WEIGHTED INTERPOLATION OF WEIGHTED $\ell^p$ SEQUENCES AND CARLESON INEQUALITY

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(Communicated by Ryskul Oinarov)

Abstract. Let \( \{z_j\} \) be a sequence in the open unit disc \( D \) and \( \rho_n = \prod_{j \neq n} |z_n - z_j|/|1 - \overline{z}_j z_n| > 0 \). For \( 0 < p < \infty \), \( H^p \) denotes a Hardy space on \( D \). For a given \( f \) in \( H^p \), we study a sequence \( \{(1 - |z_j|^2)^{1/p} f(z_j)\} \). Then it is related to a Carleson inequality.

1. Introduction

Let \( H^p \) \( (0 < p \leq \infty) \) denote a usual Hardy space in the open unit disc. In this paper, we assume that a sequence \( \{z_j\} \) in \( D \) satisfies that \( \sum_{j=1}^{\infty} (1 - |z_j|) < \infty \) and \( z_j \neq 0 \) \( (1 \leq j < \infty) \), that is, there exists a Blaschke product

\[
B(z) = \prod_{j=1}^{\infty} \left( 1 - \frac{\overline{z}_j}{|z_j|} \frac{z - z_j}{1 - \overline{z}_j z_j} \right).
\]

Let

\[
\rho_{nk} = \prod_{j=1}^{n} \left| \frac{z_k - z_j}{1 - \overline{z}_j z_k} \right|, \quad 1 \leq k \leq n,
\]

\[
\rho_k = \prod_{j=1}^{\infty} \left| \frac{z_k - z_j}{1 - \overline{z}_j z_k} \right|.
\]

Then \( \rho_{nk} \geq \rho_{n+1,k} \) and \( \lim_{n \to \infty} \rho_{nk} = \rho_k \) for \( k \geq 1 \). In this paper, we assume that \( \rho_j > 0 \) \( (1 \leq j < \infty) \). This hypothesis is equivalent to that there exists a function \( f_k \) in \( H^p(D) \) such that \( f_k(a_j) = \delta_{jk} \).


Keywords and phrases: Hardy space, weighted sequence, weighted interpolation, Carleson inequality.

This research was supported by Grant-in-Aid for Scientific Research, No. 20540148.
Put $\ell = \{\{w_j\}; w_j \in \mathbb{C}, 1 \leq j < \infty\}$ and $a = \{a_j\}$ where $a_j > 0$ $(1 \leq j < \infty)$. For $0 < p < \infty$, suppose

$$\ell^p(a) = \{\{w_j\} \in \ell; \sum_{j=0}^{\infty} a_j |w_j|^p < \infty\}$$

and

$$\ell^\infty(a) = \{\{w_j\} \in \ell; \sup_{0 \leq j < \infty} a_j |w_j| < \infty\}.$$

For $a = \{a_j\}$ and $-\infty < t < \infty$, $a^t$ denotes $\{a^t_j\}$. For $\rho = \{\rho_j\}, a = \{a_j\}$ and $-\infty < t, s < \infty$, $a^t \rho^s$ denotes $\{a^t_j \rho^s_j\}$.

Given a sequence $\{z_j\}$ in $D$, let $T_p$ be the linear operator on $H^p$ $(0 < p \leq \infty)$ defined by

$$T_p(f) = \{(1 - |z_j|^2)^{1/p} f(z_j)\}$$

with $1/p = 0$ for $p = \infty$. It is known (cf. [1]) that $\inf \rho_j > 0$ if and only if $T_p(H^p) = \ell^p$.

J. P. Earl [2] showed that $\ell^\infty(\rho^{-2}) \subset T_{\infty}(H^\infty)$. This was pointed out by A. M. Gleason (see [3]). The author [5] showed that $\ell^\infty(\rho^{-1}) \subset T_{\infty}(H^\infty)$ if and only if $\{a_n\}$ is the union of a finite number of uniformly separated sequences. J. Garnett [3] showed that $T_{\infty}(H^\infty)$ contains $\ell^\infty(\rho^{-1-\varepsilon})$ for any $\varepsilon > 0$. Hence by Lemma 2 in §2

$$\ell^1(\rho^{-1}) \subset T_1(H^1) \subset \ell^1(\rho^\varepsilon)$$

for any $\varepsilon > 0$.

In this paper, we are interested in the range of $T_p$ for $1 < p < \infty$. We could not generalize results of $p = \infty$ and $p = 1$ to $1 < p < \infty$. However we show that $T_p(H^p) \subset \ell^p(\rho^{2+\varepsilon})$ for any $\varepsilon > 0$ if $\rho_j > 0$ $(1 \leq j < \infty)$. As a result, if $p \neq \infty$ and $1/p + 1/q = 1$ then $\ell^p(\rho^{-(2+\varepsilon+q)p/q}) \subset T_p(H^p)$ for any $\varepsilon > 0$.

It should be noted that $T_p(H^p) \subset \ell^p(\rho^{2+\varepsilon})$ if and only if a Carleson inequality holds, that is,

$$\sum_{j=1}^{\infty} \rho_j^{2+\varepsilon} (1 - |z_j|^2) |f(z_j)|^p \leq \gamma ||f||_p^p \quad (f \in H^p)$$

for some finite constant $\gamma$.

2. Lemmas

In this section, we prove two lemmas in order to prove Theorem. Lemma 1 is well known (see [1, p. 142]).

For $1 \leq j \leq n$, let

$$B_n(z) = \prod_{j=1}^{n} \frac{z - z_j}{1 - z_j z} \quad \text{and} \quad B_{nj}(z) = B_n(z) \frac{1 - \bar{z}_j z}{z - z_j}$$

If $b_{nj} = B_{nj}(z_j)$ and

$$f_n(z) = \sum_{j=1}^{n} b_{nj}^{-1} w_j B_{nj}(z)$$
then $f_n$ is in $H^\infty$ and $f_n(z_j) = w_j$ $(1 \leq j \leq n)$. Put

$$\rho_{nj} = |b_{nj}| (1 \leq j \leq n).$$

**Lemma 1.** Let $1 \leq p \leq \infty$ and $1/p + 1/q = 1$. Suppose $w_j$ is a complex number for $j = 1, 2, \cdots$. There exists a function $f$ in $H^p$ such that $(1 - |z_j|^2)^{1/p} f(z_j) = w_j$ for $j = 1, 2, \cdots$ if and only if there exists a positive finite constant $\gamma$ such that for any $n \geq 1$ and for all $g$ in $H^q$,

$$\left| \sum_{j=1}^{n} \frac{w_j}{b_{nj}} (1 - |z_j|^2)^{1/q} g(z_j) \right| \leq \gamma \|g\|_q.$$  

**Lemma 2.** Let $1 \leq p \leq \infty$ and $1/p + 1/q = 1$.

(1) When $1 < p < \infty$, $T_p(H^p) \supseteq \ell^p(a)$ if and only if $T_q(H^q) \subseteq \ell^q(a^{-q/p} \rho^{-q})$.

(2) $T_1(H^1) \supseteq \ell^1(a)$ if and only if $T_\infty(H^\infty) \subseteq \ell^\infty(a^{-1} \rho^{-1})$.

(3) $T_\infty(H^\infty) \supseteq \ell^\infty(a)$ if and only if $T_1(H^1) \subseteq \ell^1(a^{-1} \rho^{-1})$.

**Proof.** (1) For the ‘only if’ part, since $\sup_{\|w\| \leq 1} \left| \sum_{j=1}^{n} \frac{w_j}{b_{nj}} (1 - |z_j|^2)^{1/q} g(z_j) \right| \leq \gamma \|g\|_q \quad (g \in H^q)$

where $w = \{w_j\}$ and $\|w\| = (\sum_{j=1}^{\infty} a_j |w_j|^p)^{1/p}$. Hence for any $n \geq 1$

$$\left( \sum_{j=1}^{n} (a_j^{1/p} \rho_{nj})^{-q} (1 - |z_j|^2) |g(z_j)|^q \right)^{1/q} \leq \gamma \|g\|_q \quad (g \in H^q).$$

Assuming $\|g\|_q = 1$,

$$\sum_{j=1}^{n} (a_j^{1/p} \rho_{nj})^{-q} (1 - |z_j|^2) |g(z_j)|^q \leq \gamma^q.$$  

For any $\varepsilon > 0$, there exists a positive integer $s(j)$ for each $j$ such that for all $k \geq s(j)$

$$(a_j^{1/p} \rho_j)^{-q} (1 - |z_j|^2) |g(z_j)|^q - \frac{\varepsilon}{2j} \leq (a_j^{1/p} \rho_{kj})^{-q} (1 - |z_j|^2) |g(z_j)|^q$$

because $\lim_{n \to \infty} \rho_{nj} = \rho_j$. Hence for any $n \geq 1$

$$\sum_{j=1}^{n} (a_j^{1/p} \rho_j)^{-q} (1 - |z_j|^2) |g(z_j)|^q - \sum_{j=1}^{n} \frac{\varepsilon}{2j} \leq \sum_{j=1}^{n} (a_j^{1/p} \rho_{s(j),j})^{-q} (1 - |z_j|^2) |g(z_j)|^q$$

$$\leq \sum_{j=1}^{k} (a_j^{1/p} \rho_{kj})^{-q} (1 - |z_j|^2) |g(z_j)|^q \leq \gamma^q.$$
where $k = \max(s(1), \cdots, s(n))$. Thus for any $\varepsilon > 0$

$$
\sum_{j=1}^{\infty} (a_j^{1/p} \rho_j)^{-q} (1 - |z_j|^2) |g(z_j)|^q - \varepsilon \leq \gamma^q.
$$

This implies the ‘only if’ part. For the ‘if’ part, by Lemma 1 it is sufficient to show that there exists a finite positive constant $\gamma$ such that for all $n \geq 1$

$$
\sup_{w \in \ell^p(a)} \sup_{||w|| \leq 1} \left| \sum_{j=1}^{n} \frac{w_j}{b_{nj}} (1 - |z_j|^2)^{1/q} g(z_j) \right| \leq \gamma < \infty
$$

In fact, for all $n \geq 1$

$$
\sup_{w \in \ell^p(a)} \sup_{||w|| \leq 1} \left| \sum_{j=1}^{n} \frac{w_j}{b_{nj}} (1 - |z_j|^2)^{1/q} g(z_j) \right|
\leq \sup_{||g||_q \leq 1} \left( \sum_{j=1}^{n} (a_j^{1/p} \rho_{nj})^{-q} (1 - |z_j|^2) |g(z_j)|^q \right)^{1/q}
\leq \sup_{||g||_q \leq 1} \left( \sum_{j=1}^{\infty} (a_j^{1/p} \rho_j)^{-q} (1 - |z_j|^2) |g(z_j)|^q \right)^{1/q}.
$$

(2) For the ‘only if’ part, since $\{(1 - |z_j|) f(z_j) ; f \in H^1\} \supset \ell^1(a)$, by Lemma 1 there exists a positive finite constant $\gamma$ such that for any $n \geq 1$ and $\|g\|_{\infty} = 1$

$$
\max_{1 \leq j \leq n} \frac{1}{a_j \rho_{nj}} |g(z_j)| \leq \gamma.
$$

For any $\varepsilon > 0$, there exists a positive integer $s(j)$ for each $j$ such that for all $k \geq s(j)$

$$
\frac{1}{a_j \rho_j} |g(z_j)| - \varepsilon \leq \frac{1}{a_j \rho_{kj}} |g(z_j)| \leq \gamma
$$

because $\lim_{n \to \infty} \rho_{nj} = \rho_j$. This implies that $\{(f(z_j)) ; f \in H^\infty\} \subset \ell^\infty(a^{-1} \rho^{-1})$. For the ‘if’ part, we can prove it as in (1).

(3) We can prove (3) as in (1). □

**Lemma 3.** For $k = 1, 2, \cdots$

$$
\sum_{j=1}^{\infty} \frac{(1 - |z_j|^2)(1 - |z_k|^2)}{|1 - \bar{z_j}z_k|^2} \leq 1 - 2 \log \rho_k.
$$

**Proof.** See [1, p. 150]. □
Lemma 4. Let \( a_{jk} \) (\( j, k = 1, 2, \cdots \)) be complex numbers such that \( a_{kj} = \overline{a_{jk}} \) and

\[
\sum_{j=1}^{n} |a_{jk}| \leq M \quad (1 \leq k \leq n).
\]

Then for any numbers \( x_1, \cdots, x_n \),

\[
\left| \sum_{j,k=1}^{n} a_{jk}x_jx_k \right| \leq M \sum_{j=1}^{n} |x_j|^2 \quad (1 \leq k \leq n).
\]

Proof. See ([1, p. 150], [7, p. 42]). \( \square \)

3. Theorem

By (1) of Lemma 2, when \( 1 < p < \infty \) and \( 1/p + 1/q = 1 \), \( T_p(H^p) \supseteq \ell^p(\rho^{-p}) \) if and only if \( T_q(H^q) \subseteq \ell^q \). \( T_\infty(H_\infty) \supseteq \ell^\infty(\rho^{-1}) \) if and only if \( T_1(H^1) \subseteq \ell^1 \). \( T_1(H^1) \subseteq \ell^1(\rho^{-1}) \) if and only if \( T_\infty(H_\infty) \subseteq \ell^\infty \). Hence for \( 1 < p \leq \infty \), this is equivalent to that \( \{z_j\}_{j=1}^\infty \) is the union of a finite number of uniformly separated sequence. See [6] for \( 1 < p < \infty \) and [5] for \( p = \infty \). It is known [3] that \( T_\infty(H_\infty) \supseteq \ell^\infty(\rho^{-1-\varepsilon}) \) for any \( \varepsilon > 0 \). It is interesting to know that for \( 1 < p < \infty, T_p(H^p) \supseteq \ell^p(\rho^{-p-\varepsilon}) \) for any \( \varepsilon > 0 \). Unfortunately we can not prove it. In this section, we show that \( T_2(H^2) \supseteq \ell^2(\rho^{-4-\varepsilon}) \) for any \( \varepsilon > 0 \).

Theorem.
1. \( \ell^2(\rho^{-4-\varepsilon}) \subset T_2(H^2) \) for any \( \varepsilon > 0 \).
2. \( T_2(H^2) \subset \ell^2(\rho^{2+\varepsilon}) \) for any \( \varepsilon > 0 \).

Proof. (1) Suppose \( w = (w_k) \in \ell^2(\rho^{-4-\varepsilon}) \) for any \( \varepsilon > 0 \). Put

\[
F_{nk}(z) = (1 - |z_k|^2)^{3/2}[B_n(z)]^2(z - z_k)^{-2}
\]

and

\[
f_n(z) = \sum_{k=1}^{n} w_k[B_{nk}(z_k)]^{-2}F_{nk}(z).
\]

Then \( (1 - |z_k|^2)^{1/2}f_n(z_k) = w_k \quad (1 \leq k \leq n) \) and for any \( \varepsilon > 0 \)

\[
\|f_n\|_2^2 = (f_n, f_n) = \sum_{j,k=1}^{n} w_j[B_{nj}(z_j)]^{-2} \cdot \overline{w_k[B_{nk}(z_k)]^{-2}} \cdot (F_{nj}, F_{nk})
\]

\[
\leq \sum_{j,k=1}^{n} \rho_j^{-2} |w_j| \cdot \rho_k^{-2} |w_k| \cdot |(F_{nj}, F_{nk})|
\]

\[
= \sum_{j,k=1}^{n} \rho_j^{-2-\varepsilon} |w_j| \cdot \rho_k^{-2-\varepsilon} |w_k| \cdot |(F_{nj}, F_{nk})| \rho_j^\varepsilon \rho_k^\varepsilon.
\]
Since \(|(F_{nj}, F_{nk})| \leq 2(1 - |z_j|^2)(1 - |z_k|^2)|1 - z_j \bar{z}_k|^2\), by Lemma 3
\[
\sum_{j=1}^{n} |(F_{nj}, F_{nk})|\rho_j^\epsilon \rho_k^\epsilon \leq 2\rho_k^\epsilon (1 - 2 \log \rho_k) \quad (1 \leq k \leq n)
\]
By Lemma 4
\[
\|f_n\|_2^2 \leq \left( \max_{1 \leq k \leq n} 2\rho_k^\epsilon (1 - 2 \log \rho_k) \right) \sum_{j=1}^{n} \rho_j^{-4-2\epsilon} |w_j|^2.
\]
and \(\sup_{1 \leq k \leq \infty} \rho_k^\epsilon (1 - 2 \log \rho_k) < \infty\). This implies that if \(w = \{w_k\} \in \ell^2(\rho^{-4-2\epsilon})\) then \((f_n)\) is a normal family and so a subsequence tends uniformly in each disc \(|z| \leq r < 1\) to a function \(f \in H^2\) for which \(T_2(f) = w\).

(2) Put \(a_j = \rho_j^{-4-2\epsilon}\) then \(a_j^{-1} \rho_j^{-2} = \rho_j^{4+2\epsilon}, \rho_j^{-2} = \rho_j^{2+2\epsilon}\), then Lemma 2 implies (2). \(\square\)

**Corollary 1.** Let \(0 < p < \infty\), then \(T_p(H^p) \subset \ell^p(\rho^{2+\epsilon})\) for any \(\varepsilon > 0\).

**Proof.** If \(f \in H^p\) then \(f = Bg^{2/p}\) where \(B\) is a Blaschke product and \(g\) is nonvanishing \(H^2\) function. By (2) of Theorem
\[
\sum_{j=1}^{\infty} \rho_j^{2+\epsilon} (1 - |z_j|^2)|f(z_j)||^p \leq \sum_{j=1}^{\infty} \rho_j^{2+\epsilon} (1 - |z_j|^2)|g(z_j)|^2
\]
\[
\leq \gamma ||g||_2^2 = \gamma ||f||_p^p
\]
where \(\gamma\) is a finite positive constant depending \(p\). \(\square\)

**Corollary 2.** Let \(1 \leq p < \infty\), then \(\ell^p(\rho^{-(2+\epsilon+q)p/q}) \subset T_p(H^p)\) for any \(\varepsilon > 0\).

**Proof.** This is a result of Lemma 2 and Corollary 1. \(\square\)

**References**


(Received December 17, 2010)