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AN INEQUALITY IN THE COMPLEX DOMAIN

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Abstract. We prove the inequality $|1+z_1|+|1+z_2|+|1+z_1z_2| \geqslant |z_1|+|z_2|$, where z_1, z_2 are two arbitrary complex numbers. Consequently, it results that, if $z_1, \ldots, z_n$ are n arbitrary complex number, then $\sum_{k=1}^{n} |1+z_k| + \frac{1}{n-1} \sum_{1 \leqslant i<j \leqslant n} |1+z_iz_j| \geqslant \sum_{k=1}^{n} |z_k|$.

1. Introduction

An interesting class of inequalities are those that bound the sum of the moduli of two or more than two complex numbers. An example is represented by the inequality $|z_1|+|z_2| \leqslant |1+z_1z_2|$, which is true when $|z_1|$ and $|z_2|$ satisfy the condition $(|z_1|-1)(|z_2|-1) > 1$ (see [1], 3.8.13), but is not true when $(|z_1|-1)(|z_2|-1) \leqslant 1$. The aim of this paper is to prove the following general result:

THEOREM 1. *If z_1, z_2 are two arbitrary complex numbers, then it results that*

$$|1+z_1|+|1+z_2|+|1+z_1z_2| \geqslant |z_1|+|z_2|. \tag{1.1}$$

We denote with $\mathbb{C}^*$ the field of the complex numbers which are different from zero and with $|z|$ the modulus of the complex number z. If $z \in \mathbb{C}^*$, then there exists an angle φ, called the argument of z, which is unique then we restrict ourselves to the interval $[0, 2\pi)$, so that z can be written as $z = |z|(\cos\varphi + i\sin\varphi)$. As we can easily see, if one of the two numbers is zero, the inequality is true, so we will assume that the two numbers are different from zero. Then, using the above trigonometric form, the inequality can be reformulated in the following form: if $z_1 = a(\cos\alpha + i\sin\alpha)$, $z_2 = b(\cos\beta + i\sin\beta)$, with $|z_1| = a > 0$, $|z_2| = b > 0$ and $\arg(z_1) = \alpha \in [0, 2\pi)$, $\arg(z_2) = \beta \in [0, 2\pi)$, then it results that

$$\begin{aligned} &\sqrt{a^2+2a\cos\alpha+1} + \sqrt{b^2+2b\cos\beta+1} + \\ &+ \sqrt{(ab)^2+2ab\cos(\alpha+\beta)+1} \geqslant a+b. \end{aligned} \tag{1.2}$$

Squaring the above inequality, we get the following equivalent form:

$$\begin{aligned} &(1-ab)^2 + 2(1+a\cos\alpha)(1+b\cos\beta) - 2ab\sin\alpha\sin\beta + \\ &+ 2|1+z_1z_2|\big(|1+z_1|+|1+z_2|\big) + 2|1+z_1||1+z_2| \geqslant 0. \end{aligned} \tag{1.3}$$

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Let us remind the following important inequalities: if $z \in \mathbb{C}^*$, $z = \rho(\cos\varphi + \sin\varphi)$, then it results that

$$|1+z| = \sqrt{\rho^2 + 2\rho\cos\varphi + 1} \geqslant \max\{|\rho + \cos\varphi|, |1+\rho\cos\varphi|, \rho|\sin\varphi|\}. \tag{1.4}$$

In the next paragraph we prove some lemmas, which we will use in the third paragraph, where we will prove the result.

2. Lemmas

LEMMA 1. *Let us assume* $\alpha, \beta \in [0, 2\pi)$ *and* $\cos\alpha \cdot \cos\beta \leqslant 0$. *It results that*

$$\cos\alpha + \cos\beta + |\sin(\alpha+\beta)| \geqslant 0.$$

Proof. We suppose that $\cos\alpha + \cos\beta < 0$, otherwise the inequality is obvious. Let us fix $\cos\alpha \geqslant 0$, $\cos\beta \leqslant 0$. It results that $\alpha \in \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right)$ and $\beta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. We consider first the case $\alpha \in \left[0, \frac{\pi}{2}\right]$. Then $\cos\alpha < -\cos\beta = \cos(\pi - \beta)$, consequently $-\alpha < \pi - \beta < \alpha$ and therefore $|\sin(\alpha+\beta)| = -\sin(\alpha+\beta)$. Let us denote $x = \cos\alpha \in [0,1]$ and $y = \cos\beta \in [-1,0]$. Next, let us suppose that $\beta \in \left[\frac{\pi}{2}, \pi\right]$. We get $\sin\beta \geqslant 0$, so the above inequality can be reformulated: if $x \in [0,1], y \in [-1,0]$ and $x + y < 0$, then

$$x + y - (x\sqrt{1-y^2} + y\sqrt{1-x^2}) \geqslant 0.$$

Squaring the inequality and taking into account that $xy < 0$, we get the following equivalent form $1 + xy \geqslant \sqrt{1-x^2} \cdot \sqrt{1-y^2}$. Squaring again, we get $(x+y)^2 \geqslant 0$, which is obvious. Suppose that $\beta \in \left[\pi, \frac{3\pi}{2}\right]$. Then $\sin\beta \leqslant 0$ and the inequality has the equivalent form: if $x \in [0,1]$, $y \in [-1,0]$ and $x + y < 0$, then

$$x + y - (y\sqrt{1-x^2} - x\sqrt{1-y^2}) \geqslant 0,$$

which is equivalent to $xy + x^2y^2 \leqslant -xy\sqrt{1-x^2} \cdot \sqrt{1-y^2}$, which is obvious, because the left-hand side of the inequality is negative and the right-hand side is positive. We consider the case $\alpha \in \left[\frac{3\pi}{2}, 2\pi\right)$. Denote $\alpha_1 = 2\pi - \alpha, \beta_1 = 2\pi - \beta$. Then $\alpha_1 \in \left[0, \frac{\pi}{2}\right]$, $\beta_1 \in \left[\frac{\pi}{2} \frac{3\pi}{2}\right]$, so we obtain

$$\begin{aligned}\cos\alpha + \cos\beta + |\sin(\alpha+\beta)| &= \cos\alpha_1 + \cos\beta_1 + |\sin(4\pi - (\alpha_1+\beta_1))| \\ &= \cos\alpha_1 + \cos\beta_1 + |\sin(\alpha_1+\beta_1)| \geqslant 0. \qquad \square\end{aligned}$$

LEMMA 2. *Let us assume that the complex numbers* $z_1 = a(\cos\alpha + i\sin\alpha)$, $z_2 = b(\cos\beta + i\sin\beta)$ *satisfy the condition* $1 + 2b\cos\beta < 0$. *Then, if* $a \leqslant 1 + (1-ab)^2$, *the inequality (1.1) is true.*

Proof. If $a \leqslant 1 + (1-ab)^2$, then $-\frac{1+(1-ab)^2}{a} \leqslant -1 \leqslant \cos\alpha$, so we get $1 + a\cos\alpha \geqslant -(1-ab)^2$. Taking into account that $1 + b\cos\beta \geqslant 0$, we get $2(1+a\cos\alpha)(1+b\cos\beta) \geqslant -2(1-ab)^2(1+b\cos\beta)$. On the other hand, the condition $1 + 2b\cos\beta < 0$

leads to $-2(1+b\cos\beta) > -1$. Multiplying this inequality with $(1-ab)^2$, we get $-2(1-ab)^2(1+b\cos\beta) \geqslant -(1-ab)^2$, so we obtain

$$(*)(1-ab)^2 + 2(1+a\cos\alpha)(1+b\cos\beta) \geqslant 0.$$

Let us consider the inequality (1.3); the left-hand side of the inequality can be arranged in the following form

$$\begin{gathered}\big((1-ab)^2 + 2(1+a\cos\alpha)(1+b\cos\beta)\big)+\\ +2\big(|1+z_1||1+z_2| - ab\sin\alpha\sin\beta\big)+\\ +2|1+z_1z_2|\big(|1+z_1|+|1+z_2|\big).\end{gathered}$$

Taking into account $(*)$ and (1.4), we get that the left-hand side of the inequality is positive, so the inequality (1.3) is true and the inequality (1.1) is true. □

LEMMA 3. *Let us assume that the complex numbers* $z_1 = a(\cos\alpha + i\sin\alpha)$, $z_2 = b(\cos\beta + i\sin\beta)$ *satisfy the following conditions:*

$$\begin{gathered} ab < 1,\ a \geqslant 1,\ 1 \geqslant b > \frac{1}{2},\\ \alpha, \beta \in \left[\frac{\pi}{2}, \pi\right],\\ 1 + 2b\cos\beta < 0,\\ \cos\alpha < -\frac{1+(1-ab)^2}{a}. \end{gathered}$$

It results that

$$|1+z_1z_2| \geqslant \sqrt{2+3(1-ab)^2} \geqslant \sqrt{2}.$$

Proof. Let us denote $ab = c$. First, let us notice that the conditions $c < 1$, $b > \frac{1}{2}$ and $-1 < -\frac{1+(1-c)^2}{a}$ leads to $1+(1-c)^2 < a < 2c$. Taking into account that $\cos\beta < -\frac{1}{2b} = -\frac{a}{2c}$, we get $\cos\alpha\cos\beta > \frac{1+(1-c)^2}{2c}$. Then, taking into account that $0 \leqslant \sin\alpha \leqslant \sqrt{\frac{a^2-\left(1+(1-c)^2\right)^2}{a^2}}$ and $0 \leqslant \sin\beta \leqslant \sqrt{\frac{4c^2-a^2}{4c^2}}$, we get

$$\cos(\alpha+\beta) \geqslant \frac{1+(1-c)^2}{2c} - \frac{\sqrt{4c^2-a^2}\cdot\sqrt{a^2-\left(1+(1-c)^2\right)^2}}{2ac}.$$

It results that

$$|1+z_1z_2|^2 = c^2+2c\cos(\alpha+\beta)+1 \geqslant c^2+2+(1-c)^2 - \frac{\sqrt{4c^2-a^2}\cdot\sqrt{a^2-\left(1+(1-c)^2\right)^2}}{a}.$$

Let us prove the following inequality:

$$\frac{\sqrt{4c^2-a^2}\cdot\sqrt{a^2-\left(1+(1-c)^2\right)^2}}{a}\leqslant(2c-1)-(1-c)^2.$$

Indeed, squaring the above inequality we can reduce it to

$$\begin{aligned}4a^2c^2-4c^2\left(1+(1-c)^2\right)^2-a^4+a^2+2a^2(1-c)^2\\ \leqslant a^2(2c-1)^2-2a^2(2c-1)(1-c)^2,\end{aligned}$$

which is equivalent to $\left(a^2-2c\left(1+(1-c)^2\right)\right)^2\geqslant 0$. It results that

$$|1+z_1z_2|^2\geqslant c^2+2+(1-c)^2-\left((2c-1)-(1-c)^2\right)=2+3(1-c)^2. \qquad \square$$

3. Proof of the Theorem 1

The proof will be divided in three cases: (A) $ab\geqslant 1$, (B) $ab<1$ and $\cos\alpha\cdot\cos\beta\leqslant 0$ and (C) $ab<1$ and $\cos\alpha\cdot\cos\beta\geqslant 0$, where $\alpha,\beta\in[0,2\pi)$. First, let us observe that in order to prove the inequality (1.2) it is sufficient to prove the inequality

$$\cos\alpha+\cos\beta+\sqrt{(ab)^2+2ab\cos(\alpha+\beta)+1}\geqslant 0, \tag{3.1}$$

subject to the condition $\cos\alpha+\cos\beta<0$, otherwise it is obvious. Indeed, if (3.1) is true, then, taking into account (1.4), we get

$$\begin{aligned}\sqrt{a^2+2a\cos\alpha+1}+\sqrt{b^2+2b\cos\beta+1}+\sqrt{(ab)^2+2ab\cos(\alpha+\beta)+1}\\ \geqslant a+b+\left(\cos\alpha+\cos\beta+\sqrt{(ab)^2+2ab\cos(\alpha+\beta)+1}\right)\geqslant a+b.\end{aligned}$$

(A) Let us suppose that $ab\geqslant 1$. We prove that, if $\cos\alpha+\cos\beta<0$ and $\alpha,\beta\in[0,2\pi)$, then $\cos\frac{\alpha+\beta}{2}<0$. Indeed, let us suppose that, on the contrary, $\cos\frac{\alpha+\beta}{2}>0$ and let us assume that $\alpha\geqslant\beta$. Then, because $\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}<0$, it would result that $\cos\frac{\alpha-\beta}{2}<0$. The angles α and β would satisfy $\frac{\pi}{2}<\frac{\alpha-\beta}{2}<\frac{3\pi}{2}$ and $\frac{\alpha+\beta}{2}\in\left[0,\frac{\pi}{2}\right)\cup\left(\frac{3\pi}{2},2\pi\right]$. Consequently, we would get either $\beta<0$ or $\alpha>2\pi$, which are both not true. Let us notice that the condition $\alpha,\beta\in[0,2\pi)$ is necessary, because, if $\cos\alpha+\cos\beta<0$ and $\alpha,\beta\in\mathbb{R}$, then it does not result that $\cos\frac{\alpha+\beta}{2}<0$.

Next, let us consider the function $f:[0,\infty)\to\mathbb{R}$,

$$f(x)=\sqrt{x^2+2x\cos(\alpha+\beta)+1} \tag{3.2}$$

which is increasing on the interval $[1,\infty)$. As $ab\geqslant 1$, it results that $f(ab)\geqslant f(1)=2\left|\cos\frac{\alpha+\beta}{2}\right|=-2\cos\frac{\alpha+\beta}{2}$. Also we have

$$\cos\alpha+\cos\beta-2\cos\frac{\alpha+\beta}{2}=2\cos\frac{\alpha+\beta}{2}\left(\cos\frac{\alpha-\beta}{2}-1\right)\geqslant 0.$$

It results that the inequality (3.1) is true and so (1.1) is true.

(B) Let us suppose that $ab < 1$ and $\cos\alpha \cdot \cos\beta \leqslant 0$. Suppose that $\cos\alpha + \cos\beta < 0$. If $\cos(\alpha+\beta) \geqslant 0$, it results that $\sqrt{(ab)^2 + 2ab\cos(\alpha+\beta)+1} \geqslant 1$. Taking into account the fact that one of the number $\cos\alpha$ or $\cos\beta$ is positive, it results that $\cos\alpha + \cos\beta + \sqrt{(ab)^2 + 2ab\cos(\alpha+\beta)+1} \geqslant \cos\alpha + \cos\beta + 1 \geqslant 0$, so the inequality (3.1)is true. Let us suppose that $\cos(\alpha+\beta) < 0$. The function (3.2) has the minimal value $f(-\cos(\alpha+\beta)) = |\sin(\alpha+\beta)|$. According to Lemma 1, we get $\cos\alpha + \cos\beta + \sqrt{(ab)^2 + 2ab\cos(\alpha+\beta)+1} \geqslant \cos\alpha + \cos\beta + |\sin(\alpha+\beta)| \geqslant 0$ and the inequality (3.1) is true and so (1.1) is true.

(C) Let us suppose that $ab = c < 1$ and $\cos\alpha \cdot \cos\beta > 0$.

First, let us prove the inequality (1.1), subject to the conditions $|z_1| = a \in (0,1]$, $|z_2| = b \in (0,1]$. Taking into account that $\frac{1}{|z_1|} \geqslant 1, \frac{1}{|z_2|} \geqslant 1$ and the case (A), we get $|z_2||1+z_1| + |z_1||1+z_2| + |1+z_1z_2| \geqslant |z_1| + |z_2|$, which proves the inequality (1.1).

Let us suppose that one of the modulus is greater than 1 and one is not. Let us fix $a \geqslant 1$, $b \leqslant 1$ with $c < 1$ and $cos\alpha < 0, \cos\beta < 0$.

Suppose that $\alpha \in \left[\frac{\pi}{2}, \pi\right]$ and $\beta \in \left[\pi, \frac{3\pi}{2}\right]$ or $\beta \in \left[\frac{\pi}{2}, \pi\right]$ and $\alpha \in \left[\pi, \frac{3\pi}{2}\right]$. Then it results

$$|z_1 - z_2| + |1 + z_1z_2| \geqslant |z_1| + |z_2|.$$

Indeed, squaring the above inequality, we get the following inequality

$$(1-c)^2 - 4c\sin\alpha\sin\beta + 2|z_1 - z_2||1 + z_1z_2| \geqslant 0,$$

which is true, because $\sin\alpha\sin\beta \leqslant 0$. But $|1+z_1| + |1+z_2| \geqslant |z_1 - z_2|$, so the inequality (1.1) is true.

Next we suppose $\alpha, \beta \in \left[\frac{\pi}{2}, \pi\right]$. Let us notice that in this conditions it results that $|1 - z_1| \geqslant 1$ and $|1 - z_2| \geqslant 1$. This fact has two important consequences. First, let us observe that if $b \leqslant \frac{1}{2}$, then (1.1) is true. Indeed, we get

$$\begin{aligned} |1+z_1| + |1+z_2| + |1+z_1z_2| &\geqslant |1+z_1z_2 - 1 - z_1| + |1+z_2| \\ &\geqslant |z_1||1-z_2| + (1-|z_2|) \geqslant |z_1| + |z_2|, \end{aligned}$$

which is true, because $1 - |z_2| \geqslant |z_2|$. Secondly, let us observe that if $|1+z_2| \geqslant |z_2|$, then $|1+z_1| + |1+z_2| + |1+z_1z_2| \geqslant |z_2| + |z_1||1-z_2| \geqslant |z_1| + |z_2|$, so (1.1) is true. We get the same conclusion if $|1+z_1| \geqslant |z_1|$.

Next, we suppose that $|1+z_2| < |z_2|$ and $b \in (\frac{1}{2}, 1]$. Replacing in the trigonometric form of the complex numbers, we get the following set of conditions

$$ab < 1,\ a \geqslant 1,\ 1 \geqslant b > \tfrac{1}{2},$$

$$\alpha, \beta \in \left[\frac{\pi}{2}, \pi\right],$$

$$1 + 2b\cos\beta < 0.$$

If $a \leqslant 1 + (1-c)^2$, then, according to Lemma 2, it results that (1.1) is true. Suppose that $a > 1 + (1-c)^2$, which can be rewritten as $-1 < -\frac{1+(1-c)^2}{a}$. If $-\frac{1+(1-c)^2}{a} \leqslant \cos\alpha$, then, following the same line as in the proof of the Lemma 2, we get that the inequality (1.1) is true.

Let us suppose that $\cos\alpha < -\frac{1+(1-c)^2}{a}$. Then, according to the Lemma 3, we deduce $|1+z_1z_2| \geqslant \sqrt{2}$. The left-hand side of the inequality (1.3) can be arranged in the following form

$$(1-ab)^2+2\Big(|1+z_1||1+z_2|+(1+a\cos\alpha)(1+b\cos\beta)\Big)+ \\ +2\Big(|1+z_1z_2|\big(|1+z_1|+|1+z_2|\big)-ab\sin\alpha\sin\beta\Big).$$

Taking into account (1.4), we get $|1+z_1||1+z_2|+(1+a\cos\alpha)(1+b\cos\beta) \geqslant 0$. Taking into account (1.4) and $|1+z_1z_2| \geqslant \sqrt{2}$, we get

$$\begin{aligned}|1+z_1z_2|\big(|1+z_1|+|1+z_2|\big)-ab\sin\alpha\sin\beta &\geqslant \sqrt{2}(a\sin\alpha+b\sin\beta)-ab\sin\alpha\sin\beta \\ &= a\sin\alpha(\sqrt{2}-b\sin\beta)+\sqrt{2}b\sin\beta \geqslant 0.\end{aligned}$$

It results that the left-hand side of the inequality (1.3) is positive, so it is true and so (1.1) is true.

We suppose that $\alpha,\beta \in \left[\pi,\frac{3\pi}{2}\right]$. Let us consider the complex conjugate of the complex numbers z_1 and z_2, denoted $\bar{z}_1$ and $\bar{z}_2$. Then their arguments, denoted $\arg(\bar{z}_1)=\alpha'=2\pi-\alpha$ and $\arg(\bar{z}_2)=\beta'=2\pi-\beta$ have the property that $\alpha',\beta' \in \left[\frac{\pi}{2},\pi\right]$. It results that

$$\begin{aligned}|1+z_1|+|1+z_2|+|1+z_1z_2| &= |1+\bar{z}_1|+|1+\bar{z}_2|+|1+\bar{z}_1\cdot\bar{z}_2| \\ &\geqslant |\bar{z}_1|+|\bar{z}_1| = |z_1|+|z_2|.\end{aligned}$$

This concludes the proof of the Theorem 1.

If we put $z_1=z_2$ in the inequality (1.1) we obtain:

PROPOSITION 1. *If z is an arbitrary complex number, then*

$$|1+z^2| \geqslant 2\big(|z|-|1+z|\big). \tag{3.3}$$

A generalization of the inequality (1.1) is represented by the next result.

PROPOSITION 2. *If $z_1,z_2,\ldots z_n$ are n arbitrary complex numbers, then*

$$\sum_{k=1}^{n}|1+z_k|+\frac{1}{n-1}\sum_{1\leqslant i<j\leqslant n}|1+z_iz_j| \geqslant \sum_{k=1}^{n}|z_k|. \tag{3.4}$$

Proof. Applying the inequality (1.1) to each pair (i,j) with $i<j$, where $i,j \in \{1,\ldots,n\}$, we get:

$$\begin{gathered}|1+z_1|+|1+z_2|+|1+z_1z_2| \geqslant |z_1|+|z_2|,\ldots,|1+z_1|+|1+z_n|+|1+z_1z_n| \geqslant |z_1|+|z_n|, \\ |1+z_2|+|1+z_3|+|1+z_2z_3| \geqslant |z_2|+|z_3|,\ldots,|1+z_2|+|1+z_n|+|1+z_2z_n| \geqslant |z_2|+|z_n|, \\ \vdots \\ |1+z_{n-1}|+|1+z_n|+|1+z_{n-1}z_n| \geqslant |z_{n-1}|+|z_n|.\end{gathered}$$

Summing up the inequalities, one obtains

$$(n-1)\sum_{k=1}^{n}|1+z_k| + \sum_{1\leqslant i<j\leqslant n}|1+z_iz_j| \geqslant (n-1)\sum_{k=1}^{n}|z_k|.$$

It results that (3.4) is true. □

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