Some New Inequalities for an Interior Point of a Triangle

Jian Liu

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Abstract. In this paper we establish three new inequalities involving an arbitrary point of a triangle. Some related conjectures and problems are put forward.

1. Introduction

Let $P$ be an arbitrary point in the plane of triangle $ABC$ and let $D$, $E$, $F$ be the feet of the perpendiculars from $P$ to $BC$, $CA$, $AB$, respectively. In [1], the author gave the following identity:

$$
\vec{S}_{\triangle PBC} \cdot PA^2 + \vec{S}_{\triangle PCA} \cdot PB^2 + \vec{S}_{\triangle PAB} \cdot PC^2 = 4R^2\vec{S}_{\triangle DEF},
$$

(1.1)

where $R$ is the circumradius of $\triangle ABC$ and $\vec{S}_{\triangle PBC}$, $\vec{S}_{\triangle PCA}$, $\vec{S}_{\triangle PAB}$, $\vec{S}_{\triangle DEF}$ denote directed areas of $\triangle PBC$, $\triangle PCA$, $\triangle PAB$, $\triangle DEF$. The directed area of a triangle is defined as follows: Given a triangle $XYZ$, if the orientation around the vertexes $X$, $Y$, $Z$ in sequence is counterclockwise, then its directed area $\vec{S}_{\triangle XYZ}$ is positive and $\vec{S}_{\triangle XYZ} = S_{\triangle XYZ}$. If that one is clockwise, then the directed area $\vec{S}_{\triangle XYZ}$ is negative and $\vec{S}_{\triangle XYZ} = -S_{\triangle XYZ}$.

In particular, when $P$ lies inside triangle $ABC$, identity (1.1) becomes

$$
S_aR_1^2 + S_bR_2^2 + S_cR_3^2 = 4R^2S_p,
$$

(1.2)

where $R_1 = PA$, $R_2 = PB$, $R_3 = PC$ and $S_a$, $S_b$, $S_c$ denote the areas of the $\triangle PBC$, $\triangle PCA$, $\triangle PAB$ respectively, and $S_p$ is the area of the pedal triangle $DEF$.

It is well known that the following inequality holds between the area $S$ of the triangle $ABC$ and the area $S_p$ of the pedal triangle $DEF$:

$$
S_p \leq \frac{1}{4}S,
$$

(1.3)

with equality if and only if $P$ is the circumcenter of the triangle $ABC$ (see Figure 1). Therefore, it follows from (1.2) that (see Figure 2)

$$
S_aR_1^2 + S_bR_2^2 + S_cR_3^2 \leq SR^2.
$$

(1.4)

This inequality inspires the author to find the similar conclusion:


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THEOREM 1.1. Let $P$ be an arbitrary interior point of the triangle $ABC$. Then

$$S_a R_1^3 + S_b R_2^3 + S_c R_3^3 \leq SR^3,$$  \hspace{1cm} (1.5)

with equality if and only if $P$ is the circumcenter of the triangle $ABC$.

If $P$ coincides with the centroid of $\triangle ABC$, then $S_a = S_b = S_c = \frac{1}{3}S$, $R_1 = \frac{2}{3}m_a$, $R_2 = \frac{2}{3}m_b$, $R_3 = \frac{2}{3}m_c$ (the three medians of $\triangle ABC$) and it follows from (1.5) that

$$m_a^3 + m_b^3 + m_c^3 \leq \frac{81}{8}R^3,$$  \hspace{1cm} (1.6)

which was conjectured by Ji Chen in [2].

At the same time when inequality (1.5) has been proven, we obtain the following two interesting geometric inequalities:

THEOREM 1.2. Let $P$ be an arbitrary point of triangle $ABC$ with circumradius $R$ and inradius $r$. Let $r_p$ be the inradius of the pedal triangle of $P$ with respect to triangle $ABC$. Then

$$\frac{1}{2r_p} \geq \frac{1}{R} + \frac{1}{2r},$$  \hspace{1cm} (1.7)

with equality if and only if triangle $ABC$ is equilateral and $P$ is its center.

THEOREM 1.3. Let $P$ be an arbitrary interior point of $\triangle ABC$ and let $D$, $E$, $F$ denote the feet of the perpendiculars from $P$ to $BC$, $CA$, $AB$ respectively. Let $r_p$ be the inradius of the pedal triangle $DEF$ and let $PA = R_1$, $PB = R_2$, $PC = R_3$, $PD = r_1$, $PE = r_2$, $PF = r_3$. Then

$$R_1 + R_2 + R_3 \geq r_1 + r_2 + r_3 + 6r_p,$$  \hspace{1cm} (1.8)

with equality if and only if $\triangle ABC$ is equilateral and $P$ is its center.

In this note we will prove the above three theorems and propose some related conjectures and problems.
2. Some lemmas

To prove our theorems, we need several lemmas.

**Lemma 2.1.** Let $P$ be an arbitrary point with barycentric coordinates $(x,y,z)$ in the plane of the triangle $ABC$. Then

$$(x+y+z)^2PA^2 = (x+y+z)(yc^2 + zb^2) - (yza^2 + zxb^2 + yxc^2),$$  \hspace{1cm} (2.1)

where $a$, $b$, $c$ are the lengths of the edges $BC$, $CA$, $AB$ respectively.

The above formulae is well known (see e.g. [3] $P_{278}$).

**Lemma 2.2.** For any point $P$ inside triangle $ABC$, we have

$$cr_2 + br_3 \leq aR_1.$$  \hspace{1cm} (2.2)

If $AO$ ($O$ is the circumcenter of $\triangle ABC$) cuts $BC$ at $X$, then the equality if and only if $P$ lies on the segment $AX$.

Analogously to (2.2) we also have two inequalities. Lemma 2.2 is a simple important proposition and it has various proofs, see [4]–[10]. Next, we give a crucial lemma which is substantially equivalent to Lemma 2.2.

**Lemma 2.3.** For any point $P$ inside triangle $ABC$, we have

$$R_1 \geq \frac{R^2}{2R} + \frac{2RS_p}{S},$$  \hspace{1cm} (2.3)

with equality as in (2.2).

**Proof.** Note that $S_a = \frac{1}{2}ar_1$, $S_b = \frac{1}{2}br_2$, $S_c = \frac{1}{2}cr_3$, $S_a + S_b + S_c = S$, applying Lemma 2.1 and 2.2 we have

$$(S_a + S_b + S_c)^2R_1^2 = (S_a + S_b + S_c)(S_b c^2 + S_c b^2) - (S_b S_c a^2 + S_c S_a b^2 + S_a S_b c^2)$$

$$= \frac{1}{2}(br_2 c^2 + cr_3 b^2)S - \frac{1}{4}(bcr_2r_3 a^2 + car_3 r_1 b^2 + abr_1 r_2 c^2)$$

$$= \frac{1}{2}bc(br_2 + cr_3)S - \frac{1}{4}abc(ар_2 r_3 + br_3 r_1 + cr_1 r_2)$$

$$\leq \frac{1}{2}abcR_1 S - \frac{1}{2}abc(r_2 r_3 \sin A + r_3 r_1 \sin B + r_1 r_2 \sin C)$$

$$= \frac{1}{2}abcR_1 S - abc(S_{\Delta PEF} + S_{\Delta PFD} + S_{\Delta PDE})$$

$$= \frac{1}{2}abc(R_1 S - 2RS_p).$$
Then make use of \( S_a + S_b + S_c = S \) and \( abc = 4SR \), we get
\[
R_1S - 2RS_p \geq \frac{SR_1^2}{2R},
\]
this yields inequality (2.3). Clearly, the condition of the equality in (2.3) is the same as (2.2). □

**Lemma 2.4.** For any point \( P \) inside triangle \( ABC \), we have
\[
S_aR_1 + S_bR_2 + S_cR_3 \geq 4RS_p,
\]
(2.4)
with equality if and only if \( P \) is the circumcenter of the triangle \( ABC \).

**Proof.** Since the area of the quadrilateral is less than or equal to the half product of two diagonals, so we have
\[
S_b + S_c \leq \frac{1}{2}aR_1, \quad S_c + S_a \leq \frac{1}{2}bR_2, \quad S_a + S_b \leq \frac{1}{2}cR_3,
\]
(2.5)
with equalities if and only if \( PA \perp BC, PB \perp CA, PC \perp AB \) respectively. Adding up these inequalities and using identity \( S_a + S_b + S_c = S \), we obtain
\[
aR_1 + bR_2 + cR_3 \geq 4S.
\]
(2.6)
Equality holds if and only if \( P \) is the orthocenter of \( \triangle ABC \).

Applying inequality (2.6) to the pedal triangle \( \triangle DEF \) (see Figure 1), we get
\[
EF \cdot r_1 + FD \cdot r_2 + DE \cdot r_3 \geq 4S_p,
\]
Observe that \( EF = \frac{ar_1}{2R} \), \( ar_1 = 2S_a \), etc., then inequality (2.4) follows at once. According to the equality condition of (2.6), we conclude easily that the equality in (2.4) holds if and only if \( P \) is the circumcenter of the triangle \( ABC \). □

**Remark 2.1.** Inequality (2.4) can also be proven easily by using Lemma 2.2 and the identity (1.2).

**Lemma 2.5.** Suppose that \( P \) is any point in the plane of the triangle \( ABC \). Then
\[
aR_1^2 + bR_2^2 + cR_3^2 \geq abc,
\]
(2.7)
with equality holds if and only if \( P \) is the incenter of \( \triangle ABC \).

Inequality (2.7) is given first by M. K. Lamkin (see [3]). The author [11] generalized its equivalent form:
\[
R_1^2 \sin A + R_2^2 \sin B + R_3^2 \sin C \geq 2S
\]
(2.8)
to the polygon. I proved the following result: For any polygon $A_1A_2\cdots A_n$ and an arbitrary point $P$
\[
\sum_{i=1}^{n} PA_i^2 \sin A_i \geq 2F, \tag{2.9}
\]
where $F$ is the area of the polygon. Later, the author further generalized inequality (2.9) into the case involving two arbitrary points $P, Q$ (see [12]):
\[
\sum_{i=1}^{n} PA_i \cdot QA_i \sin A_i \geq 2F. \tag{2.10}
\]

**Lemma 2.6.** For any point $P$ inside triangle $ABC$, we have
\[
\frac{R_1^2 + R_2^2 + R_3^2}{r_1 + r_2 + r_3} \geq 2R, \tag{2.11}
\]
with equality if and only if $\triangle ABC$ is equilateral and $P$ is its center.

Inequality (2.11) was first posed by the author, and it was first proved by Xiao-Guang Chu and Zhen-Gang Xiao [13]. Xue-Zhi Yang gave a simple proof in his book [14, P15]. We introduce a brief sketch of his proof as follows:

Applying the Cosine Law, one gets easily
\[
4S^2 R_1^2 = b^2 c^2 (r_2^2 + r_3^2) + bcr_2r_3(b^2 + c^2 - a^2). \tag{2.12}
\]
Then we use $abc = 4SR$ and the identity:
\[
ar_1 + br_2 + cr_3 = 2S, \tag{2.13}
\]
we obtain
\[
4S^2 (\sum R_1^2 - 2R \sum r_1) = \sum [b^2 c^2 (r_2^2 + r_3^2) + bcr_2r_3(b^2 + c^2 - a^2)] - abc \sum ar_1 \sum r_1. \tag{2.14}
\]
where $\sum$ denotes cyclic sums. From this we can obtain the identity:
\[
4S^2 (\sum R_1^2 - 2R \sum r_1) = \sum bcr_2r_3(b - c)^2 + \frac{1}{2} \sum a [c(r_3 + r_1) - b(r_1 + r_2)]^2, \tag{2.15}
\]
which implies inequality (2.11).

3. The proofs of the Theorems

3.1. The proof of Theorem 1.1

*Proof.* We multiply both sides of inequality (2.3) by $S_aR_1$, then
\[
\frac{S_a R_3^3}{2R} + \frac{2RS_p S_a R_1}{S} \leq S_a R_1^2.
\]
Analogously, we have

\[
\frac{S_b R_2^3}{2R} + \frac{2RS_p S_b R_2}{S} \leq S_b R_2^2, \quad \frac{S_c R_3^3}{2R} + \frac{2RS_p S_c R_3}{S} \leq S_c R_3^2.
\]

By adding up three inequalities and then using identity (1.2) one has

\[
\frac{S_a R_1^3 + S_b R_2^3 + S_c R_3^3}{2R} + \frac{2RS_p}{S}(S_a R_1 + S_b R_2 + S_c R_3) \leq 4R^2 S_p.
\]

So, it follows from Lemma 2.4 that

\[
\frac{S_a R_1^3 + S_b R_2^3 + S_c R_3^3}{2R} + \frac{8R^2 S_p^2}{S} \leq 4R^2 S_p,
\]

Namely

\[
S_a R_1^3 + S_b R_2^3 + S_c R_3^3 \leq 8R^3 \left( S_p - \frac{2S_p^2}{S} \right) = S R^3 - \frac{(S - 4S_p)^2 R^3}{S} \leq S R^3.
\]

This completes the proof of (1.5). According to the conditions of equality (2.3) and (1.3), we conclude that the equality in (1.5) occurs if and only if \( P \) is the circumcenter of the triangle \( ABC \). \( \square \)

**Remark 3.1.** By applying the inequality of Theorem 1.1 and the weighted power mean inequality, we can get the following generalization of inequality (1.5):

\[
S_a R_1^k + S_b R_2^k + S_c R_3^k \leq S R^k, \quad (3.1)
\]

where \( 0 < k \leq 3 \). In addition, by using Radon inequality, we can prove that if \( k < 0 \) then (3.1) holds inversely.

### 3.2. The proof of Theorem 1.2

*Proof.* From Lemma 2.3 and Lemma 2.5, we have

\[
aR_1 + bR_2 + cR_3 \geq \frac{1}{2R}(aR_1^2 + bR_2^2 + cR_3^2) + \frac{2RS_p}{S}(a + b + c) \\
\geq \frac{abc}{2R} + \frac{2RS_p}{S}(a + b + c).
\]

Since \( abc = 4SR, a + b + c = 2s \), it follows that

\[
aR_1 + bR_2 + cR_3 \geq 2S + \frac{4R}{r} S_p.
\]

(3.2)
Equality occurs if and only if the point $P$ coincide with the circumcenter and the incenter of $\triangle ABC$. This means that $\triangle ABC$ is equilateral and $P$ is its center.

From (3.2) we have

\[
\frac{aR_1 + bR_2 + cR_3}{8RS_p} \geq \frac{S}{4RS_p} + \frac{1}{2r}.
\]

As

\[
r_p = \frac{4RS_p}{aR_1 + bR_2 + cR_3},
\]

we get

\[
\frac{1}{2r_p} \geq \frac{S}{4RS_p} + \frac{1}{2r}.
\]

Hence, the inequality (1.7) follows immediately from (3.4) and (1.3). □

3.3. The proof of Theorem 1.3

Proof. First, by adding up the inequality of Lemma 2.3 and its analogues we get

\[
R_1 + R_2 + R_3 \geq \frac{R_1^2 + R_2^2 + R_3^2}{2R} + \frac{6RS_p}{S}.
\]

Form inequality (2.6) and identity (24), we know (2.6) is equivalent to

\[
\frac{S_p}{S} \geq \frac{r_p}{R},
\]

with equality as in (2.6). Therefore, the inequality (1.8) of Theorem 1.3 follows immediately from (3.5), (3.6) and (2.11). Clearly equality holds in (1.8) if and only if $\triangle ABC$ is equilateral and $P$ is its center. □

REMARK 3.2. From inequality (3.5) and the following identity (we omit its proof)

\[
\frac{R_1^2 + R_2^2 + R_3^2}{2R} + \frac{6RS_p}{S} = r_1 \left( \frac{b}{c} + \frac{c}{b} \right) + r_2 \left( \frac{c}{a} + \frac{a}{c} \right) + r_3 \left( \frac{a}{b} + \frac{b}{a} \right),
\]

we get

\[
R_1 + R_2 + R_3 \geq r_1 \left( \frac{b}{c} + \frac{c}{b} \right) + r_2 \left( \frac{c}{a} + \frac{a}{c} \right) + r_3 \left( \frac{a}{b} + \frac{b}{a} \right).
\]

Further, we have the following famous Erdős-Mordell inequality (see [5]–[10]):

\[
R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3).
\]

Recently, the author gave this new proof in [15].
4. Several conjectures and problems

Euler inequality in the triangle is well known, it states that

\[ R \geq 2r. \]  \hspace{1cm} (4.1)

From this we consider the stronger inequality of Theorem 1.2. After being checked by the computer, we pose the following stronger conjecture:

**Conjecture 1.** For any arbitrary interior point \( P \), we have

\[ \frac{1}{2r_p} \geq \frac{1}{\sqrt{2Rr}} + \frac{1}{2r}. \]  \hspace{1cm} (4.2)

From (1.7) and the arithmetic-geometric mean inequality, it is easy to prove:

\[ 8r_p \leq R + 2r. \]  \hspace{1cm} (4.3)

For this inequality, we have the following unsolved problem:

**Problem 1.** Find the maximum value \( k \) such that the inequality

\[ 2(k + 2)r_p \leq R + kr \]  \hspace{1cm} (4.4)

is valid for arbitrary interior point \( P \) of \( \triangle ABC \).

**Remark 4.1.** From Euler inequality (4.1) we see that the inequality which takes the maximum value \( k \) is the strongest in all inequalities whose type is as (4.4). With the help of the computer, the author finds the maximum value \( k \) is about 7.88 ···.

Next, we denote the circumradius of the pedal triangle \( DEF \) by \( R_p \), note that \( R_p \geq 2r_p \), we first suppose

\[ R_p + 6r_p \leq R + 2r, \]  \hspace{1cm} (4.5)

which is stronger than the inequality (4.3). Further the following with one parameter conjecture is posed:

**Conjecture 2.** If real number \( k \) satisfies 1.8 \( \leq k \leq 7.8 \), then the inequality

\[ R_p + 2kr_p \leq R + (k - 1)r \]  \hspace{1cm} (4.6)

holds for an arbitrary interior point \( P \) of \( \triangle ABC \).

Also, we can put forward the following problem:

**Problem 2.** Find the maximum and the minimum value of \( k \) such that the inequality (4.6) holds for an arbitrary interior point \( P \) of \( \triangle ABC \).
When \( k = 2 \), inequality (4.6) becomes

\[
R_p + 4r_p \leq R + r. \tag{4.7}
\]

This inequality has not yet been proved. The author thinks it has the following exponential generalization:

**Conjecture 3.** If \( k \geq \frac{3}{4} \) is a real number, then the following inequality

\[
R^k_p + (4r_p)^k \leq R^k + r^k \tag{4.8}
\]

holds for an arbitrary interior point \( P \) of \( \triangle ABC \).

Another similar difficult conjecture is

**Conjecture 4.** If \( k \geq \frac{1}{2} \) is a real number, then the following inequality

\[
\frac{1}{R^k_p} + \frac{1}{r^k_p} \geq \frac{1}{r^k} + \frac{4^k}{R^k} \tag{4.9}
\]

holds for an arbitrary interior point \( P \) of \( \triangle ABC \).

It is possible that the inequality similar to (4.9) holds true for Cevian triangles. So we propose the following dual conjecture:

**Conjecture 5.** Let \( P \) be an interior point of \( \triangle ABC \). Let \( LMN \) denotes the Cevian triangle of \( P \) respect to \( \triangle ABC \) and let \( R_q, r_q \) denote its circumradius and inradius respectively. If \( k \geq \frac{1}{2} \) is a real number, then the following inequality holds:

\[
\frac{1}{R^k_q} + \frac{1}{r^k_q} \geq \frac{1}{r^k} + \frac{4^k}{R^k}. \tag{4.10}
\]

Considering the exponential generalization of Theorem 1.3, the following conjecture is brought forward:

**Conjecture 6.** If \( k \) is a positive number, then the inequality:

\[
R^k_1 + R^k_2 + R^k_3 - (r^k_1 + r^k_2 + r^k_3) \geq 3 \cdot 2^k (2^k - 1) r^k_p \tag{4.11}
\]

holds for an arbitrary interior point \( P \) of \( \triangle ABC \).

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(Received May 20, 2011)

Jian Liu
East China Jiaotong University
Jiangxi province Nanchang City
330013, China

e-mail: China99jian@163.com