Journal of Mathematical Inequalities
Volume 6, Number 2 (2012), 161–166 doi:10.7153/jmi-06-16

WEIGHTED INTERPOLATION OF WEIGHTED ℓ^p SEQUENCES AND CARLESON INEQUALITY

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(*Communicated by Ryskul Oinarov*)

Abstract. Let $\{z_j\}$ be a sequence in the open unit disc D and $\rho_n = \prod_{j\neq n} |z_n - z_j|/|1-\bar{z}_j z_n| > 0$. For $0 < p < \infty$, H^p denotes a Hardy space on D. For a given f in H^p, we study a sequence $\{(1-|z_j|^2)^{1/p} f(z_j)\}$. Then it is related to a Carleson inequality.

1. Introduction

Let H^p $(0 < p \leqslant \infty)$ denote a usual Hardy space in the open unit disc. In this paper, we assume that a sequence $\{z_j\}$ in D satisfies that $\sum_{j=1}^{\infty}(1-|z_j|) < \infty$ and $z_j \neq 0$ $(1 \leqslant j < \infty)$, that is, there exists a Blaschke product

$$B(z) = \prod_{j=1}^{\infty}\left(-\frac{\bar{z}_j}{|z_j|}\,\frac{z - z_j}{1-\bar{z}_j z}\right).$$

Let

$$\rho_{nk} = \prod_{\substack{j=1 \\ j\neq k}}^{n}\left|\frac{z_k - z_j}{1-\bar{z}_j z_k}\right|, \qquad 1 \leqslant k \leqslant n,$$

$$\rho_k = \prod_{\substack{j=1 \\ j\neq k}}^{\infty}\left|\frac{z_k - z_j}{1-\bar{z}_j z_k}\right|.$$

Then $\rho_{nk} \geqslant \rho_{n+1,k}$ and $\lim_{n\to\infty}\rho_{nk} = \rho_k$ for $k \geqslant 1$. In this paper, we assume that $\rho_j > 0$ $(1 \leqslant j < \infty)$. This hypothesis is equivalent to that there exists a function f_k in $H^p(D)$ such that $f_k(a_j) = \delta_{jk}$.

Mathematics subject classification (2010): 30D55, 30E05.
Keywords and phrases: Hardy space, weighted sequence, weighted interpolation, Carleson inequality.
This research was supported by Grant-in-Aid for Scientific Research, No. 20540148.

Put $\ell = [\{w_j\}; w_j \in \mathbb{C}, 1 \leqslant j < \infty\}]$ and $a = \{a_j\}$ where $a_j > 0$ $(1 \leqslant j < \infty)$. For $0 < p < \infty$, suppose

$$\ell^p(a) = [\{w_j\} \in \ell\,;\ \sum_{j=0}^{\infty} a_j |w_j|^p < \infty]$$

and

$$\ell^\infty(a) = [\{w_j\} \in \ell\,;\ \sup_{0 \leqslant j < \infty} a_j |w_j| < \infty].$$

For $a = \{a_j\}$ and $-\infty < t < \infty, a^t$ denotes $\{a_j^t\}$. For $\rho = \{\rho_j\}, a = \{a_j\}$ and $-\infty < t, s < \infty, a^s \rho^t$ denotes $\{a_j^s \rho_j^t\}$.

Given a sequence $\{z_j\}$ in D, let T_p be the linear operator on H^p $(0 < p \leqslant \infty)$ defined by

$$T_p(f) = \{(1 - |z_j|^2)^{1/p} f(z_j)\}$$

with $1/p = 0$ for $p = \infty$. It is known (cf. [1]) that $\inf_j \rho_j > 0$ if and only if $T_p(H^p) = \ell^p$.

J. P. Earl [2] showed that $\ell^\infty(\rho^{-2}) \subset T_\infty(H^\infty)$. This was pointed out by A. M. Gleason (see [3]). The author [5] showed that $\ell^\infty(\rho^{-1}) \subset T_\infty(H^\infty)$ if and only if $\{a_n\}$ is the union of a finite number of uniformly separated sequences. J. Garnett [3] showed that $T_\infty(H^\infty)$ contains $\ell^\infty(\rho^{-1-\varepsilon})$ for any $\varepsilon > 0$. Hence by Lemma 2 in §2

$$\ell^1(\rho^{-1}) \subseteq T_1(H^1) \subseteq \ell^1(\rho^\varepsilon)$$

for any $\varepsilon > 0$.

In this paper, we are interested in the range of T_p for $1 < p < \infty$. We could not generalize results of $p = \infty$ and $p = 1$ to $1 < p < \infty$. However we show that $T_p(H^p) \subset \ell^p(\rho^{2+\varepsilon})$ for any $\varepsilon > 0$ if $\rho_j > 0$ $(1 \leqslant j < \infty)$. As a result, if $p \neq \infty$ and $1/p + 1/q = 1$ then $\ell^p(\rho^{-(2+\varepsilon+q)p/q}) \subset T_p(H^p)$ for any $\varepsilon > 0$.

It should be noted that $T_p(H^p) \subset \ell^p(\rho^{2+\varepsilon})$ if and only if a Carleson inequality holds, that is,

$$\sum_{j=1}^{\infty} \rho_j^{2+\varepsilon}(1 - |z_j|^2)|f(z_j)|^p \leqslant \gamma \|f\|_p^p \quad (f \in H^p)$$

for some finite constant γ.

2. Lemmas

In this section, we prove two lemmas in order to prove Theorem. Lemma 1 is well known (see [1, p. 142]).

For $1 \leqslant j \leqslant n$, let

$$B_n(z) = \prod_{j=1}^{n} \frac{z - z_j}{1 - \bar{z}_j z} \qquad \text{and} \qquad B_{nj}(z) = B_n(z) \frac{1 - \bar{z}_j z}{z - z_j}$$

If $b_{nj} = B_{nj}(z_j)$ and

$$f_n(z) = \sum_{j=1}^{n} b_{nj}^{-1} w_j B_{nj}(z)$$

then f_n is in H^∞ and $f_n(z_j) = w_j$ $(1 \leqslant j \leqslant n)$. Put

$$\rho_{nj} = |b_{nj}| \ (1 \leqslant j \leqslant n).$$

LEMMA 1. *Let* $1 \leqslant p \leqslant \infty$ *and* $1/p + 1/q = 1$. *Suppose* w_j *is a complex number for* $j = 1, 2, \cdots$. *There exists a function* f *in* H^p *such that* $(1-|z_j|^2)^{1/p} f(z_j) = w_j$ *for* $j = 1, 2, \cdots$ *if and only if there exists a positive finite constant* γ *such that for any* $n \geqslant 1$ *and for all* g *in* H^q,

$$\left| \sum_{j=1}^{n} \frac{w_j}{b_{nj}} (1-|z_j|^2)^{1/q} g(z_j) \right| \leqslant \gamma \|g\|_q.$$

LEMMA 2. *Let* $1 \leqslant p \leqslant \infty$ *and* $1/p + 1/q = 1$.
(1) When $1 < p < \infty$, $T_p(H^p) \supseteq \ell^p(a)$ *if and only if* $T_q(H^q) \subseteq \ell^q(a^{-q/p}\rho^{-q})$.
(2) $T_1(H^1) \supseteq \ell^1(a)$ *if and only if* $T_\infty(H^\infty) \subseteq \ell^\infty(a^{-1}\rho^{-1})$.
(3) $T_\infty(H^\infty) \supseteq \ell^\infty(a)$ *if and only if* $T_1(H^1) \subseteq \ell^1(a^{-1}\rho^{-1})$.

Proof. (1) For the 'only if' part, since $[\{(1-|z_j|^2)^{1/p} f(z_j)\} \, ; \, f \in H^p] \supset \ell^p(a)$, by Lemma 1 there exists a positive finite constant γ such that for any $n \geqslant 1$

$$\sup_{\substack{w \in \ell^p(a) \\ \|w\| \leqslant 1}} \left| \sum_{j=1}^{n} \frac{w_j}{b_{nj}} (1-|z_j|^2)^{1/q} g(z_j) \right| \leqslant \gamma \|g\|_q \quad (g \in H^q)$$

where $w = \{w_j\}$ and $\|w\| = (\sum_{j=1}^{\infty} a_j |w_j|^p)^{1/p}$. Hence for any $n \geqslant 1$

$$\left(\sum_{j=1}^{n} (a_j^{1/p} \rho_{nj})^{-q} (1-|z_j|^2) |g(z_j)|^q \right)^{1/q} \leqslant \gamma \|g\|_q \quad (g \in H^q).$$

Assuming $\|g\|_q = 1$,

$$\sum_{j=1}^{n} (a_j^{1/p} \rho_{nj})^{-q} (1-|z_j|^2) |g(z_j)|^q \leqslant \gamma^q.$$

For any $\varepsilon > 0$, there exists a positive integer $s(j)$ for each j such that for all $k \geqslant s(j)$

$$(a_j^{1/p} \rho_j)^{-q} (1-|z_j|^2) |g(z_j)|^q - \frac{\varepsilon}{2^j} \leqslant (a_j^{1/p} \rho_{kj})^{-q} (1-|z_j|^2) |g(z_j)|^q$$

because $\lim_{n\to\infty} \rho_{nj} = \rho_j$. Hence for any $n \geqslant 1$

$$\begin{aligned} &\sum_{j=1}^{n} (a_j^{1/p} \rho_j)^{-q} (1-|z_j|^2) |g(z_j)|^q - \sum_{j=1}^{n} \frac{\varepsilon}{2^j} \\ &\leqslant \sum_{j=1}^{n} (a_j^{1/p} \rho_{s(j)j})^{-q} (1-|z_j|^2) |g(z_j)|^q \\ &\leqslant \sum_{j=1}^{k} (a_j^{1/p} \rho_{kj})^{-q} (1-|z_j|^2) |g(z_j)|^q \leqslant \gamma^q \end{aligned}$$

where $k = \max(s(1), \cdots, s(n))$. Thus for any $\varepsilon > 0$

$$\sum_{j=1}^{\infty} (a_j^{1/p}\rho_j)^{-q}(1-|z_j|^2)|g(z_j)|^q - \varepsilon \leqslant \gamma^q.$$

This implies the 'only if' part. For the 'if' part, by Lemma 1 it is sufficient to show that there exists a finite positive constant γ such that for all $n \geqslant 1$

$$\sup_{\substack{w\in\ell^p(a)\\ \|w\|\leqslant 1}} \sup_{\|g\|_q\leqslant 1} \left| \sum_{j=1}^{n} \frac{w_j}{b_{nj}}(1-|z_j|^2)^{1/q} g(z_j) \right| \leqslant \gamma < \infty$$

In fact, for all $n \geqslant 1$

$$\begin{aligned}
&\sup_{\substack{w\in\ell^p(a)\\ \|w\|\leqslant 1}} \sup_{\|g\|_q\leqslant 1} \left| \sum_{j=1}^{n} \frac{w_j}{b_{nj}}(1-|z_j|^2)^{1/q} g(z_j) \right| \\
&\leqslant \sup_{\|g\|_q\leqslant 1} \left(\sum_{j=1}^{n} (a_j^{1/p}\rho_{nj})^{-q}(1-|z_j|^2)|g(z_j)|^q \right)^{1/q} \\
&\leqslant \sup_{\|g\|_q\leqslant 1} \left(\sum_{j=1}^{\infty} (a_j^{1/p}\rho_{j})^{-q}(1-|z_j|^2)|g(z_j)|^q \right)^{1/q}.
\end{aligned}$$

(2) For the 'only if' part, since $[\{(1-|z_j|)f(z_j)\}\ ;\ f \in H^1] \supset \ell^1(a)$, by Lemma 1 there exists a positive finite constant γ such that for any $n \geqslant 1$ and $\|g\|_\infty = 1$

$$\max_{1\leqslant j\leqslant n} \frac{1}{a_j\rho_{nj}}|g(z_j)| \leqslant \gamma\,.$$

For any $\varepsilon > 0$, there exists a positive integer $s(j)$ for each j such that for all $k \geqslant s(j)$

$$\frac{1}{a_j\rho_j}|g(z_j)| - \varepsilon \leqslant \frac{1}{a_j\rho_{kj}}|g(z_j)| \leqslant \gamma$$

because $\lim_{n\to\infty}\rho_{nj} = \rho_j$. This implies that $[\{(f(z_j)\}\ ;\ f \in H^\infty] \subset \ell^\infty(a^{-1}\rho^{-1})$. For the 'if' part, we can prove it as in (1).

(3) We can prove (3) as in (1). □

LEMMA 3. *For* $k = 1, 2, \cdots$

$$\sum_{j=1}^{\infty} \frac{(1-|z_j|^2)(1-|z_k|^2)}{|1-\bar{z}_j z_k|^2} \leqslant 1 - 2\log\rho_k.$$

Proof. See [1, p. 150]. □

LEMMA 4. *Let a_{jk} $(j,k=1,2,\cdots)$ be complex numbers such that $a_{kj}=\overline{a_{jk}}$ and*

$$\sum_{j=1}^{n}|a_{jk}|\leqslant M \quad (1\leqslant k\leqslant n).$$

Then for any numbers $x_1,\cdots,x_n$,

$$\left|\sum_{j,k=1}^{n}a_{jk}x_j\bar{x}_k\right|\leqslant M\sum_{j=1}^{n}|x_j|^2 \quad (1\leqslant k\leqslant n).$$

Proof. See ([1, p. 150], [7, p. 42]). □

3. Theorem

By (1) of Lemma 2, when $1<p<\infty$ and $1/p+1/q=1$, $T_p(H^p)\supseteq \ell^p(\rho^{-p})$ if and only if $T_q(H^q)\subseteq \ell^q$. $T_\infty(H^\infty)\supseteq \ell^\infty(\rho^{-1})$ if and only if $T_1(H^1)\subseteq \ell^1$. $T_1(H^1)\supseteq \ell^1(\rho^{-1})$ if and only if $T_\infty(H^\infty)\subseteq \ell^\infty$. Hence for $1<p\leqslant\infty$, this is equivalent to that $\{z_j\}_{j=1}^{\infty}$ is the union of a finite number of uniformly separated sequence. See [6] for $1<p<\infty$ and [5] for $p=\infty$. It is known [3] that $T_\infty(H^\infty)\supseteq \ell^\infty(\rho^{-1-\varepsilon})$ for any $\varepsilon>0$. It is interesting to know that for $1<p<\infty, T_p(H^p)\supseteq \ell^p(\rho^{-p-\varepsilon})$ for any $\varepsilon>0$. Unfortunately we can not prove it. In this section, we show that $T_2(H^2)\supseteq \ell^2(\rho^{-4-\varepsilon})$ for any $\varepsilon>0$.

THEOREM.
(1) $\ell^2(\rho^{-4-\varepsilon})\subset T_2(H^2)$ for any $\varepsilon>0$.
(2) $T_2(H^2)\subset \ell^2(\rho^{2+\varepsilon})$ for any $\varepsilon>0$.

Proof. (1) Suppose $w=(w_k)\in \ell^2(\rho^{-4-\varepsilon})$ for any $\varepsilon>0$. Put

$$F_{nk}(z)=(1-|z_k|^2)^{3/2}[B_n(z)]^2(z-z_k)^{-2}$$

and

$$f_n(z)=\sum_{k=1}^{n}w_k[B_{nk}(z_k)]^{-2}F_{nk}(z).$$

Then $(1-|z_k|^2)^{1/2}f_n(z_k)=w_k \quad (1\leqslant k\leqslant n)$ and for any $\varepsilon>0$

$$\begin{aligned}\|f_n\|_2^2=(f_n,f_n)&=\sum_{j,k=1}^{n}w_j[B_{nj}(z_j)]^{-2}\cdot\bar{w}_k\overline{[B_{nk}(z_k)]}^{-2}\cdot(F_{nj},F_{nk})\\ &\leqslant\sum_{j,k=1}^{n}\rho_j^{-2}|w_j|\cdot\rho_k^{-2}|w_k|\cdot|(F_{nj},F_{nk})|\\ &=\sum_{j,k=1}^{n}\rho_j^{-2-\varepsilon}|w_j|\cdot\rho_k^{-2-\varepsilon}|w_k|\cdot|(F_{nj},F_{nk})|\rho_j^{\varepsilon}\rho_k^{\varepsilon}.\end{aligned}$$

Since $|(F_{nj},F_{nk})| \leqslant 2(1-|z_j|^2)(1-|z_k|^2)|1-z_j\bar{z}_k|^2$, by Lemma 3

$$\sum_{j=1}^{n} |(F_{nj},F_{nk})|\rho_j^{\varepsilon}\rho_k^{\varepsilon} \leqslant 2\rho_k^{\varepsilon}(1-2\log\rho_k) \quad (1 \leqslant k \leqslant n)$$

By Lemma 4

$$\|f_n\|_2^2 \leqslant (\max_{1\leqslant k\leqslant n} 2\rho_k^{\varepsilon}(1-2\log\rho_k)) \sum_{j=1}^{n} \rho_j^{-4-2\varepsilon}|w_j|^2.$$

and $\sup\limits_{1\leqslant k<\infty} \rho_k^{\varepsilon}(1-2\log\rho_k) < \infty$. This implies that if $w=\{w_k\} \in \ell^2(\rho^{-4-2\varepsilon})$ then (f_n) is a normal family and so a subsequence tends uniformly in each disc $|z| \leqslant r < 1$ to a function $f \in H^2$ for which $T_2(f)=w$.

(2) Put $a_j = \rho_j^{-4-2\varepsilon}$ then $a_j^{-1}\rho_j^{-2} = \rho_j^{4+2\varepsilon}\cdot\rho_j^{-2} = \rho_j^{2+2\varepsilon}$, then Lemma 2 implies (2). □

COROLLARY 1. *Let* $0<p<\infty$, *then* $T_p(H^p) \subset \ell^p(\rho^{2+\varepsilon})$ *for any* $\varepsilon>0$.

Proof. If $f \in H^p$ then $f = Bg^{2/p}$ where B is a Blaschke product and g is nonvanishing H^2 function. By (2) of Theorem

$$\sum_{j=1}^{\infty} \rho^{2+\varepsilon}(1-|z_j|^2)|f(z_j)|^p \leqslant \sum_{j=1}^{\infty} \rho^{2+\varepsilon}(1-|z_j|^2)|g(z_j)|^2 \leqslant \gamma\|g\|_2^2 = \gamma\|f\|_p^p$$

where γ is a finite possitive constant depending p. □

COROLLARY 2. *Let* $1 \leqslant p < \infty$, *then* $\ell^p(\rho^{-(2+\varepsilon+q)p/q}) \subset T_p(H^p)$ *for any* $\varepsilon>0$.

Proof. This is a result of Lemma 2 and Corollary 1. □

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(Received December 17, 2010)

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