FOURIER TRANSFORM AND $L_p$–MIXED CENTROID BODIES

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Abstract. In this paper we introduce the concept of $L_p$-mixed centroid body of a convex body and consider a Busemann-Petty type problem whether $\Gamma_{-p, \mathcal{K}} \subseteq \Gamma_{-p, \mathcal{L}}$ implies $W_i(\mathcal{K}) \leq W_i(\mathcal{L})$.

1. Introduction

The nature of the duality between the Brunn-Minkowski theory and the dual Brunn-Minkowski theory is subtle. Lutwak, Yang and Zhang [13] showed that there exists a new ellipsoid (John ellipsoid) $\Gamma_{-2\mathcal{K}}$ associated with convex body $\mathcal{K}$ in the dual Brunn-Minkowski theory, which is the dual analog of the classical Legendre ellipsoid $\Gamma_2\mathcal{K}$ in the Brunn-Minkowski theory. More generally, they [14] introduced the concept of $L_p$-John ellipsoids. If $\mathcal{K}$ is a convex body which contains the origin in its interior and real $p > 0$, the $L_p$-John ellipsoid $\Gamma_{-p}\mathcal{K}$ is defined by

$$\rho(\Gamma_{-p}\mathcal{K}, u)^{-p} = \frac{1}{\text{vol}_n(\mathcal{K})} \int_{S^{n-1}} |u \cdot v|^p dS_p(\mathcal{K}, v), \quad u \in S^{n-1},$$

(1.1)

where $S_p(\mathcal{K}, \cdot)$ is the $L_p$-surface area measure. If $p \geq 1$, the body $\Gamma_{-p}\mathcal{K}$ is a convex body.

The main object of this article is the $i$-th $L_p$-mixed centroid body $\Gamma_{-p,i}\mathcal{K}$. Let $\Gamma_{-p,i}\mathcal{K}, i = 0, 1, \cdots, n-1, p > 0$, denote the star body whose radial function is given by

$$\rho(\Gamma_{-p,i}\mathcal{K}, \theta)^{-p} = \frac{1}{W_i(\mathcal{K})} \int_{S^{n-1}} |\theta \cdot u|^p dS_{p,i}(\mathcal{K}, u), \forall \theta \in S^{n-1}.$$

(1.2)

Here $S_{p,i}(\mathcal{K}, \cdot)$ is the $i$-th $L_p$-mixed surface area measure with $n - i - 1$ copies of $\mathcal{K}$ and $i$ copies of $B$ (the unit ball). More precisely, the Borel measure $S_{p,i}(\mathcal{K}, \cdot)$, on $S^{n-1}$, is defined by [12]

$$S_{p,i}(\mathcal{K}, \omega) = \int_{\omega} h_K^{1-p}(u)dS_i(\mathcal{K}, u),$$

(1.3)

for each Borel $\omega \subset S^{n-1}$.


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If $i = 0$, $S_{p,i}(K, \cdot)$ is just $S_p(K, \cdot)$. Obviously, $\Gamma_{-p,0}K = \Gamma_{-p}K$. In this article, we consider the following Busemann-Petty type problem for $L_p$-mixed centroid bodies. Let $K$ and $L$ be origin-symmetric convex bodies in $\mathbb{R}^n$ and $i = 0, 1, \cdots, n - 1, p \geq 1$. Suppose

$$\Gamma_{-p,i}K \subseteq \Gamma_{-p,i}L.$$  

Does it follow that

$$W_i(K) \leq W_i(L)?$$

By using the Fourier analytic formula for the $L_p$-mixed centroid body, we will obtain the following results.

**Theorem 1.** Let $K$ and $L$ be origin-symmetric convex bodies in $\mathbb{R}^n$, $i = 0, 1, \cdots, n - 1$ and $p \geq 1, p \neq n - i, p$ is not an even integer. Suppose that the support function $h_K$ is infinitely smooth and the functions $C_p \hat{h}_K^p(\theta) \geq 0$ for all $\theta \in S^{n-1}$. If

$$\Gamma_{-p,i}K \subseteq \Gamma_{-p,i}L,$$

then

$$W_i(K) \leq W_i(L).$$

**Theorem 2.** Let $K$ be an origin-symmetric convex body in $\mathbb{R}^n$, and $i = 0, 1, \cdots, n - 1, p \geq 1, p \neq n - i, p$ is not an even integer. If the mixed curvature function $f_i(K, \cdot)$ is positive on $S^{n-1}$ and $C_p \hat{h}_K^p(\theta)$ is negative on an open subset of $S^{n-1}$, then there exists an origin-symmetric convex body $D$ so that

$$\Gamma_{-p,i}D \subseteq \Gamma_{-p,i}K,$$

but

$$W_i(D) > W_i(K).$$

### 2. Notation and preliminaries

#### 2.1. $L_p$-mixed quermassintegrals and $L_p$-mixed curvature functions

The setting for this paper is $n$-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors, we write $\mathcal{K}_0^n$. A compact convex set $K$ is uniquely determined by its support function $h(K, \cdot)$ on the Euclidean unit sphere $S^{n-1}$, defined by

$$h(K, u) = h_K(u) = \max\{u \cdot x : x \in K\}.$$  

(2.1)

The radial function $\rho(L, \cdot)$ of a compact, star-shaped $L$ (about the origin) is defined by

$$\rho(L, u) = \max\{\lambda \geq 0 : \lambda u \in L\}, \quad u \in S^{n-1}.$$  

(2.2)
We call $L$ a star body if $\rho(L, \cdot)$ is continuous on $S^{n-1}$ and $L$ contains the origin in its interior.

For $K, L \in \mathcal{K}^n$, and $\varepsilon > 0$, the Minkowski linear combination $K + L \in \mathcal{K}^n$ is defined by

$$K + L = \{x + y | x \in K, y \in L\}. \quad (2.3)$$

It is easy to check that

$$h(K + \varepsilon L, \cdot) = h(K, \cdot) + \varepsilon h(L, \cdot). \quad (2.4)$$

For $K, L \in \mathcal{K}_0^n, p \geq 1$, and $\varepsilon > 0$, the Firey $L_p$-combination $K + p \varepsilon \cdot L \in \mathcal{K}_0^n$ is defined by (see [1,12])

$$h(K + p \varepsilon \cdot L, \cdot)^p = h(K, \cdot)^p + \varepsilon h(L, \cdot)^p, \quad (2.5)$$

where “$\cdot$” in $\varepsilon \cdot L$ denotes the Firey scalar multiplication, i.e., $\varepsilon \cdot L = \frac{\varepsilon}{p} L$.

Let $W_i(K,L)$ denote the mixed volume $V(K,\ldots, K,L,B)$ with $n-i-1$ copies of $K$, $i$ copies of the unit ball $B$, and one $L$ ($i = 0, 1, \ldots, n-1$). In particular, $W_i(K,K)$ is just the quermassintegral $W_i(K)$.

For $K \in \mathcal{K}^n$ and $i = 0, 1, \ldots, n-1$, let $S_i(K, \cdot)$ denote the mixed surface area measure $S(K, \ldots, K,B,\ldots,B,\cdot)$ with $n-i-1$ copies of $K$, $i$ copies of $B$ (see [11]). The mixed quermassintegral $W_i(K,L)$ has the following integral representation:

$$W_i(K,L) = \frac{1}{n} \int_{Sn-1} h(L,u)dS_i(K,u) \quad (2.6)$$

for all $L \in \mathcal{K}^n$.

Suppose that $\mathbb{R}$ is the set of real numbers. A convex body $K \in \mathcal{K}^n$ is said to have a continuous $i$-th curvature function $f_i(K, \cdot) : S^{n-1} \rightarrow \mathbb{R}$, if its mixed surface area measure $S_i(K, \cdot)$ is absolutely continuous with respect to spherical Lebesgue measure $S$, and has the Radon-Nikodym derivative

$$\frac{dS_i(K,\cdot)}{dS} = f_i(K,\cdot). \quad (2.7)$$

For $K,L \in \mathcal{K}_0^n$ and $p \geq 1, i = 0, 1, \ldots, n-1$, the $i$-th $L_p$-mixed quermassintegral $W_{p,i}(K,L)$ with $n-i-1$ copies of $K$, $i$ copies of $B$ is defined by [12]

$$\frac{n-i}{p} W_{p,i}(K,L) = \lim_{\varepsilon \rightarrow 0^+} \frac{W_i(K + p \varepsilon \cdot L) - W_i(K)}{\varepsilon}. \quad (2.8)$$

Moreover, Lutwak [12] proved there exists a regular Borel measure $S_{p,i}(K,\cdot)$, such that the $L_p$-mixed quermassintegral $W_{p,i}(K,L)$ has the following integral representation:

$$W_{p,i}(K,L) = \frac{1}{n} \int_{Sn-1} h(L,u)^p dS_{p,i}(K,u) \quad (2.8)$$

for all $L \in \mathcal{K}_0^n$. And the measure $S_{p,i}(K,\cdot)$ is absolutely continuous with respect to $S_i(K,\cdot)$, and has Radon-Nikodym derivative

$$\frac{dS_{p,i}(K,\cdot)}{dS_i(K,\cdot)} = h(K,\cdot)^{1-p}. \quad (2.9)$$
For $K, L \in \mathcal{K}_0^n$ and $p \geq 1, i = 0, 1, \cdots, n - 1$, the $L_p$-mixed curvature function $f_{p,i}(K, \cdot)$ is defined by
\[
f_{p,i}(K, \cdot) = \frac{dS_{p,i}(K, \cdot)}{dS}.
\]
(2.10)

If the mixed surface area measure $S_i(K, \cdot)$ is absolutely continuous with respect to spherical Lebesgue measure $S$, we have
\[
f_{p,i}(K, u) = f_i(K, u)h(K, u)^{1-p}.
\]
(2.11)

### 2.2. Fourier transform and Parseval’s formula

Koldobsky’s book [7] is an excellent general reference for the Fourier transform. Some basic notions and the background material are required. As usual, we denote by $S(\mathbb{R}^n)$ the space of rapidly decreasing infinitely differentiable functions (test functions) on $\mathbb{R}^n$, and by $S'(\mathbb{R}^n)$ the space of distributions over $S(\mathbb{R}^n)$. Every locally integrable real valued function $f$ on $\mathbb{R}^n$ with power growth at infinity represents a distribution acting by integration: for any $\phi \in S(\mathbb{R}^n)$, $\langle f, \phi \rangle = \int_{\mathbb{R}^n} f(x)\phi(x)dx$.

The Fourier transform $\hat{f}$ of a distribution $f \in S'(\mathbb{R}^n)$ is defined by $\langle \hat{f}, \phi \rangle = (2\pi)^n \langle f, \phi \rangle$ for every test function $\phi$, where
\[
\hat{\phi}(y) = \int \phi(x)\exp(-i\langle x, y \rangle)dx.
\]
(2.12)

A distribution $f$ is called even homogeneous of degree $p \in \mathbb{R}$ if $\langle f, \phi(\cdot/\alpha) \rangle = |\alpha|^{n+p}\langle f, \phi \rangle$ for every $\alpha \in \mathbb{R}$, $\alpha \neq 0$. The Fourier transform of an even homogeneous distribution of degree $p$ is an even homogeneous distribution of degree $-n-p$. A distribution $f$ is called positive if $\langle f, \phi \rangle \geq 0$ for every $\phi \geq 0$, implying that $f$ is necessarily a non-negative Borel measure on $\mathbb{R}^n$. We use Schwartz’s generalization of Bochner’s theorem (see [3]) as a definition, and call a homogeneous distribution positive definite if its Fourier transform is a positive distribution.

Let $\mu$ be a finite Borel measure on the unit sphere $S^{n-1}$. We extend $\mu$ to a homogeneous distribution of degree $-n-p$. A distribution $\mu_{p,e}$ is called the $L_p$ extended measure of $\mu$ if, for every even test function $\phi \in S(\mathbb{R}^n)$,
\[
\langle \mu_{p,e}, \phi \rangle = \int_{S^{n-1}} \langle r_+^{-1-p}, \phi(r\xi) \rangle d\mu(\xi).
\]
(2.13)

In most cases we are only interested in even test functions supported outside of the origin, for which
\[
\langle r_+^{-1-p}, \phi(r\xi) \rangle = \int_\mathbb{R} r_+^{-1-p} \phi(r\xi)dr = \frac{1}{2} \int_\mathbb{R} |r|^{-1-p} \phi(r\xi)dr,
\]
(2.14)
(see [2]) for the general definition of $\langle r_+^{-1-p}, \phi(r\xi) \rangle$.

If $\mu$ is absolutely continuous with density $g \in L_1(S^{n-1})$, we define the extension $g(x), x \in \mathbb{R}^n \setminus \{0\}$ as a homogeneous function of degree $-n-p$:
\[
g(x) = |x|^{-n-p}g(x/|x|),
\]
and identify $\mu_{p,e}$ with $\hat{g}$.

Since Koldobsky found the Fourier analytic characterization of intersection bodies, the Fourier analytic approach to Busemann-Petty type problems has recently been developed and has led to many results (see [4–9, 15–18]).
3. Main results

In order to prove our main results, the following results are required.

**Lemma 3.1.** [7] Let $p > -1$, $p \neq 2k$, $k \in \mathbb{N} \cup \{0\}$. For every $\theta \in S^{n-1}$,

$$
\hat{\mu}_{p,e}(\theta) = \frac{1}{4\pi C_p} \int_{S^{n-1}} |\theta \cdot y|^p d\mu(y),
$$

(3.1)

where the constant

$$
C_p = \frac{2^{p+1}\sqrt{\pi} \Gamma((p+1)/2)}{\Gamma(-p/2)}
$$

is positive for each $p \in (4k-2, 4k)$ and negative for each $p \in (4k, 4k+2)$.

**Lemma 3.2.** [12] If $K, L \in \mathcal{K}_0^n$, $i = 0, 1, \cdots, n-1$ and $p > 1$, then

$$
W_{p,i}(K,L)^{n-i} \geq W_i(K)^{n-i-p}W_i(L)^p,
$$

(3.2)

with equality if and only if $K$ and $L$ are dilates.

The following statement follows from (1.2) and Lemma 3.1.

**Lemma 3.3.** Let $p \geq 1$, $p$ is not an even integer and $i = 0, 1, \cdots, n-1$. Then for every $\theta \in S^{n-1}$,

$$
\hat{S}_{p,i}(K,\cdot)(\theta) = \frac{W_i(K)}{4\pi C_p} \rho(\Gamma_{-p,i}K, \theta)^{-p},
$$

(3.3)

where $C_p$ is as above.

In particular, if $S_{p,i}(K,\cdot)$ is absolutely continuous with respect to the spherical Lebesgue measure, then

$$
\frac{f_{p,i}(K,\cdot)(\theta)}{W_i(K)} = \frac{1}{4\pi C_p} \rho(\Gamma_{-p,i}K, \theta)^{-p}.
$$

(3.4)

Taking $i = 0$ to Lemma 3.3, we immediately obtain that

**Corollary 3.1.** Let $p \geq 1$, $p$ is not an even integral. If $S_p(K,\cdot)$ is absolutely continuous with respect to the spherical Lebesgue measure, then for every $\theta \in S^{n-1}$,

$$
\frac{f_p(K,\cdot)(\theta)}{\text{vol}_n(K)} = \frac{1}{4\pi C_p} \rho(\Gamma_{-p}K, \theta)^{-p}.
$$

(3.5)

**Theorem 3.1.** Let $K$ and $L$ be origin-symmetric convex bodies in $\mathbb{R}^n$, $i = 0, 1, \cdots, n-1$ and $p \geq 1$, $p \neq n-i$, $p$ is not an even integer. If

$$
\Gamma_{-p,i}K = \Gamma_{-p,i}L,
$$

then

$$
K = L.
$$
Proof. Applying (1.2) and the uniqueness theorem of the Fourier transform, we have $S_{p,i,e}(K, \cdot) = S_{p,i,e}(L, \cdot)$. By homogeneity, $S_{p,i}(K, \cdot) = S_{p,i}(L, \cdot)$. It remains to use the uniqueness property of $L_p$-mixed surface area measures for $p \neq n - i$ (see [12]). □

Remark. In the case $p = n - i$ and $p$ is not an even integer, it follows that $\Gamma_{-(n-i),i}K = \Gamma_{-(n-i),i}L$ implies $K$ and $L$ are dilates. Theorem 3.1 is not true for even values of $p$. Indeed, one can perturb $S_{p,i}(K, \cdot)$ (i.e., to perturb a body $K$) without changing $\rho(\Gamma_{-p,i}K, \cdot)$ (see the following theorem).

Theorem 3.2. Let $K$ be an origin-symmetric convex body in $\mathbb{R}^n$, $i = 0, 1, \cdots, n - 1$ and $p \geq 1$, $p \neq n - i$. If $p$ is an even integer, then there exists an origin-symmetric convex body $L$, such that

$$\Gamma_{-p,i}K = \Gamma_{-p,i}L,$$

but

$$W_i(K) \neq W_i(L).$$

Proof. Then there exists a nonzero continuous even function $g$ on $S^{n-1}$ such that

$$\int_{S^{n-1}} |x \cdot \xi|^p g(x) dx = 0, \quad \xi \in S^{n-1}.$$  \hfill (3.6)

Indeed, if $p = 2k$, then $|x \cdot \xi|^{2k}$ is a polynomial of degree $2k$ with coefficients depending on $\xi$. So, it is enough to construct a nontrivial even function $g$, satisfying

$$\int_{S^{n-1}} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} g(x) dx = 0,$$  \hfill (3.7)

for all integer power $0 \leq i_j \leq 2k$ such that $\sum_{j=1}^{n} i_j = 2k$.

Taking $g(x) = \sum_{l=1}^{m} c_l \chi_1^{2l}$ and solving the system of linear equations, one can find a nontrivial solution $c_1, \cdots, c_m$ provided $m$ is big enough. Consider an origin-symmetric convex body $K$ in $\mathbb{R}^n$ with a strictly positive $i$-th $L_p$-mixed curvature function (i.e., $f_{p,i}(K, \xi) > 0$, for all $\xi \in S^{n-1}$). We may assume that

$$\int_{S^{n-1}} h_{i}^p(\xi) g(\xi) d\xi \geq 0,$$  \hfill (3.8)

(otherwise consider $-g(\xi)$ instead of $g(\xi)$). Choose $\varepsilon > 0$ such that

$$\frac{f_{p,i}(K, \xi)}{W_i(K)} - \varepsilon g(\xi) > 0.$$  \hfill (3.9)

Then we may use the existence theorem for $L_p$-mixed curvature functions to conclude that there exists an origin-symmetric convex body $L$ in $\mathbb{R}^n$ such that

$$\frac{f_{p,i}(L, \xi)}{W_i(L)} = \frac{f_{p,i}(K, \xi)}{W_i(K)} - \varepsilon g(\xi).$$  \hfill (3.10)
Applying (1.2) (2.10) (3.6) and (3.10), we obtain that

\[
\rho(\Gamma_{-p,i}L, \xi)^{-p} = \frac{1}{W_i(L)} \int_{S^{n-1}} |\theta \cdot \xi|^p dS_{p,i}(L, \xi)
\]

\[
= \frac{1}{W_i(L)} \int_{S^{n-1}} |\theta \cdot \xi|^p f_{p,i}(L, \xi) dS(\xi)
\]

\[
= \frac{1}{W_i(K)} \int_{S^{n-1}} |\theta \cdot \xi|^p f_{p,i}(K, \xi) dS(\xi) - \epsilon \int_{S^{n-1}} |\theta \cdot \xi|^p g(\xi) dS(\xi)
\]

\[
= \frac{1}{W_i(K)} \int_{S^{n-1}} |\theta \cdot \xi|^p dS_{p,i}(K, \xi)
\]

\[
= \rho(\Gamma_{-p,i}K, \xi)^{-p}.
\]

It is just to say

\[
\Gamma_{-p,i}L = \Gamma_{-p,i}K. \quad (3.11)
\]

But

\[
1 = \frac{W_{p,i}(K,K)}{W_i(K)}
\]

\[
= \frac{1}{n} \int_{S^{n-1}} h^p_K(\xi) \frac{f_{p,i}(K, \xi)}{W_i(K)} d\xi
\]

\[
= \frac{1}{n} \int_{S^{n-1}} h^p_K(\xi) \frac{f_{p,i}(L, \xi)}{W_i(L)} d\xi + \frac{\epsilon}{n} \int_{S^{n-1}} h^p_K(\xi) g(\xi) d\xi \quad (3.12)
\]

\[
\geq \frac{1}{n} \int_{S^{n-1}} h^p_K(\xi) \frac{f_{p,i}(L, \xi)}{W_i(L)} d\xi
\]

\[
= \frac{W_{p,i}(L,K)}{W_i(L)}.
\]

Hence

\[
W_i(L) \geq W_{p,i}(L,K).
\]

It follows from Lemma 3.2 that

\[
W_i(L) \geq W_i(K).
\]

So if \(W_i(L) = W_i(K)\), then there is an equality in (3.2) and then \(L\) and \(K\) are dilates. This contradicts the construction of the body \(L\). \(\square\)
Proof of Theorem 1. Noting that
\[
\Gamma_{-p,iK} \subseteq \Gamma_{-p,iL} \Rightarrow Cp \frac{\hat{f}_{p,i}(K,\cdot)(\theta)}{W_i(K)} \geq Cp \frac{\hat{f}_{p,i}(L,\cdot)(\theta)}{W_i(L)}. \tag{3.13}
\]
From \(C_p \frac{\hat{f}_{p,i}(K,\cdot)(\theta)}{W_i(K)} \geq C_p \frac{\hat{f}_{p,i}(L,\cdot)(\theta)}{W_i(L)}\) and \(C_p \hat{h}_K^p(\theta) \geq 0, \ \forall \theta \in S^{n-1}\), we get
\[
\int_{S^{n-1}} \hat{h}_K^p(\theta) \frac{\hat{f}_{p,i}(K,\cdot)(\theta)}{W_i(K)} d\theta \geq \int_{S^{n-1}} \hat{h}_K^p(\theta) \frac{\hat{f}_{p,i}(L,\cdot)(\theta)}{W_i(L)} d\theta = (*) \tag{3.14}
\]
Using Parseval’s formula on the sphere, one can have
\[
(*) = \frac{(2\pi)^n}{W_i(L)} \int_{S^{n-1}} h_K^p(\theta) f_{p,i}(L,\theta) d\theta \tag{3.15}
\]
\[
= \frac{(2\pi)^n}{W_i(L)} \int_{S^{n-1}} h_K^p(\theta) dS_{p,i}(L,\theta)
\]
\[
= n(2\pi)^n \frac{W_{p,i}(L,K)}{W_i(L)}
\]
But
\[
\int_{S^{n-1}} \hat{h}_K^p(\theta) \frac{f_{p,i}(K,\cdot)(\theta)}{W_i(K)} d\theta = \frac{(2\pi)^n}{W_i(K)} \int_{S^{n-1}} h_K^p(\theta) f_{p,i}(K,\theta) d\theta
\]
\[
= n(2\pi)^n \frac{W_{p,i}(K,K)}{W_i(K)} = n(2\pi)^n. \tag{3.16}
\]
Thus
\[
W_{p,i}(L,K) \leq W_i(L). \tag{3.17}
\]
Applying the Lemma 3.2, we get
\[
W_i(K) \leq W_i(L). \tag{3.18}
\]

Proof of Theorem 2. Let \(\Omega = \{\theta \in S^{n-1} : C_p \hat{h}_K^p(\theta) < 0\}\). Consider a function \(\nu \in C^\infty(S^{n-1})\) such that \(C_p \nu\) is a positive even function supported on \(\Omega\), \(\nu\) is not identically zero. We extend \(\nu\) to a homogeneous function \(r^n \nu(\theta)\) of degree \(p\) on \(\mathbb{R}^n\). Then the Fourier transform of \(r^n \nu(\theta)\) is a homogeneous function of degree \(-n-p\) : \(\hat{r^n \nu(\theta)} = r^{-n-p} g(\theta)\), where \(g\) is an infinitely smooth function on \(S^{n-1}\). Since \(g\) is
bounded on $S^{n-1}$ and $f_{p,i}(K, \theta) = h^1_p(K) f_i(K, \theta) > 0$, one can choose a small $\varepsilon > 0$ so that, for every $\theta \in S^{n-1}$ and $r > 0$,

$$\frac{f_{p,i}(D, r\theta)}{W_i(D)} = \frac{f_{p,i}(K, r\theta)}{W_i(K)} + \varepsilon r^{-n-p} g(\theta) > 0.$$  \hspace{1cm} (3.19)

By Lutwak’s extension of the Minkowski’s existence theorem, $f_{p,i}(D, \theta)$ defines a convex body $D \in \mathbb{R}^n$. By the definition of the function $\nu$, one can obtain that

$$C_p \frac{f_{p,i}(D, \cdot)(r\theta)}{W_i(D)} = C_p \frac{f_{p,i}(K, \cdot)(r\theta)}{W_i(K)} + \varepsilon r^n C_p \nu(\theta) \geq C_p \frac{f_{p,i}(K, \cdot)(r\theta)}{W_i(K)},$$ or equivalently

$$\Gamma_{-p,i} D \subseteq \Gamma_{-p,i} K.$$  

Next, since $C_p \nu$ is supported and is positive in the set where $C_p \hat{h}^p_K < 0$,

$$\int_{S^{n-1}} \hat{h}^p_K(\theta) \frac{f_{p,i}(D, \cdot)(\theta)}{W_i(D)} d\theta$$

$$= \int_{S^{n-1}} \hat{h}^p_K(\theta) \frac{f_{p,i}(K, \cdot)(\theta)}{W_i(K)} d\theta + \int_{S^{n-1}} \hat{h}^p_K(\theta) \varepsilon \nu(\theta) d\theta$$  \hspace{1cm} (3.21)

$$< \int_{S^{n-1}} \hat{h}^p_K(\theta) \frac{f_{p,i}(K, \cdot)(\theta)}{W_i(K)} d\theta = (*)$$

Now the Parseval’s formula gives

$$(*) = \frac{(2\pi)^n}{W_i(K)} \int_{S^{n-1}} h^p_K(\theta) f_{p,i}(K, \theta) d\theta$$

$$= \frac{n(2\pi)^n}{W_i(K)} \int_{S^{n-1}} h^p_K(\theta) dS_{p,i}(K, \theta)$$  \hspace{1cm} (3.22)

$$= n(2\pi)^n \frac{W_{p,i}(K, K)}{W_i(K)} = n(2\pi)^n.$$

And

$$\int_{S^{n-1}} \hat{h}^p_K(\theta) \frac{f_{p,i}(D, \cdot)(\theta)}{W_i(D)} d\theta$$

$$= \frac{(2\pi)^n}{W_i(D)} \int_{S^{n-1}} h^p_K(\theta) f_{p,i}(D, \theta) d\theta$$

$$= \frac{(2\pi)^n}{W_i(D)} \int_{S^{n-1}} h^p_K(\theta) dS_{p,i}(D, \theta)$$  \hspace{1cm} (3.23)

$$= n(2\pi)^n \frac{W_{p,i}(D, K)}{W_i(D)}.$$
Thus
\[ W_{p,i}(D, K) < W_i(D). \] (3.24)

As in the previous lemma, this implies
\[ W_i(K) < W_i(D). \] \(\square\)

Taking \(i = 0\) to Theorem 1 and Theorem 2, respectively, we obtain that

**Corollary 3.2.** [15] Let \( K \) and \( L \) be origin-symmetric convex bodies in \( \mathbb{R}^n \), and \( p \geq 1, p \neq n, \) \( p \) is not an even integer. Suppose that the support function \( h_K \) is infinitely smooth and the functions \( C_p h_K^p(\theta) \geq 0 \) for all \( \theta \in S^{n-1}. \) If
\[ \Gamma_{-p} K \subseteq \Gamma_{-p} L, \]
then
\[ \text{vol}_n(K) \leq \text{vol}_n(L). \]

**Corollary 3.3.** [15] Let \( K \) be an origin-symmetric convex body in \( \mathbb{R}^n \), and \( p \geq 1, p \neq n, \) \( p \) is not an even integer. If the curvature function \( f(K, \cdot) \) is positive on \( S^{n-1} \) and \( C_p h_K^p(\theta) \) is negative on an open subset of \( S^{n-1}, \) then there exists an origin-symmetric convex body \( D \) so that
\[ \Gamma_{-p} D \subseteq \Gamma_{-p} K, \]
but
\[ \text{vol}_n(D) > \text{vol}_n(K). \]

**References**


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