Journal of Mathematical Inequalities
Volume 6, Number 2 (2012), 205–214 doi:10.7153/jmi-06-21

FURTHER DEVELOPMENT AND REFINING SOME INEQUALITIES FOR POLYNOMIAL ZEROS

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(*Communicated by A. Horwitz*)

Abstract. In the article, we extend bounds for the roots of polynomials with complex coefficients obtained by Rică Zamfir [R. Zamfir, Refining Some Inequalities, Journal of Inequalities in Pure and Applied Mathematics, no. 3, Article 77, vol. 9, 2008 available online at http://jipam.vu.edu.au/v1n2/011_99.html]

1. Introduction

In the paper [7] Zamfir improved two well known bounds [1], [6] for the roots of polynomials with complex coefficients.

THEOREM 1.1. [6] *If* $f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 \in C[X]$, $a_n \neq 0$ *and* z *is an arbitrary root of* f, *then:*

$$|z|^2 \leqslant 1 + \left|\frac{a_0}{a_n}\right|^2 + \left|\frac{a_1 - a_0}{a_n}\right|^2 + \dots + \left|\frac{a_n - a_{n-1}}{a_n}\right|^2. \tag{1}$$

THEOREM 1.2. [1. p. 151] *If* f *is polynomial like in Theorem* 1.1 *and* $p \in \{1,2,\dots,n\}$, *then at least* p *roots of* f *are within in the disk:*

$$|z| \leqslant 1 + \left(\sum_{j=0}^{p-1} \left|\frac{a_j}{a_n}\right|^2\right)^{\frac{1}{2}}. \tag{2}$$

In [7] Zamfir established refinements of the inequalities (1) and (2).

THEOREM 1.3. [7] *If* $f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 \in C[X]$, *let* $b_0 = a_0$, $b_1 = a_1 - a_0$, ... $b_n = a_n - a_{n-1}$. *Then, for any root* z *of* f, *we have:*

$$|z|^2 \leqslant 1 + \sum_{j=0}^{n} \left|\frac{b_j}{a_n}\right|^2 - \frac{(\Re(b_0\bar{b}_1 + b_1\bar{b}_2 + \dots + b_{n-1}\bar{b}_n - b_n\bar{a}_n))^2}{(|b_0|^2 + |b_1|^2 + \dots + |b_n|^2)\,|a_n|^2}. \tag{3}$$

Mathematics subject classification (2010): 12D10.
Keywords and phrases: Inequality, polynomial, roots.

THEOREM 1.4. [7] *If* $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \in C[X]$, *and* $p \in \{1,2,\ldots,n\}$, *then at least* p *roots of* f *are within in the disk:*

$$|z| \leqslant 1 + \left(\sum_{j=0}^{p-1} \left| \frac{a_j}{a_n} \right|^2 - \frac{(\Re(a_0\bar{a}_1 + a_1\bar{a}_2 + \ldots + a_{p-1}\bar{a}_p))^2}{(|a_0|^2 + |a_1|^2 + \ldots + |a_p|^2)\,|a_n|^2} \right)^{\frac{1}{2}}. \tag{4}$$

REMARK 1.1. [7] If $\Re(b_0\bar{b}_1 + b_1\bar{b}_2 + \ldots + b_{n-1}\bar{b}_n - b_n\bar{a}_n) \neq 0$, then inequality (3) is better than inequality (1).

We note, that Theorem 1.3 contains a typographical error. The correct version of Theorem 1.3 is:

THEOREM 1.3. *If* $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \in C[X]$, *let* $b_0 = a_0$, $b_1 = a_1 - a_0$, ... $b_n = a_n - a_{n-1}$. *Then, for any root* z *of* f, *we have*

$$|z|^2 \leqslant 1 + \sum_{j=0}^{n} \left| \frac{b_j}{a_n} \right|^2 - \frac{(\Re(b_0\bar{b}_1 + b_1\bar{b}_2 + \ldots + b_{n-1}\bar{b}_n - b_n\bar{a}_n))^2}{(|b_0|^2 + |b_1|^2 + \ldots + |b_n|^2 + |a_n|^2)\,|a_n|^2}. \tag{5}$$

In this paper, we establish refinements of (1), (2) in the case $\Re(b_0\bar{b}_1 + b_1\bar{b}_2 + \ldots + b_{n-1}\bar{b}_n - b_n\bar{a}_n) = 0$ with the exception of $a_i = a_j$, $i,j \in \{0,1,\ldots,n\}$. We also establish a new refinement of (1), (5). We use the same method of proofs as in [7]. This method was used among others by L. Panaitopol, D. Stefănescu [5] and R. Zamfir [7].

2. The main results

In this section, we present Theorems 2.1, 2.2, which establish refinements of inequalities (1), (2), Theorem 2.3 which is a refinement of (1), (5), and Lemma 2.1.

THEOREM 2.1. *If* $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \in C[X]$, *let* $b_0 = a_0$, $b_1 = a_1 - a_0$, ... $b_n = a_n - a_{n-1}$, $a_n \neq 0$, *and* $k \in \{1,2,\ldots,n\}$. *Then, for any root* z *of* f, *we have*

$$|z|^2 \leqslant 1 + \sum_{j=0}^{n} \left| \frac{b_j}{a_n} \right|^2 - \frac{(\Re(b_0\bar{b}_k + b_1\bar{b}_{k+1} + \ldots + b_{n-k}\bar{b}_n - b_{n-k+1}\bar{a}_n))^2}{(|b_0|^2 + |b_1|^2 + \ldots + |b_n|^2 + |a_n|^2)\,|a_n|^2}. \tag{6}$$

REMARK 2.1. We note, that Theorem 1.3 is a special case of Theorem 2.1. If $\Re(b_0\bar{b}_1 + b_1\bar{b}_2 + \ldots + b_{n-1}\bar{b}_n - b_n\bar{a}_n) = 0$ and $\Re(b_0\bar{b}_k + b_1\bar{b}_{k+1} + \ldots + b_{n-k}\bar{b}_n - b_{n-k+1}\bar{a}_n) \neq 0$ for some $k \in \{2,\ldots,n\}$ then inequality (6) is better than inequalities (1), (5).

THEOREM 2.2. *If* $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \in C[X]$, $a_m = 0$, $m > n$ *and* $p \in \{1,2,\ldots,n\}$ *such that* $k+1 \leqslant p$ *for some* $k \in \{2,\ldots,n-1\}$. *Then at least* p *roots of* f *are within in the disk:*

$$|z| \leqslant 1 + \left(\sum_{j=0}^{p-1} \left| \frac{a_j}{a_n} \right|^2 - \frac{(\Re(a_0\bar{a}_k + a_1\bar{a}_{k+1} + \ldots + a_{p-1}\bar{a}_{p-1+k}))^2}{(|a_0|^2 + |a_1|^2 + \ldots + |a_{p-1+k}|^2)\,|a_n|^2} \right)^{\frac{1}{2}}. \tag{7}$$

REMARK 2.2. If $\Re(a_0\bar{a}_1+a_1\bar{a}_2+\ldots+a_{p-1}\bar{a}_p)=0$ and $\Re(a_0\bar{a}_k+a_1\bar{a}_{k+1}+\ldots+a_{p-1}\bar{a}_{p-1+k})\neq 0$ then inequality (7) is better than inequalities (2), (4).

THEOREM 2.3. *If* $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0\in C[X]$, *let* $B_k=\Re(b_0\bar{b}_k+b_1\bar{b}_{k+1}+\ldots+b_{n-k}\bar{b}_n-b_{n-k+1}\bar{a}_n)$, $k\in\{1,\ldots,n\}$, $A=|b_0|^2+|b_1|^2+\ldots+|b_n|^2$. *Then, for any root* z *of* f, *we have*

$$|z|^2\leqslant 1+\frac{1}{|a_n|^2}\left((A+|a_n|^2)(\alpha_0^2+\alpha_1^2)+2\alpha_0B_2+2\alpha_1B_1+2\alpha_0\alpha_1B_1+A\right), \tag{8}$$

where

$$\alpha_0=\frac{B_1^2-B_2(A+|a_n|^2)}{(A+|a_n|^2)^2-B_1^2},\qquad \alpha_1=\frac{B_1B_2-B_1(A+|a_n|^2)}{(A+|a_n|^2)^2-B_1^2}.$$

REMARK 2.3. Inequality (8) is better than inequalities (1), (5). It follows from Remark 3.1.

LEMMA 2.1. *If* $B_k=\Re(b_0\bar{b}_k+b_1\bar{b}_{k+1}+\ldots+b_{n-k}\bar{b}_n-b_{n-k+1}\bar{a}_n)=0$, $k\in\{1,\ldots,n\}$. *Then* $a_i=a_j$, $i,j\in\{0,1,\ldots,n\}$.

3. Proofs of main results

Proof of Theorem 2.1. Similarly as in [7] we consider the polynomial $F_k(x)=(x^k-\alpha)f(x)$, where α is a real number. We get

$$\begin{aligned}F_k(x)&=a_nx^{n+k}+\ldots+a_{n-k+1}x^{n+1}+(a_{n-k}-\alpha a_n)x^n+\ldots+(a_0-\alpha a_k)x^k\\&\quad-\alpha a_{k-1}x^{k-1}-\ldots-\alpha a_0.\end{aligned}$$

By applying (1) to F_k, we have for any root z of F_k

$$|z|^2\leqslant 1+\sum_{i=0}^{k-1}\left|\frac{\alpha b_i}{a_n}\right|^2+\sum_{i=0}^{n-k}\left|\frac{b_i-\alpha b_{i+k}}{a_n}\right|^2+\left|\frac{b_{n-k+1}+\alpha a_n}{a_n}\right|^2+\sum_{i=n-k+2}^{n}\left|\frac{b_i}{a_n}\right|^2.$$

Using the formula $|x-y|^2=|x|^2+|y|^2-2\Re(x\bar{y})$ we obtain that $|b_i-\alpha b_{i+j}|^2=|b_i|^2+\alpha^2|b_{i+j}|^2-2\alpha\Re(b_i\bar{b}_{i+j})$. From this we have for any root z of f

$$|z|^2\leqslant 1+\frac{1}{|a_n|^2}\left((A+|a_n|^2)\alpha^2-2B_k\alpha+A\right)$$

where A, B_k is the same as in Theorem 2.3. The minimal value $A-\frac{B_k}{A+|a_n|^2}$ of the function $g(\alpha)=(A+|a_n|^2)\alpha^2-2B_k\alpha+A$ is obtained for $\alpha=\frac{B_k}{A+|a_n|^2}$. The proof is complete. □

Proof of Theorem 2.2. Similarly as in [7] we put $F_k(x)=(x^k-\alpha)f(x)$, $k\leqslant p-1$. The coefficients of F_k are $c_m=\alpha a_m$ for $m\in\{0,\ldots,k-1\}$, $c_m=a_{m-k}-\alpha a_m$ for

$m \in \{k,...,n\}$, $c_m = a_{m-k}$ for $m \in \{n+1,...,n+k\}$. By applying (2) to F_k, at least p roots of F_k are within the disk

$$| z | \leqslant 1 + \left(\sum_{i=0}^{p-1} \left| \frac{c_i}{a_n} \right|^2 \right)^{\frac{1}{2}}.$$

Denote $A_m = | a_0 |^2 + ... + | a_m |^2$, $B_{k,m} = \Re(a_0 \bar{a}_k + ... + a_{m-k} \bar{a}_m)$. Then

$$\begin{aligned} \sum_{i=0}^{p-1} \left| \frac{c_i}{a_n} \right|^2 &= \frac{1}{| a_n |^2} \left(\alpha^2 \sum_{i=0}^{p-1} | a_i |^2 - 2\alpha \Re\Big(\sum_{i=0}^{p-k-1} a_i \bar{a}_{i+k} \Big) + \sum_{i=0}^{p-1-k} | a_i |^2 \right) \\ &= \frac{1}{| a_n |^2} \left(A_{p-1} \alpha^2 - 2B_{k,p-1} \alpha + A_{p-1-k} \right) = h(\alpha). \end{aligned}$$

It is evident that h is minimal for $\alpha = \frac{B_{k,p-1}}{A_{p-1}}$ and the minimal value is:

$$\frac{1}{| a_n |^2} \left(A_{p-1-k} - \frac{B_{k,p-1}^2}{A_{p-1}} \right).$$

Since $| B_{k,p-1} | \leqslant A_{p-1}$ we deduce that at least $p-k$ roots of f are within the same disk. It implies that at least p roots of f are within the disk

$$| z | \leqslant 1 + \left(\sum_{k=0}^{p-1} | a_k |^2 - \frac{B_{k,p-1+k}^2}{| a_n |^2 A_{p-1+k}} \right)^{\frac{1}{2}}.$$

The proof is complete. □

Proof of Theorem 2.3. We put $F_m(x) = (x^m + \alpha_{m-1}x^{m-1} + ... + \alpha_1 x + \alpha_0) f(x)$, $1 < m \leqslant n$. The coefficients of F_m are $c_j = \sum_{i=0}^{m} \alpha_i a_{j-i}$ for $j \in \{0,...,m+n\}$, where $a_{n+k} = 0$, $\alpha_{m+k} = 0$ for $k \geqslant 1$, $a_k = 0$, $\alpha_k = 0$ for $k < 0$, $\alpha_m = 1$. By applying (1) to F_m we get that if z is a root of F_m then

$$| z |^2 \leqslant 1 + \left| \frac{c_0}{c_{m+n}} \right|^2 + \sum_{j=1}^{m+n} \left| \frac{c_j - c_{j-1}}{c_{m+n}} \right|^2.$$

From

$$c_k - c_{k-1} = \sum_{i=0}^{m} \alpha_i a_{k-i} - \sum_{i=0}^{m} \alpha_i a_{k-i-1} = \sum_{i=0}^{m} \alpha_i b_{k-i}$$

and by using mathematical induction we obtain

$$| z |^2 \leqslant 1 + \frac{1}{| a_n |^2} \left((A + | a_n |^2) \sum_{i=0}^{m-1} \alpha_i^2 + 2 \sum_{i=0}^{m-1} \alpha_i B_{m-i} + 2 \sum_{i=0}^{m-2} \sum_{j=1,i<j}^{m-1} \alpha_i \alpha_j B_{j-i} + A \right).$$

For $m = 2$ we have

$$|z|^2 \leqslant 1 + \frac{1}{|a_n|^2}\left((A + |a_n|^2)(\alpha_0^2 + \alpha_1^2) + 2\alpha_0 B_2 + 2\alpha_1 B_1 + 2\alpha_0\alpha_1 B_1 + A\right).$$

Denote

$$g(\alpha_0, \alpha_1) = (A + |a_n|^2)(\alpha_0^2 + \alpha_1^2) + 2\alpha_0 B_2 + 2\alpha_1 B_1 + 2\alpha_0\alpha_1 B_1 + A.$$

The minimal value of the function $g(\alpha_0, \alpha_1)$ is obtained for

$$\alpha_0 = \frac{B_1^2 - B_2(A + |a_n|^2)}{(A + |a_n|^2)^2 - B_1^2}, \quad \alpha_1 = \frac{B_1(B_2 - (A + |a_n|^2))}{(A + |a_n|^2)^2 - B_1^2}.$$

It follows from $(A + |a_n|^2)^2 > B_1^2$. This inequality we obtain from

$$B_1 = A + |a_n|^2 - \frac{1}{2}\left(\sum_{i=0}^{n-1} |b_i - b_{i+1}|^2 + |b_0|^2 + |a_n + b_n|^2 + |a_n|^2\right),$$

and from

$$\begin{aligned} A + |a_n|^2 + B_1 &= 2(A + |a_n|^2) - \frac{1}{2}\left(\sum_{i=0}^{n-1} |b_i - b_{i+1}|^2 + |b_0|^2 + |a_n + b_n|^2 + |a_n|^2\right) \\ &= \frac{1}{2}(|b_0|^2 + |a_n|^2) + \sum_{i=0}^{n-1} |b_i|^2 - \frac{1}{2}|b_i - b_{i+1}|^2 + |b_{i+1}|^2 \\ &\quad + |a_n|^2 - \frac{1}{2}|a_n + b_n|^2 + |b_n|^2 > 0. \end{aligned}$$

The proof is complete. □

REMARK 3.1. If we put $\alpha_0 = 0$, $\alpha_1 = \frac{-B_1}{A + |a_n|^2}$ then

$$g(\alpha_0, \alpha_1) = A - \frac{B_1^2}{A + |a_n|^2}.$$

It implies (8) is better than (5)

Proof of Lemma 2.1. Using mathematical induction we prove the following formula:

$$\sum_{j=1}^{n} B_j = -\frac{1}{2}\left(\sum_{i=0}^{n-1} |a_i - a_{i+1}|^2 + |a_n - a_0|^2\right). \tag{9}$$

If $n = 1$ then

$$B_1 = \Re(a_0(\bar{a}_1 - \bar{a}_0) - (a_1 - a_0)\bar{a}_1) = -|a_0 - a_1|^2.$$

Suppose (9) is fulfilled for all $i,\ i \leqslant k < n$. For $j \in \{1,...,k\}$ we have

$$
\begin{aligned}
B_j &= \Re(b_0\bar{b}_j) + \Re(b_1\bar{b}_{j+1}) + ... + \Re(b_{k+1-j}\bar{b}_{k+1}) - \Re(b_{k-j+2}\bar{a}_{k+1}) \\
&= \Re(b_0\bar{b}_j) + ... + \Re(b_{k-j}\bar{b}_k) - \Re(b_{k+1-j}\bar{a}_k) + \Re(b_{k-j+1}\bar{a}_k) \\
&\quad + \Re(b_{k+1-j}\bar{b}_{k+1}) - \Re(b_{k+2-j}\bar{a}_{k+1}). \\
B_{k+1} &= \Re(b_0\bar{b}_{k+1}) - \Re(b_1\bar{a}_{k+1}).
\end{aligned}
$$

It implies

$$
\begin{aligned}
\sum_{j=1}^{k+1} B_j &= -\frac{1}{2}\left(\sum_{i=0}^{k-1} \mid a_i - a_{i+1} \mid^2 + \mid a_k - a_0 \mid^2\right) \\
&\quad + \sum_{j=1}^{k}\Big(\Re(b_{k+1-j}\bar{a}_k) + \Re(b_{k+1-j}\bar{b}_{k+1}) - \Re(b_{k+2-j}\bar{a}_{k+1})\Big) \\
&\quad + \Re(b_0\bar{b}_{k+1}) - \Re(b_1\bar{a}_{k+1}).
\end{aligned}
$$

From

$$
\begin{aligned}
&\sum_{j=1}^{k} \Re(b_{k+1-j}\bar{a}_k) + \Re(b_{k+1-j}\bar{b}_{k+1}) - \Re(b_{k+2-j}\bar{a}_{k+1}) \\
&= \sum_{j=1}^{k} \Re(b_{k+1-j}\bar{a}_{k+1}) - \Re(b_{k+2-j}\bar{a}_{k+1}) \\
&= \Re(b_1\bar{a}_{k+1}) - \Re(b_{k+1}\bar{a}_{k+1})
\end{aligned}
$$

we get

$$
\begin{aligned}
\sum_{j=1}^{k+1} B_j &= -\frac{1}{2}\left(\sum_{i=0}^{k-1} \mid a_i - a_{i+1} \mid^2 + \mid a_k - a_0 \mid^2\right) + \Re(b_0\bar{b}_{k+1}) - \Re(b_{k+1}\bar{a}_{k+1}) \\
&= -\frac{1}{2}\sum_{i=0}^{k-1} \mid a_i - a_{i+1} \mid^2 - \frac{1}{2} \mid a_k \mid^2 - \frac{1}{2} \mid a_0 \mid^2 + \Re(a_0\bar{a}_k) + \Re(a_0\bar{a}_{k+1}) - \Re(a_0\bar{a}_k) \\
&\quad - \Re(a_{k+1}\bar{a}_{k+1}) + \Re(a_k\bar{a}_{k+1}) \\
&= -\frac{1}{2}\left(\sum_{i=0}^{k} \mid a_i - a_{i+1} \mid^2 + \mid a_{k+1} - a_0 \mid^2\right).
\end{aligned}
$$

The proof is complete. □

4. Conjecture and examples

We denote

$$
g_m(x_0,...,x_{m-1}) = (A + \mid a_n \mid^2)\sum_{i=0}^{m-1} x_i^2 + 2\sum_{i=0}^{m-1} x_i B_{m-i} + 2\sum_{i=0}^{m-2}\sum_{j=1,i<j}^{m-1} x_i x_j B_{j-i} + A, \quad (10)
$$

$D_m = \det(d)$ where d is the matrix

$$d_{i,j} = \begin{cases} 2(A+ \mid a_n \mid^2) & \text{for} \quad i = j, \quad i,j \in \{0,...,m-1\}, \\ 2B_{j-i} & \text{for} \quad i < j, \quad i,j \in \{0,...,m-1\}, \\ d_{j,i} & \text{for} \quad i,j \in \{0,...,m-1\}, \end{cases} \tag{11}$$

$m \in \{3,...,n\}$. We conjecture that $D_m > 0$ and all main minors of D_m are positive. This implies if $\alpha_{0,m},\dots,\alpha_{m-1,m}$ are solutions of the equations:

$$x_i(A+ \mid a_n \mid^2) + \sum_{j=0}^{i-1} x_j B_{i-j} + \sum_{j=i+1}^{m-1} x_j B_{j-i} = -B_{m-i}, \quad i \in \{0,...,m-1\} \tag{12}$$

then if z is a root of f we have

$$\mid z \mid^2 \leqslant 1 + \frac{1}{\mid a_n \mid^2} g_m(\alpha_{0,m},\dots,\alpha_{m-1,m})$$

and $g_l(\alpha_{0,l},\dots,\alpha_{l-1,l}) \geqslant g_m(\alpha_{0,m},\dots,\alpha_{m-1,m})$, $2 \leqslant l < m$. So, if the conjecture holds then we obtain a new refinements of (2).

Next, we recompute all examples in Zamfir's paper [7] and compare our new bounds (6), (8) for moduli of polynomial zeros to correct Zamfir's bound (5) and also to Cauchy's bound.

Cauchy's bound is given by

$$\mid z \mid < 1 + \max\left\{ \left|\frac{a_0}{a_n}\right| - 1, \left|\frac{a_1}{a_n}\right|, \dots, \left|\frac{a_{n-1}}{a_n}\right| \right\}.$$

We use the following notations:

- zbe – Zamfir's bound with typographical error,
- zbc – correct Zamfir's bound,
- cb – Cauchy's bound,
- mb_2, mb_3,...,mb_k – our new bounds (Theorem 6),
- nb – our new bound (Theorem 8).

EXAMPLE 4.1. Let $f(z) = 20z^4 - 2z^3 + 2z^2 - z + 1$. Using the mathematical program MATLAB we find the roots of f :

$$z_{1,2} = -0.2717 \pm 0.4173\imath, \quad z_{3,4} = -0.3217 \pm 0.3133\imath.$$

The parameters for calculation are:

$$b_0 = 1, \quad b_1 = -2, \quad b_2 = 3, \quad b_3 = -4, \quad b_4 = 22, \quad A = 514,$$
$$B_1 = -548, \quad B_2 = 157, \quad B_3 = -108, \quad B_4 = 62,$$
$$\alpha_0 = 0.2930, \quad \alpha_1 = 0.7753.$$

The bounds are:

$$zbe = 0.907, \quad zbc = 1.2098, \quad mb_2 = 1.4892, \quad mb_3 = 1.5010, \quad mb_4 = 1.5081,$$
$$nb = 1.1567, \quad cb = 1.1000.$$

We obtained, that Cauchy's bound is the best. The corrected Zamfir's bound is better than all other bounds mb_k $k = 2,3,4$. Our new bound is better than Cauchy's bound.

EXAMPLE 4.2. Let $f(z) = 6z^4 + 35z^3 + 31z^2 + 35z + 6$. Similarly, we find the roots of f:

$$z_{1,2} = -0.3029 \pm 0.9530\imath, \quad z_3 = -5.0287, \quad z_4 = -0.1989.$$

The parameters for calculation are:

$$b_0 = 6, \quad b_1 = 29, \quad b_2 = -4, \quad b_3 = 4, \quad b_4 = -29, \quad A = 1750,$$
$$B_1 = 100, \quad B_2 = 184, \quad B_3 = -793, \quad B_4 = -348,$$
$$\alpha_0 = -0.1002, \quad \alpha_1 = -0.0504.$$

The bounds are:

$$zbe = 7.032, \quad zbc = 7.0325, \quad mb_2 = 7.0060, \quad mb_3 = 6.3111, \quad mb_4 = 6.9085,$$
$$nb = 6.9971, \quad cb = 6.8333.$$

We obtained, that Cauchy's bound is the best. All other bounds mb_k $k = 2,3,4$ are better than the corrected Zamfir's bound.

EXAMPLE 4.3. Let $f(z) = 7z^5 - 20z^3 + z + 1$. Similarly, we find the roots of f:

$$z_{1,2} = -0.2025 \pm 0.2815\imath, \quad z_3 = 0.4235, \quad z_4 = -1.6842, \quad z_5 = 1.6658.$$

The parameters for calculation are:

$$b_0 = 1, \quad b_1 = 0, \quad b_2 = -1, \quad b_3 = -20, \quad b_4 = 20, \quad b_5 = 7, A = 851,$$
$$B_1 = -289, \quad B_2 = -301, \quad B_3 = 113, \quad B_4 = 27, \quad B_5 = 7,$$
$$\alpha_0 = 0.4879, \quad \alpha_1 = 0.4778.$$

The bounds are:

$$zbe = 4.048, \quad zbc = 4.0587, \quad mb_2 = 4.0389, \quad mb_3 = 4.2518,$$
$$mb_4 = 4.2838, \quad mb_5 = 4.2856, \quad nb = 3.5430, \quad cb = 3.8571.$$

We obtained, that our new bound (Theorem 8) is the best. Cauchy's bound is better than all other bounds mb_k $k = 1,2,3,4,5$.

EXAMPLE 4.4. Let $f(z) = 10z^5 + z^4 + 100z^3 + 10z^2 + 90z + 1$. Similarly, we find the roots of f :

$$z_{1,2} = 0.0055 \pm 3.0002\imath, \quad z_{3,4} = -0.05 \pm 0.9981\imath, \quad z_5 = -0.0111.$$

The parameters for calculation are:

$$b_0 = 1, \quad b_1 = 89, \quad b_2 = -80, \quad b_3 = 90, \quad b_4 = -99, \quad b_5 = 9, \quad A = 32304,$$
$$B_1 = -24122, \quad B_2 = 17650, \quad B_3 = -10341, \quad B_4 = 1502, \quad B_5 = -881,$$
$$\alpha_0 = 0.0212, \quad \alpha_1 = 0.7602.$$

The bounds are:

$$zbe = 11.9965, \quad zbc = 12.0197, \quad mb_2 = 15.0965, \quad mb_3 = 17.0599,$$
$$mb_4 = 17.9818, \quad mb_5 = 17.9945, \quad nb = 12.0170, \quad cb = 11.$$

We obtained, that Cauchy's bound is the best. Zamfir's bound is better than all other bounds mb_k $k = 2,3,4,5$.

EXAMPLE 4.5. Let $f(z) = z^5 + 7z^4 + 55z^3 + 112z^2 + z + 1$. Similarly, we find the roots of f :

$$z_{1,2} = -2.2170 \pm 6.2199\imath, \quad z_{3,4} = -0.0023 \pm 0.0946\imath, \quad z_5 = -2.5615.$$

The parameters for calculation are:

$$b_0 = 1, \quad b_1 = 0, \quad b_2 = 111, \quad b_3 = -57, \quad b_4 = -48, \quad b_5 = -6,$$
$$A = 17911, \quad B_1 = -3297, \quad B_2 = -4827, \quad B_3 = -666, \quad B_4 = -159, \quad B_5 = -6,$$
$$\alpha_0 = 0.3140, \quad \alpha_1 = 0.2419.$$

The bounds are:

$$zbe = 131.5488, \quad zbc = 131.5490, \quad mb_2 = 128.8844, \quad mb_3 = 133.7432,$$
$$mb_4 = 133.8304, \quad mb_5 = 133.8357, \quad nb = 124.8955, \quad cb = 113.$$

We obtained, that Cauchy's bound is the best. mb_2 is better than Zamfir's bound which is better than other bounds mb_k $k = 3,4,5$.

EXAMPLE 4.6. Let $f(z) = z^3 - z^2 - 2z - 1.5$. Similarly, we find the roots of f :

$$z_1 = -0.6056 + 0.5582\imath, \quad z_2 = -0.6056 - 0.5582\imath, \quad z_3 = 2.2112.$$

The parameters for calculation are:

$$b_0 = -1.5, \quad b_1 = -0.5, \quad b_2 = 1, \quad b_3 = 2, \quad A = 7.5, \quad B_1 = 0.25,$$
$$B_2 = -3.5, \quad B_3 = -2.5, \quad \alpha_0 = 0.4130, \quad \alpha_1 = -0.0416.$$

The bounds are:

$$zbe = 2.9140, \quad zbc = 2.9142, \quad mb_2 = 2.6568, \quad mb_3 = 2.7865,$$
$$nb = 2.6541, \quad cb = 3.$$

We obtained, that all new bounds are better than Cauchy's bound.

EXAMPLE 4.7. Let $f(z) = -4z^5 - 10z^4 + z^3 + 30z^2 + 40z + 3$. Similarly, we find the roots of f:

$$z_{1,2} = -1.1294 \pm 1.1890\imath, \quad z_3 = 1.7910, \quad z_4 = -1.9526, \quad z_5 = -0.0797.$$

The parameters for calculation are:

$$b_0 = 3, \quad b_1 = 37, \quad b_2 = -10, \quad b_3 = -20, \quad b_4 = -11, \quad b_5 = 6, \quad A = 2476,$$
$$B_1 = 308, \quad B_2 = -1211, \quad B_3 = -670, \quad B_4 = 149, \quad B_5 = 166,$$
$$\alpha_0 = 0.5090, \quad \alpha_1 = -0.1865.$$

The bounds are:

$$zbe = 12.3837, \quad zbc = 12.3847, \quad mb_2 = 10.9073, \quad mb_3 = 12.0205,$$
$$mb_4 = 12.4577, \quad mb_5 = 12.4523, \quad nb = 10.6599, \quad cb = 11.$$

We obtained, that our new bound and mb_2 bound are better than Cauchy's and Zamfir's bounds.

Acknowledgements. The author is deeply grateful to the anonymous referee for his valuable remarks and suggestions which improved this paper.

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(Received July 26, 2011)

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