SCHUR–CONVEXITY OF THE WEIGHTED ČEBYŠEV FUNCTIONAL II

V. ČULJAK

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Abstract. In this paper the weighted Čebyshev functional $T(p; f, g; a, b)$ is regarded as a function of two variables

$$T(p; f, g; x, y) = \int_x^y p(t) f(t) g(t) dt \int_x^y p(t) dt - \left( \int_x^y p(t) f(t) dt \int_x^y p(t) dt \right) \left( \int_x^y p(t) g(t) dt \int_x^y p(t) dt \right), \quad (x, y) \in [a, b] \times [a, b]$$

where $f, g$ and $p > 0$ are Lebesgue integrable functions. For a function

$$K(p; f, g; x, y) = \left( \int_x^y p(t) dt \right)^2 T(p; f, g; x, y) \quad (x, y) \in [a, b] \times [a, b]$$

the property of Schur-convexity, Schur-geometric convexity, Schur-harmonic convexity and $(1,1)$-convexity is proved.

1. Introduction

Let $f, g$ and $p > 0$ be Lebesgue integrable functions on the interval $I = [a, b] \subseteq \mathbb{R}$. In this paper the weighted Čebyshev functional $T(p; f, g; a, b)$ is regarded as a function of two variables

$$T(p; f, g; x, y) = \frac{\int_x^y p(t) f(t) g(t) dt}{\int_x^y p(t) dt} - \left( \frac{\int_x^y p(t) f(t) dt}{\int_x^y p(t) dt} \right) \left( \frac{\int_x^y p(t) g(t) dt}{\int_x^y p(t) dt} \right), \quad (x, y) \in I^2.$$

In [4] we proved Schur-convexity of a function $T(1; f, g; x, y)$ with $(x, y) \in I^2$.

**Theorem A1.** Let $f$ and $g$ be Lebesgue integrable functions on $I = [a, b] \subseteq \mathbb{R}$. If they are monotone in the same sense (in the opposite sense) then $T(x, y) := T(1; f, g; x, y)$, $(x, y) \in I^2$ is Schur-convex (Schur-concave) on $I$.

Using the following notations:

$$P(x, y) := \int_x^y p(t) dt, \quad \overline{f}_p(x, y) := \frac{1}{\int_x^y p(t) dt} \int_x^y p(t) f(t) dt \quad \text{and} \quad \overline{g}_p(x, y) := \frac{1}{\int_x^y p(t) dt} \int_x^y p(t) g(t) dt$$

we obtained next result for the weighted Čebyshev functional (see [5]):


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THEOREM A2. Let $f$ and $g$ be Lebesgue integrable functions on $I = [a, b] \subseteq \mathbb{R}$ and let $p$ be a positive continuous weight on $I$ such that $pf$ and $pg$ are also Lebesgue integrable functions on $I$. Then $T(x, y) := T(p; f, g; x, y)$ is Schur-convex (Schur-concave) on $I^2$ if and only if the inequality

$$T(x, y) \leq \frac{p(x)(f(x) - f(y))(g(x) - g(y))}{p(x) + p(y)}$$

holds (reverses) for all $x, y$ in $I$.

For a function $K : I^2 \subseteq \mathbb{R}^2 \to \mathbb{R}$ defined by

$$K(p; f, g; x, y) = \left( \int_x^y p(t) dt \right)^2 \cdot T(p; f, g; x, y), \ (x, y) \in I^2$$

the author in [17] proved the following statement:

THEOREM A3. Let $f, g : [a, b] \to \mathbb{R}$ be Lebesgue integrable functions, and let $p : [a, b] \to \mathbb{R}^+_0$ be a Lebesgue integrable function. A function $K(x, y) := K(p; f, g; x, y)$ defined as (2) is increasing (decreasing) with $y$ on $I = [a, b]$ and decreasing (increasing) with $x$ on $I$ if $f$ and $g$ are monotone in the same sense (in the opposite sense).

In this paper we prove the property of Schur-convexity, Schur-geometrically convexity and Schur-harmonic convexity of a function $K(x, y)$ with $(x, y)$ depending of monotonicity and simultan ordering of the functions $f$ and $g$. We also show $(1, 1)$-convexity of a function $K$.

2. Definitions and properties

The concepts of majorizations and Schur-convex functions involve convex functions and measure of the diversity of the components of an $n$-tuple in $\mathbb{R}^n$. Most of the basic results are given in Marshall and Olkin’s book [8]. In the recently references [1], [2], [3], [10], [11], [13], [15], [16], we can find the definitions and applications of the Schur-convex, Schur-geometrically convex and Schur-harmonic convex functions.

In this section we will recall usefull definitions, lemmas and theorems:

DEFINITION 1. Let $x, y$ be in $E \subseteq \mathbb{R}^n$ and let $x[i], y[i]$ denote the $i$th largest component in $x$ and $y$. We say $y$ majorizes $x$, denote $x \prec y$ if

$$\sum_{i=1}^k x[i] \leq \sum_{i=1}^k y[i], \ k = 1, ..., n - 1,$$

$$\sum_{i=1}^n x[i] = \sum_{i=1}^n y[i].$$
DEFINITION 2. Let $x, y$ be in $E \subseteq \mathbb{R}^n$. A function $F : E \to \mathbb{R}$ is called a Schur-convex function on $E$ if
\[ F(x_1, x_2, \ldots, x_n) \leq F(y_1, y_2, \ldots, y_n) \]
for each $x$ and $y$ in $E$ such that $x \prec y$.
A function $F$ is Schur-concave if and only if $-F$ is a Schur-convex function.

DEFINITION 3. Let $x, y$ be in $E \subseteq \mathbb{R}^n_+$. A function $F : E \to [0, \infty)$ is called a Schur-geometrically convex function on $E$ if
\[ F(x_1, x_2, \ldots, x_n) \leq F(y_1, y_2, \ldots, y_n) \]
for each two positive $x$ and $y$ in $E$ such that $(\ln x_1, \ln x_2, \ldots, \ln x_n) \prec (\ln y_1, \ln y_2, \ldots, \ln y_n)$ i.e $y$ logarithm majorizes $x$.
A function $F$ is Schur-geometrically concave if and only if $-F$ is a Schur-geometrically convex function.

DEFINITION 4. Let $x, y$ be in $E \subseteq \mathbb{R}^n_+$. A function $F : E \to [0, \infty)$ is called a Schur-harmonic convex function on $E$ if
\[ F(x_1, x_2, \ldots, x_n) \leq F(y_1, y_2, \ldots, y_n) \]
for each two positive $x$ and $y$ on $E$ such that it holds $(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}) \prec (\frac{1}{y_1}, \frac{1}{y_2}, \ldots, \frac{1}{y_n})$.
A function $F$ is Schur-harmonic concave if and only if $-F$ is a Schur-harmonic convex function.

The next lemmas gives us the characterisations of Schur-convexity, Schur-geometrically convexity and Schur-harmonic convexity (see [8, p. 57], [11, p. 333], [14], [15, p. 108], [16]):

**Lemma A1.** Let $E \subseteq \mathbb{R}^n$ be a symmetric convex set with a nonempty interior. Let $F : E \to \mathbb{R}$ be a continuous function on $E$ and differentiable on the interior of $E$. Then $F$ is Schur-convex (Schur-concave) if and only if it is symmetric and
\[ \left( \frac{\partial F}{\partial x_2} - \frac{\partial F}{\partial x_1} \right) (x_2 - x_1) \geq 0 \ (\leq 0) \] (3)
holds for all $x$ in the interior of $E$, $x_1 \neq x_2$.

**Lemma A2.** Let $E \subseteq \mathbb{R}^n_+$ be a symmetric logarithm convex set with a nonempty interior, i.e. $\ln E = \{ \ln x = (\ln x_1, \ldots, \ln x_n) : x \in E \}$ is a convex set. Let $F : E \to [0, \infty)$ be a continuous function on $E$ and differentiable on the interior of $E$. Then $F$ is Schur-geometrically convex (Schur-geometrically concave) if it is symmetric and the inequality
\[ \left( x_2 \frac{\partial F}{\partial x_2} - x_1 \frac{\partial F}{\partial x_1} \right) (\ln x_2 - \ln x_1) \geq 0 \ (\leq 0) \] (4)
holds for all $x$ in the interior of $E$, $x_1 \neq x_2$. 
**Lemma A3.** Let $E \subseteq \mathbb{R}^n_+$ be a symmetric harmonic convex set with a nonempty interior, i.e., $1/E = \{1/x = (1/x_1, \ldots, 1/x_n) : x \in E \}$ is a convex set. Let $F : E \to [0, \infty)$ be a continuous function on $E$ and differentiable on the interior of $E$. Then $F$ is Schur-harmonic convex (Schur-harmonic concave) if it is symmetric and

$$\left( x_2^2 \frac{\partial F}{\partial x_2} - x_1^2 \frac{\partial F}{\partial x_1} \right) (x_2 - x_1) \geq 0 \quad (\leq 0)$$

holds for all $x$ in the interior of $E$, $x_1 \neq x_2$.

**Definition 5.** The functions $f$ and $g : [a, b] \to \mathbb{R}$ are similarly ordered if

$$(f(x_1, x_2, \ldots, x_n) - f(y_1, y_2, \ldots, y_n)) \cdot (g(x_1, x_2, \ldots, x_n) - g(y_1, y_2, \ldots, y_n)) \geq 0,$$

for each two $n$-tuples $x$ and $y$ on $[a, b]$.

Function $f$ and $g$ are oppositely ordered if $f$ and $-g$ are similarly ordered.

We recall the well-known Čebyšev inequality for monotone functions (see [9, p. 239], [11, p. 197]) and for similar ordered functions (see [7, p. 168], [9, p. 252]):

**Theorem A4.** Let $f$ and $g$ be Lebesgue integrable on an interval $I = [a, b] \subseteq \mathbb{R}$ and let $p$ be a positive continuous weight on $I$ such that $p f g$, $p f$ and $pg$ are also Lebesgue integrable functions on $I$. If $f$ and $g$ are monotone in the same sense (in the opposite sense) then it holds

$$T(p; f, g; a, b) \geq 0 \quad (\leq 0).$$

**Theorem A5.** Let $f$ and $g$ be Lebesgue integrable functions on $I = [a, b] \subseteq \mathbb{R}$ and let $p$ be a positive continuous weight on $I$ such that $p f g$, $p f$ and $pg$ are also Lebesgue integrable functions on $I$. If $f$ and $g$ are similarly (oppositely) ordered then it holds

$$T(p; f, g; a, b) \geq 0 \quad (\leq 0).$$

Popoviciu in ([12, p. 60]) used the $(n, m)$-divided difference of the function in the definition of the $(n, m)$-convexity (concavity) (see also [11, p. 18]):

**Definition 6.** A function $F : I^2 \to \mathbb{R}$ is $(n, m)$-convex (concave) if for all distinct points $x_0, x_1, \ldots, x_n \in I$ and $y_0, y_1, \ldots, y_m \in I$ yields

$$\left[ x_0, x_1, \ldots, x_n \atop y_0, y_1, \ldots, y_m \right] F = \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{F(x_i, x_j)}{\omega'(x_i) \cdot w'(y_j)} \geq 0 \quad (\leq 0),$$

where $\omega(x) = \prod_{i=0}^{n} (x - x_i)$, $w(y) = \prod_{j=0}^{m} (y - y_j)$.

The next lemma gives the necessary and sufficient conditions for verifying the $(n, m)$-convexity (concavity):
Lemma A4. If the partial derivative $\frac{\partial^{(n+m)} F}{\partial x^n \partial y^m}$ exists then $F : I^2 \to \mathbb{R}$ is $(n,m)$-convex (concave) if and only if
\[ \frac{\partial^{(n+m)} F}{\partial x^n \partial y^m} \geq 0 \ (\leq 0). \]

3. Results

Theorem 3.1. Let $f$ and $g$ be Lebesgue integrable on interval $I = [a, b] \subseteq \mathbb{R}$. Let $p$ be a positive continuous weight on $I$ such that $pfg$, $pf$ and $pg$ are also Lebesgue integrable functions on $I$. If $f$ and $g$ are monotone in the same sense (in the opposite sense) then for a function $K(x, y) := K(p; f, g; x, y)$ defined by (2) holds

(i) $K(x, y) \geq 0 \ (\leq 0)$, for $(x, y) \in I^2$;
(ii) $K(x, y)$ is Schur-convex (Schur-concave) with $(x, y)$ on $I^2 \subseteq \mathbb{R}^2$;
(iii) $K(x, y)$ Schur-geometrical convex (Schur-geometrical concave) with $(x, y)$ on $I^2 \subseteq \mathbb{R}^2$;
(iv) $K(x, y)$ is Schur-harmonic convex (Schur-harmonic concave) with $(x, y)$ on $I^2 \subseteq \mathbb{R}^2$;
(v) $K(x, y) := K(p; f, g; x, y)$ is an $(1,1)$-concave (convex) function on $I^2 \subseteq \mathbb{R}^2$.

Proof. Let $f$ and $g$ be monotone in the same sense (in the opposite sense). Let $p$ be a positive continuous weight on $I$ such that $pfg$, $pf$ and $pg$ are also Lebesgue integrable functions on $I = [a, b]$.

We may assume that $x < y$ without loss of generality.

Now, we calculate $\frac{\partial K(x, y)}{\partial y}$, $\frac{\partial K(x, y)}{\partial x}$ and $\frac{\partial^2 K(x, y)}{\partial x \partial y}$:

\[
\frac{\partial K(x, y)}{\partial y} = p(y) \int_x^y p(t)(f(t) - f(y))(g(t) - g(y))dt; \quad (6)
\]
\[
\frac{\partial K(x, y)}{\partial x} = -p(x) \int_x^y p(t)(f(t) - f(x))(g(t) - g(x))dt; \quad (7)
\]
\[
\frac{\partial^2 K(x, y)}{\partial x \partial y} = -p(x)p(y)(f(y) - f(x))(g(y) - g(x)) \quad (8)
\]

(i) Applying Čebyšev inequality, Theorem A4 to the function $K(p; f, g; x, y) = [P(x, y)]^2 \cdot T(p; f, g; x, y)$ we obtain that holds $K(x, y) \geq 0 \ (\leq 0)$.

(ii) To prove Schur-convexity of $K(p; x, y)$ (or Schur-concavity) we apply Lemma A1. It is sufficient to discuss the following inequality $\left( \frac{\partial K(x, y)}{\partial y} - \frac{\partial K(x, y)}{\partial x} \right)(y - x) \geq 0 \ (\leq 0)$, for all $x, y \in [a, b]$, since the function $K(x, y)$ is evidently symmetric. According Theorem A3 we know that $K(x, y)$ is increasing (decreasing) with $y$ on $I$ and decreasing (increasing) with $x$ on $I$. So, it follows Schur-convexity (Schur-concavity) of $K$ as in the statement (ii).

(iii) The set $I^2 \subseteq \mathbb{R}^2_+$ is a symmetric logarithm convex set. By applying the condition in Lemma A2 to the function $K(x, y)$ we conclude that $\left( y \frac{\partial K(x, y)}{\partial y} - x \frac{\partial K(x, y)}{\partial x} \right).
\[(\ln y - \ln x) \geq 0 \ (\leq 0), \ (x, y) \in I^2 \subset \mathbb{R}^2_+, \text{ i.e. } K(x, y) \text{ is Schur-geometrically convex (Schur-geometrically concave) with } (x, y) \text{ on } I^2 \subset \mathbb{R}^2_+.
\]

(iv) The set \(I^2 \subset \mathbb{R}^2_+\) is a symmetric harmonic convex set. According Lemma A3 we conclude that \((\frac{\partial K(x,y)}{\partial y} - x^2 \frac{\partial K(x,y)}{\partial x}) (y-x) \geq 0 \ (\leq 0), \ (x, y) \in I^2 \subset \mathbb{R}^2_+, \text{ i.e. } K(x, y) \text{ is Schur-harmonic convex (Schur-harmonic concave) with } (x, y) \text{ on } I^2 \subset \mathbb{R}^2_+.
\]

(v) Since \(\frac{\partial^2 K(x,y)}{\partial x \partial y} \leq 0 \ (\geq 0), \) Lemma A4 implies that \(K(x,y)\) is the \((1,1)\)-concave (convex) function on \(I^2 \subset \mathbb{R}^2\). □

**Theorem 3.2.** Let \(f\) and \(g\) be Lebesgue integrable functions on \(I = [a,b] \subset \mathbb{R}_+\) and let \(p\) be a positive continuous weight on \(I\) such that \(pf, \ p\) and \(pg\) are also Lebesgue integrable functions on \(I = [a,b]\). If \(f\) and \(g\) are similarly (oppositely) ordered then for the function \(K(x,y) := K(p; f, g; x, y)\) defined by (2) holds

(i) \(K(x,y)\) is increasing (decreasing) with \(y\) on \(I\) and decreasing (increasing) with \(x\) on \(I^2\);

(ii) \(K(x,y) \geq 0 \ (\leq 0)\), for \((x, y) \in I^2 \subset \mathbb{R}^2_+\);

(iii) \(K(x,y)\) is Schur-convex (Schur-concave) with \((x, y)\) on \(I^2 \subset \mathbb{R}^2\);

(iv) \(K(x,y)\) is Schur-geometrically convex (Schur-geometrically concave) with \((x, y)\) on \(I^2 \subset \mathbb{R}^2_+\);

(v) \(K(x,y)\) is Schur-harmonic convex (Schur-harmonic concave) with \((x, y)\) on \(I^2 \subset \mathbb{R}^2_+\);

(vi) \(K(x,y)\) is an \((1,1)\)-concave (convex) function on \(I^2 \subset \mathbb{R}^2\).

**Proof.** Let \(f\) and \(g\) are similarly ordered (oppositely ordered) on \(I = [a,b] \subset \mathbb{R}_+\). Let \(p\) be a positive continuous weight on \(I\) such that \(pf, \ p\) and \(pg\) are also Lebesgue integrable functions on \(I\).

(i) From (6) and (7) we have that \(\frac{\partial K(x,y)}{\partial y} \geq 0 \ (\leq 0)\) and \(\frac{\partial K(x,y)}{\partial x} \leq 0 \ (\geq 0)\), So, it holds statement (i).

(ii) \(K(p; f, g; x, y) = [P(x,y)]^2 \cdot T(p; f, g; x, y)\) and Čebyšev inequality, Theorem A5 implies statement (ii).

(iii) The claim (i) implis that \((\frac{\partial K(x,y)}{\partial y} - \frac{\partial K(x,y)}{\partial x}) (y-x) \geq 0 \ (\leq 0)\) on \(I^2\) and according Lemma A3 it follows the property of Schur-convexity (Schur-concavity) of \(K(x,y)\) on \(I^2\).

(iv) Similarly, by statement (i) we conclude that \((y \frac{\partial K(x,y)}{\partial y} - x \frac{\partial K(x,y)}{\partial x}) (y-x) \geq 0 \ (\leq 0)\) on \(I^2\) and according Lemma A2 we obtain the property of Schurr-geometrically convexity (Schurr-geometrically concavity) of \(K\) on \(I^2\).

(v) The claim (i) implies that \((\frac{\partial^2 K(x,y)}{\partial x^2} - x^2 \frac{\partial K(x,y)}{\partial x}) (y-x) \geq 0 \ (\leq 0)\) on \(I^2\) and according Lemma A3 we obtain the property of Schurr-harmonic convexity (Schurr-harmonic concavity) of \(K\) on \(I^2\).

(vi) Applying (8) for similarly (oppositely) ordered functions \(f\) and \(g\) we have \(\frac{\partial^2 K(x,y)}{\partial x \partial y} \leq 0 \ (\geq 0)\). Lemma A4 implies that \(K(x,y)\) is an \((1,1)\)-concave (convex) function. □
REFERENCES


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V. Čuljak
Department of Mathematics
Faculty of Civil Engineering
University of Zagreb
Kaščićeva 26
10 000 Zagreb
Croatia
e-mail: vera@master.grad.hr