) := K(p;f,g;x,y)$ *defined by (2) holds*

(i) $K(x,y) \geqslant 0 \ (\leqslant 0)$*, for* $(x,y) \in I^2$*;*

(ii) $K(x,y)$ *is Schur-convex (Schur-concave) with* (x,y) *on* $I^2 \subseteq \mathbb{R}^2$*;*

(iii) $K(x,y)$ *Schur-geometrical convex (Schur-geometrical concave) with* (x,y) *on* $I^2 \subseteq \mathbb{R}^2_+$*;*

(iv) $K(x,y)$ *is Schur-harmonic convex (Schur-harmonic concave) with* (x,y) *on* $I^2 \subseteq \mathbb{R}^2_+$*;*

(v) $K(x,y) := K(p;f,g;x,y)$ *is an* $(1,1)$*-concave (convex) function on* $I^2 \subseteq \mathbb{R}^2$*.*

Proof. Let f and g be monotone in the same sense (in the opposite sense). Let p be a positive continuous weight on I such that pfg, pf and pg are also Lebesgue integrable functions on $I = [a,b]$.

We may assume that $x < y$ without loss of generality.

Now, we calculate $\frac{\partial K(x,y)}{\partial y}$, $\frac{\partial K(x,y)}{\partial x}$ and $\frac{\partial^2 K(x,y)}{\partial x \partial y}$:

$$\frac{\partial K(x,y)}{\partial y} = p(y)\int_x^y p(t)(f(t)-f(y))(g(t)-g(y))dt; \tag{6}$$

$$\frac{\partial K(x,y)}{\partial x} = -p(x)\int_x^y p(t)(f(t)-f(x))(g(t)-g(x))dt; \tag{7}$$

$$\frac{\partial^2 K(x,y)}{\partial x \partial y} = -p(x)p(y)(f(y)-f(x))(g(y)-g(x)) \tag{8}$$

(i) Applying Čebyšev inequality, Theorem A4 to the function $K(p;f,g;x,y) = [P(x,y)]^2 \cdot T(p;f,g;x,y)$ we obtain that holds $K(x,y) \geqslant 0 \ (\leqslant 0)$.

(ii) To prove Schur-convexity of $K(p;x,y)$ (or Schur-concavity) we apply Lemma A1. It is sufficient to discuss the following inequlity $\left(\frac{\partial K(x,y)}{\partial y} - \frac{\partial K(x,y)}{\partial x}\right)(y-x) \geqslant 0$ $(\leqslant 0)$, for all $x,y \in [a,b]$, since the function $K(x,y)$ is evidently symmetric. According Theorem A3 we know that $K(x,y)$ is increasing (decreasing) with y on I and decreasing (increasing) with x on I. So, it follows Schur-convexity (Schur-concavity) of K as in the statement (ii).

(iii) The set $I^2 \subseteq \mathbb{R}^2_+$ is a symmetric logarithm convex set. By applying the condition in Lemma A2 to the function $K(x,y)$ we conclude that $\left(y\frac{\partial K(x,y)}{\partial y} - x\frac{\partial K(x,y)}{\partial x}\right) \cdot$

$(\ln y-\ln x)\geqslant 0\ \ (\leqslant 0),\ (x,y)\in I^2\subseteq\mathbb{R}_+^2$, i.e. $K(x,y)$ is Schur-geometically convex (Schur - geometically concave) with (x,y) on $I^2\subseteq\mathbb{R}_+^2$.

(iv) The set $I^2\subseteq\mathbb{R}_+^2$ is a symmetric harmonic convex set. According Lemma A3 we conclude that $\left(y^2\frac{\partial K(x,y)}{\partial y}-x^2\frac{\partial K(x,y)}{\partial x}\right)(y-x)\geqslant 0\ \ (\leqslant 0),\ (x,y)\in I^2\subseteq\mathbb{R}_+^2$, i.e. $K(x,y)$ is Schur-harmonic convex (Schur-harmonic concave) with (x,y) on $I^2\subseteq\mathbb{R}_+^2$.

(v) Since $\frac{\partial^2 K(x,y)}{\partial x\partial y}\leqslant 0\ \ (\geqslant 0)$, Lemma A4 implies that $K(x,y)$ is the $(1,1)$-concave (convex) function on $I^2\subseteq\mathbb{R}^2$. $\square$

THEOREM 3.2. *Let f and g be Lebesgue integrable functions on $I=[a,b]\subseteq\mathbb{R}_+$ and let p be a positive continuous weight on I such that pfg, pf and pg are also Lebesgue integrable functions on $I=[a,b]$. If f and g are similarly (oppositely) ordered then for the function $K(x,y):=K(p;f,g;x,y)$ defined by (2) holds*

(i) $K(x,y)$ is increasing (decreasing) with y on I and decreasing (increasing) with x on I;

(ii) $K(x,y)\geqslant 0\ \ (\leqslant 0)$, for $(x,y)\in I^2\subseteq\mathbb{R}_+^2$;

(iii) $K(x,y)$ is Schur-convex (Schur-concave) with (x,y) on $I^2\subseteq\mathbb{R}^2$;

(iv) $K(x,y)$ is Schur-geometrical convex (Schur-geometrical concave) with (x,y) on $I^2\subseteq\mathbb{R}_+^2$;

(v) $K(x,y)$ is Schur-harmonic convex (Schur-harmonic concave) with (x,y) on $I^2\subseteq\mathbb{R}_+^2$;

(vi) $K(x,y)$ is an $(1,1)$-concave (convex) function on $I^2\subseteq\mathbb{R}^2$.

Proof. Let f and g are similarly ordered (oppositely ordered) on $I=[a,b]\subseteq\mathbb{R}_+$. Let p be a positive continuous weight on I such that pfg, pf and pg are also Lebesgue integrable functions on I.

(i) From (6) and (7) we have that $\frac{\partial K(x,y)}{\partial y}\geqslant 0\ \ (\leqslant 0)$ and $\frac{\partial K(x,y)}{\partial x}\leqslant 0\ \ (\geqslant 0)$ So, it holds statement (i).

(ii) $K(p;f,g;x,y)=[P(x,y)]^2\cdot T(p;f,g;x,y)$ and Čebyšev inequality, Theorem A5 implies statement (ii).

(iii) The claim (i) implises that $\left(\frac{\partial K(x,y)}{\partial y}-\frac{\partial K(x,y)}{\partial x}\right)(y-x)\geqslant 0\ \ (\leqslant 0)$ on I^2 and according Lemma A3 it follows the property of Schur-convexity (Schur-concavity) of $K(x,y)$ on I^2.

(iv) Similarly, by statement (i) we conclude that $\left(y\frac{\partial K(x,y)}{\partial y}-x\frac{\partial K(x,y)}{\partial x}\right)\cdot$ $(\ln y-\ln x)\geqslant 0\ \ (\leqslant 0)$ on I^2 and according Lemma A2 we obtain the property of Schur-geometricaly convexity (Schur-geometricaly concavity) of K on I^2.

(v) The claim (i) implies that $\left(y^2\frac{\partial K(x,y)}{\partial y}-x^2\frac{x\partial K(x,y)}{\partial x}\right)(y-x)\geqslant 0\ \ (\leqslant 0)$ on I^2 and according Lemma A3 we obtain the property of Schur-harmonic convexity (Schur-harmonic concavity) of K on I^2.

(vi) Applying (8) for similarly (opposit) ordered functions f and g we have $\frac{\partial^2 K(x,y)}{\partial x\partial y}\leqslant 0\ \ (\geqslant 0)$. Lemma A4 implies that $K(x,y)$ is an $(1,1)$-concave (convex) function. $\square$

REFERENCES

[1] G. D. ANDERSON, M. K. VAMANAMURTHY, M. VUORINEN, *Generalized convexity and inequalities*, Journal of Mathematical Analysis and Applications **335**, 2 (2007), 1294–1308.

[2] Y. CHU AND Y.-P. LV, *The Schur harmonic convexity of the Hamy symmetric function and its applications*, Journal of Inequalities and Applications, Vol. **2009** (2009), Article ID 838529, 10 pages.

[3] Y. CHU, X. ZHANG AND G. WANG, *The Schur Geometrical Convexity of the Extended Mean Values*, Journal of Convex Analysis **15** (2008), 707–718.

[4] V. ČULJAK AND J. PEČARIĆ, *Schur-convexity of Čebišev functional*, Mathematical Inequalities & Applications, accepted for publication.

[5] V. ČULJAK, *Schur-convexity of the weighted Čebišev functional*, Journal of Mathematical Inequalities **5**, 2 (2011), 213–217.

[6] H.-N. SHI AND J. ZHANG, *Schur-convexity and Schur-geometric convexity of Čebišev functional*, RGMIA **30**, 3 (2000), 853–856.

[7] G. H. HARDY, J. E. LITTLEWOOD AND G.POLYA, *Inequalities*, Cambrige, 1934, 1952.

[8] A. W. MARSHALL AND I. OLKIN, *Inequalities: Theory of Majorization and Its Applications*, Academic Press Inc., New York, 1979.

[9] D. S. MITRINOVIĆ, J. E. PEČARIĆ AND A. M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, 1993.

[10] C. P. NICULESCU AND L.-E. PERSSON, *Convex Functions and Their Applications. A Contemporary Approach*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 23, Springer, New York, NY, USA, 2006.

[11] J. E. PEČARIĆ, F. PROSCHAN, AND Y. L. TONG, *Convex functions, partial orderings, and statistical applications*, Academic Press Inc., 1992.

[12] T. POPOVICIU, *Les functions convexes*, Herman and Cie Editeurs, Paris, 1944.

[13] H.-N. SHI, Y.-M. JIANG AND W.-D. JIANG, *Schur-convexity and Schur-geometrically concavity of Gini means*, Computers & Mathematics with Applications **57**, 2 (2009), 266–274.

[14] W. XIA AND Y. CHU, *The Schur Harmonic Convexity of Lehmer Means*, International Mathematical Forum **4**, 41 (2009), 2009–2015.

[15] X. M. ZHANG, *Geometrically Convex Functions*, Hefei: An'hui University Press, 2004. (Chinese)

[16] X. M. ZHANG, *S-Geometric Convexity of a function involving MacLaurin's elementay symmetric mean*, Journal of inequalities in pure and applied mathematics **8**, 2 (2007), art. 51.

[17] L.-C. WANG, *On the Monotonicity of Difference Generated by the Inequalities of Čebyšev Type*, Journal of Sichuan University, Natural Science Edition **39**, 3 (2002), 398–403. (Chinese)

(Received June 22, 2011)

V. Čuljak
Department of Mathematics
Faculty of Civil Engineering
University of Zagreb
Kačićeva 26
10 000 Zagreb
Croatia
e-mail: `vera@master.grad.hr`