

## AN INEQUALITY FOR THE GAMMA FUNCTION CONJECTURED BY D. KERSHAW

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*Abstract.* We prove the inequality  $\Gamma(x) + \Gamma(1/x) \leq \Gamma(1+x+1/x)$ .

### 1. Introduction

It has long been known that  $\Gamma(x) + \Gamma(1/x) \geq 2$ ; a quick proof is by convexity of  $\Gamma(x)$  and  $\Gamma(1/x)$ . Gautschi [3] generalized this statement by showing that the harmonic mean of  $\Gamma(x)$  and  $\Gamma(1/x)$  is not less than 1, which of course implies also that  $\Gamma(x)\Gamma(1/x) \geq 1$ . Since then, inequalities concerning  $\Gamma(1/x)$  have been investigated in many further articles. Alzer [1] extended Gautschi's result to the power means  $M_r[\Gamma(x), \Gamma(1/x)]$ . Kershaw and Laforgia [5] showed that  $[\Gamma(1+1/x)]^x$  is decreasing, while  $x[\Gamma(1+1/x)]^x$  is increasing. Giordano and Laforgia [4] extended Gautschi's product inequality by proving that

$$\frac{1}{2} \Gamma\left(1+x+\frac{1}{x}\right) \leq \Gamma(x)\Gamma\left(\frac{1}{x}\right) \leq \Gamma\left(1+x+\frac{1}{x}\right).$$

In connection with these results, Donald Kershaw, in private communication, formulated the following conjecture: for all  $x > 0$ ,

$$\Gamma(x) + \Gamma\left(\frac{1}{x}\right) \leq \Gamma\left(1+x+\frac{1}{x}\right), \quad (1)$$

with equality only at  $x = 1$ . In a recent article [2], Alzer obtained the following result in this direction (alongside a further generalization of Gautschi's result on harmonic means):  $\Gamma(x) + \Gamma(1/x) \leq b\Gamma(x+1/x)$ , where  $b \approx 2.098$ . Since  $x+1/x \geq 2$ , this implies a version of (1) with an intervening factor  $b/2$  (which, of course, fails to reproduce equality at 1). The methods of [2] rely on a considerable number of specific values of the gamma function and higher derivatives.

Here we will prove that Kershaw's conjecture is true, without any extra factor.

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Since (1) is unchanged when  $x$  is replaced by  $1/x$ , it is enough to prove it for  $x > 1$ . We use an assortment of different methods on different parts of the domain. The inequality (indeed, a rather stronger one) is obtained quite easily for all  $x \geq 2$ . For  $\frac{5}{4} \leq x \leq 2$ , we use convexity of  $\Gamma(x)$  and  $\Gamma(1/x)$  to derive linear bounds for the two sides of (1), which then only need to be compared at the end points; we do this on two shorter intervals. Only a few specific values of  $\Gamma(x)$  are needed, to no great degree of accuracy.

The most interesting part of the problem is for  $x$  close to 1. Both sides of (1) have derivative 0 at 1, so there is no longer any chance of deducing the result from linear upper and lower bounds. However, after substituting  $1+x$  for  $x$ , a lower bound for the right-hand side of the form  $2 + c(x^2 - x^3)$  can still (as before) be derived from the tangent to  $\Gamma(x)$  at 3. To estimate the left-hand side, we now use the power series for  $\Gamma(1+x)$ . We apply the exact values of the first three coefficients, together with a bound for the remaining ones, to establish (1) for  $1 < x \leq \frac{5}{4}$ .

**2. Proof of (1) for  $x \geq 2$**

Note that  $\Gamma(1/x) < x$  for  $x > 1$ . For  $x \geq 2$ , we prove (1) with  $\Gamma(1/x)$  replaced by  $x$ .

Case  $x \geq 3$ . Since  $\Gamma(x) > \Gamma(3) = 2$  for  $x > 3$ , we have

$$\Gamma\left(1+x+\frac{1}{x}\right) - \Gamma(x) > \Gamma(1+x) - \Gamma(x) = (x-1)\Gamma(x) \geq 2(x-1) > x.$$

LEMMA 1. *Let  $a > 0$ , and let  $P_a(x) = \Gamma(1+x+a) - \Gamma(x) - x$ . Then  $P_a(x)$  is increasing for  $x \geq 2$ .*

*Proof.* Since  $\Gamma'(x)$  is increasing for all  $x > 0$ , so is  $\Gamma(x+a) - \Gamma(x)$ . Also,  $\Gamma(x)$  is increasing for  $x \geq 2$ . The statement follows, since

$$\begin{aligned} P_a(x) &= (x+a)\Gamma(x+a) - \Gamma(x) - x \\ &= x(\Gamma(x+a) - \Gamma(x)) + a\Gamma(x+a) + x\Gamma(x) - \Gamma(x) - x \\ &= x(\Gamma(x+a) - \Gamma(x)) + a\Gamma(x+a) + (\Gamma(x) - 1)(x-1) - 1. \quad \square \end{aligned}$$

Case  $2\frac{1}{3} \leq x \leq 3$ . Then  $\Gamma(1+x+\frac{1}{x}) \geq \Gamma(1+x+\frac{1}{3})$ . Our inequality follows by Lemma 1 provided that  $P_{1/3}(2\frac{1}{3}) > 0$ . We verify this:

$$P_{1/3}(2\frac{1}{3}) = \frac{8}{3}\Gamma(\frac{8}{3}) - \Gamma(\frac{7}{3}) - \frac{7}{3} \approx 4.012 - 1.191 - 2.333 = 0.488 > 0.$$

Case  $2 \leq x \leq 2\frac{1}{3}$ . We now have  $\frac{1}{x} \geq \frac{3}{7}$  on the interval, so we verify

$$P_{3/7}(2) = \frac{17}{7}\Gamma(\frac{17}{7}) - 1 - 2 \approx 3.074 - 3 > 0.$$

**3. Proof of (1) for  $\frac{5}{4} \leq x \leq 2$**

LEMMA 2.  $\Gamma(1/x)$  is a convex function of  $x$  for  $x > 0$ .

*Proof.* Let  $0 < x_1 < x_2$ , and write  $y_j = 1/x_j$  ( $j = 1, 2$ ). Choose  $a, b$  so that  $\Gamma(1 + y_j) = ay_j + b$  for  $j = 1, 2$ . Since  $\Gamma$  is convex,  $\Gamma(1 + y) \leq ay + b$  for  $y_2 \leq y \leq y_1$ . Also, since  $\Gamma(1 + y) = y\Gamma(y)$ , we have  $\Gamma(y) \leq a + b/y$  for  $y_2 \leq y \leq y_1$ , with equality at  $y_1$  and  $y_2$ . So  $\Gamma(1/x) \leq a + bx$  for  $x_1 \leq x \leq x_2$ , with equality at  $x_1$  and  $x_2$ , so that  $a + bx$  is the linear function agreeing with  $\Gamma(1/x)$  at these points.  $\square$

(In general, if a function  $f$  is convex and increasing, then  $f(1/x)$  is convex, but this is not true for decreasing  $f$ .)

*Note.* Let  $G(x) = \Gamma(x) + \Gamma(1/x)$ . It is shown in [2, Lemma 2] that  $G(x)$  is decreasing on  $(0, 1]$ . This follows at once from our Lemma 2, since  $G(x)$  is convex and  $G'(1) = 0$ . Of course, the inequality  $G(x) \geq 2$  (for all  $x$ ) follows.

LEMMA 3. For all  $x > 0$ ,

$$\Gamma\left(1 + x + \frac{1}{x}\right) \geq 2 + (3 - 2\gamma)\frac{(x - 1)^2}{x}.$$

*Proof.* By convexity of the gamma function,

$$\Gamma(3 + y) \geq \Gamma(3) + y\Gamma'(3) = 2 + (3 - 2\gamma)y$$

for all  $y > 0$ . The statement follows, since

$$1 + x + \frac{1}{x} = 3 + \frac{(x - 1)^2}{x}. \quad \square$$

*Proof of (1) for  $\frac{5}{4} \leq x \leq 2$ .* We consider the intervals  $[\frac{5}{4}, \frac{3}{2}]$  and  $[\frac{3}{2}, 2]$  separately. Write  $\Gamma(x) + \Gamma(1/x) = G(x)$ . By Lemma 2,  $G(x)$  is convex. Using Lemma 3, we define a linear function  $F(x)$  that is a lower bound for  $\Gamma(1 + x + 1/x)$  on the interval in question. The statement then follows on verification that  $F(x) > G(x)$  at the end points.

Let  $h(x) = (x - 1)^2/x$ . Then  $h'(x) = 1 - 1/x^2$ . Hence  $h(x)$  is convex, and  $h(x) \geq h(x_0) + (x - x_0)h'(x_0)$  for any  $x, x_0 > 0$ . For the interval  $[\frac{5}{4}, \frac{3}{2}]$ , take  $x_0 = \frac{5}{4}$ . We find that  $h(x) \geq h_1(x)$  on the interval, where

$$h_1(x) = \frac{1}{20} + \frac{9}{25}\left(x - \frac{5}{4}\right).$$

Our linear lower bound is  $F_1(x) = 2 + (3 - 2\gamma)h_1(x)$ . Note that  $h_1(\frac{3}{2}) = \frac{7}{50}$ . The values are

$$F_1\left(\frac{5}{4}\right) \approx 2.092, \quad G\left(\frac{5}{4}\right) = \Gamma\left(\frac{5}{4}\right) + \Gamma\left(\frac{4}{5}\right) \approx 0.906 + 1.164 = 2.070.$$

$$F_1\left(\frac{3}{2}\right) \approx 2.258, \quad G\left(\frac{3}{2}\right) = \Gamma\left(\frac{3}{2}\right) + \Gamma\left(\frac{2}{3}\right) \approx 0.886 + 1.354 = 2.240.$$

For the interval  $[\frac{3}{2}, 2]$ , take  $x_0 = \frac{3}{2}$ , giving

$$h_2(x) = \frac{1}{6} + \frac{5}{9}\left(x - \frac{3}{2}\right),$$

with corresponding  $F_2(x)$ . Clearly,  $F_2(\frac{3}{2}) > F_1(\frac{3}{2})$ . Also,  $h_2(2) = \frac{4}{9}$ , and we find

$$F_2(2) \approx 2.820, \quad G(2) = \Gamma(2) + \Gamma(\frac{1}{2}) \approx 2.772.$$

**4. The power series for  $\Gamma(1+x)$**

We write the power series for  $\Gamma(1+x)$  in the form  $\sum_{n=0}^{\infty} (-1)^n a_n x^n$ , since (as we now show) the coefficients alternate in sign. Note that  $a_0 = \Gamma(1) = 1$ . Now

$$\Gamma^{(n)}(x) = \int_0^{\infty} t^{x-1} e^{-t} (\log t)^n dt,$$

hence

$$a_n = \frac{(-1)^n}{n!} \Gamma^{(n)}(1) = \frac{1}{n!} \int_0^{\infty} e^{-t} (-\log t)^n dt.$$

The following bound is not optimal, but it is adequate for our purposes.

LEMMA 4. *With this notation, we have  $0 < a_n \leq m$  for  $n \geq 4$ , where  $m \leq \frac{13}{12}$ .*

*Proof.* We have

$$\int_0^1 e^{-t} (-\log t)^n dt < \int_0^1 (-\log t)^n dt = \int_0^{\infty} u^n e^{-u} du = n!.$$

At the same time, this integral is greater than  $e^{-1}n!$ . Also, since  $\log t < t^{1/2}$  for  $t > 1$ ,

$$\int_1^{\infty} e^{-t} (\log t)^n dt < \int_1^{\infty} e^{-t} t^{n/2} dt < \Gamma\left(\frac{n}{2} + 1\right) < \Gamma(n-1) = \frac{n!}{n(n-1)} \leq \frac{n!}{12}$$

for  $n \geq 4$ . The statement follows. □

*Note.* Using the series expansion for  $e^{-t}$ , one finds that

$$\frac{1}{n!} \int_0^1 e^{-t} (-\log t)^n dt = 1 - \frac{1}{2!2^n} + \frac{1}{3!3^n} - \dots$$

One can deduce that  $\lim_{n \rightarrow \infty} a_n = 1$  and  $a_n < 1$  for all  $n$ . The authors are grateful to Pascal Sebah for these observations, and for the calculated values of  $a_n$  for  $n \leq 20$ .

Meanwhile, explicit values for the first few coefficients can be derived more pleasantly as follows. We use the power series (convergent for  $|x| < 1$ )

$$\frac{\Gamma'(1+x)}{\Gamma(1+x)} = \sum_{n=0}^{\infty} (-1)^{n+1} c_n x^n,$$

where  $c_0 = \gamma$  and  $c_n = \zeta(n+1)$  for  $n \geq 1$  [6, p. 12]. Now equating coefficients in the identity

$$\sum_{n=0}^{\infty} (-1)^{n+1} (n+1) a_{n+1} x^n = \left( \sum_{n=0}^{\infty} (-1)^{n+1} c_n x^n \right) \left( \sum_{n=0}^{\infty} (-1)^n a_n x^n \right)$$

we see that

$$(n + 1)a_{n+1} = c_n a_0 + c_{n-1} a_1 + \dots + c_0 a_n$$

for all  $n \geq 1$ . In particular,  $a_1 = c_0 = \gamma$ ,

$$a_2 = \frac{1}{2}(c_1 a_0 + c_0 a_1) = \frac{1}{2}(\zeta(2) + \gamma^2) \approx 0.9891,$$

$$a_3 = \frac{1}{3}(c_2 a_0 + c_1 a_1 + c_0 a_2) = \frac{1}{6}(2\zeta(3) + 3\zeta(2)\gamma + \gamma^3) \approx 0.9075.$$

**5. Proof of (1) for  $1 \leq x \leq \frac{5}{4}$**

We now substitute  $1 + x$  for  $x$ , so that (1) becomes

$$\Gamma(1 + x) + \Gamma\left(\frac{1}{1 + x}\right) \leq \Gamma\left(2 + x + \frac{1}{1 + x}\right) \tag{2}$$

We have to prove (2) for  $0 \leq x \leq \frac{1}{4}$ . We continue to use Lemma 3. In the new notation, this says

$$\Gamma\left(2 + x + \frac{1}{1 + x}\right) \geq 2 + (3 - 2\gamma)\frac{x^2}{1 + x}.$$

For  $0 < x < 1$ , we have  $1/(1 + x) > 1 - x$ , hence

$$\Gamma\left(2 + x + \frac{1}{1 + x}\right) \geq 2 + (3 - 2\gamma)(x^2 - x^3). \tag{3}$$

LEMMA 5. For  $0 \leq x \leq \frac{1}{4}$ ,

$$\Gamma(1 + x) \leq 1 - \gamma x + a_2 x^2 + b_3 x^3, \tag{4}$$

where  $b_3 = -a_3 + \frac{4}{15}m \approx -0.619$ .

*Proof.* Since the terms of the power series alternate in sign,

$$\Gamma(1 + x) \leq 1 - \gamma x + a_2 x^2 - a_3 x^3 + m(x^4 + x^6 + \dots),$$

and for  $0 \leq x \leq \frac{1}{4}$ ,

$$x^4 + x^6 + \dots = \frac{x^4}{1 - x^2} = x^3 \frac{x}{1 - x^2} \leq \frac{4}{15}x^3. \quad \square$$

LEMMA 6. For  $0 \leq y \leq \frac{1}{5}$ ,

$$\Gamma(1 - y) \leq 1 + \gamma y + a_2 y^2 + c_3 y^3, \tag{5}$$

where  $c_3 = a_3 + \frac{1}{4}m \approx 1.178$ .

*Proof.* We have  $\Gamma(1 - y) = 1 + \gamma y + \sum_{n=2}^{\infty} a_n y^n$ , and for  $0 \leq y \leq \frac{1}{5}$ ,

$$a_4 y^4 + a_5 y^5 + \dots \leq m \frac{y^4}{1 - y} = m y^3 \frac{y}{1 - y} \leq \frac{1}{4} m y^3. \quad \square$$

LEMMA 7. For  $0 \leq x \leq \frac{1}{4}$ ,

$$\Gamma\left(\frac{1}{1+x}\right) \leq 1 + \gamma x + d_2 x^2 + d_3 x^3, \quad (6)$$

where

$$d_2 = a_2 - \frac{4}{5}\gamma, \quad d_3 = c_3 - \frac{36}{25}a_2 \approx -0.246.$$

*Proof.* Note that

$$\frac{1}{1+x} = 1 - \frac{x}{1+x}.$$

We apply Lemma 6, with  $y = x/(1+x)$ , using the following estimates derived from convexity of  $1/(1+x)$  and  $1/(1+x)^2$ : for  $0 \leq x \leq \frac{1}{4}$ ,

$$\frac{1}{1+x} \leq 1 - \frac{4}{5}x, \quad \frac{1}{(1+x)^2} \leq 1 - \frac{36}{25}x.$$

We obtain

$$\begin{aligned} \Gamma\left(\frac{1}{1+x}\right) &\leq 1 + \gamma x \left(1 - \frac{4}{5}x\right) + a_2 x^2 \left(1 - \frac{36}{25}x\right) + c_3 x^3 \\ &= 1 + \gamma x + \left(a_2 - \frac{4}{5}\gamma\right)x^2 + \left(c_3 - \frac{36}{25}a_2\right)x^3. \quad \square \end{aligned}$$

*Proof of (2) for  $0 \leq x \leq \frac{1}{4}$ .* By (3), (4), (6), for  $0 \leq x \leq \frac{1}{4}$ ,

$$\Gamma\left(2+x+\frac{1}{1+x}\right) - \Gamma(1+x) - \Gamma\left(\frac{1}{1+x}\right) \geq A_2 x^2 + A_3 x^3,$$

where

$$A_2 = 3 - \frac{6}{5}\gamma - 2a_2 \approx 0.329,$$

$$A_3 = -(3-2\gamma) - b_3 - d_3 \approx -1.845 + 0.619 + 0.246 = -0.980.$$

Clearly,  $A_2 x^2 + A_3 x^3 > 0$  for  $0 \leq x \leq \frac{1}{4}$ .  $\square$

*Remark.* The power series shows clearly that we cannot replace  $\Gamma[1/(1+x)]$  by  $\Gamma(1-x)$  in (2), illustrating the closeness of our inequality. Indeed, for small  $x > 0$ , we have the slightly stronger reverse inequality  $\Gamma(1+x) + \Gamma(1-x) > \Gamma(3+x^2)$ , since

$$\Gamma(1+x) + \Gamma(1-x) = 2 + 2a_2 x^2 + O(x^4),$$

$$\Gamma(3+x^2) = 2 + (3-2\gamma)x^2 + O(x^4),$$

and  $3-2\gamma < 2a_2$ .

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