Journal of
Mathematical
Inequalities
Volume 6, Number 2 (2012), 215–227
doi:10.7153/jmi-06-22

A WEAK POINCARÉ–SOBOLEV INEQUALITY FOR FUNCTIONS IN MORREY SPACES

MODESTE ESSOH AND IBRAHIM FOFANA

(*Communicated by Vladimir Dmitrievič Stepanov*)

Abstract. We prove a weak Poincaré-Sobolev type inequality for a function belonging to Morrey spaces with respect to a Hausdorff content.

1. Introduction

Poincaré Inequality allows one to obtain estimations on a function using estimations on its derivates. Such estimations are of great importance in the modern, direct methods of the calculus of variations (see [14]).

In [1, 3] Adams has established Poincaré inequalities in term of Lebesgue norms with respect to a Hausdorff content. In this paper, we control a weak Morrey norm with respect to a Hausdorff content of a function f by classical Morrey norm of its gradient.

The statement of this result needs some notations and definitions.

The Lebesgue measure of a set E is denoted by $|E|$. $Q(x,r)$ stands for the cube centered at $x \in \mathbb{R}^n$ with side length r and sides parallel to the cordinate axes. Given a locally integrable function f, we denote by $f_{Q(x,r)}$ its mean value over $Q(x,r)$ defined by: $f_{Q(x,r)} = \dfrac{1}{|Q(x,r)|}\displaystyle\int_{Q(x,r)} f(y)dy$.

If $E \subset \mathbb{R}^n$ and $0 < \delta \leqslant n$, the Hausdorff content of E of order δ is defined by

$$H^{\delta}(E) = \inf \sum_{j=1}^{\infty} l(Q_j)^{\delta}, \tag{1.1}$$

where the infimum is taken over all coverings of E by countable families of cubes Q_j. Throughout this paper, only cubes with sides parallel to the cordinate axes are considered and $l(Q)$ denotes the side length of the cube Q.

Note that, $H^n(E) = |E|$ and if in the relation (1.1) we take the infimum only on coverings of E by dyadic cubes, we get the dyadic Hausdorff content of order δ, $H^{\delta}_{\triangle}(E)$. It is well known (see [1]) that

Property P1: there exists two constants $A > 0$ and $B > 0$ such that for any $E \subset \mathbb{R}^n$, $H^{\delta}(E) \leqslant AH^{\delta}_{\triangle}(E) \leqslant BH^{\delta}(E)$;

Mathematics subject classification (2010): 28A12, 42B25.
Keywords and phrases: Poincaré-Sobolev inequality, Morrey spaces, Hausdorff content, fractional maximal operator, Riesz potential operator.

Paper JMI-06-22

Property P2: If (E_i) is a increasing sequence of arbitrary sets, then $\lim_{i\to+\infty} H^{\delta}_{\triangle}(E_i) = H^{\delta}_{\triangle}(\bigcup_i E_i)$.

Let $1 \leqslant p < +\infty$, $0 \leqslant \lambda \leqslant n$ and $0 < \delta \leqslant n$. We denote by $L^{p,\lambda}_*(H^\delta)$ the weak Morrey type space with respect to the Hausdorff content H^δ, that is the space of all functions f such as

$$\|f\|_{L^{p,\lambda}_*(H^\delta)} := \sup_{\substack{Q\\ t>0}} tH^\delta\left(\{x\in Q, |f(x)|>t\}\right)^{\frac{1}{p}} l(Q)^{-\frac{\lambda}{p}} < \infty,$$

where the supremum is taken over all $t>0$ and cubes Q of $\mathbb{R}^n$. The classical Morrey spaces $L^{p,\lambda}(dx)$ are the spaces of all locally integrable functions f satisfying

$$\|f\|_{L^{p,\lambda}(dx)} := \sup_Q \left(\frac{1}{l(Q)^\lambda}\int_Q |f(y)|^p dy\right)^{\frac{1}{p}} < \infty$$

where the supremum is taken over all cubes Q of $\mathbb{R}^n$.

Notice that $L^{p,0}_*(H^\delta)$ is the weak Lebesgue space $L^p_*(H^\delta)$ with respect to the Hausdorff content H^δ (see [2]).

We recall that the Sobolev space $W^{1,1}_{loc}$ is defined by

$$W^{1,1}_{loc} = \left\{f\in L^1_{loc} : D_jf\in L^1_{loc}\ j\in\{1,2...,n\},\right\}$$

where L^1_{loc} is the set of all locally Lebesgue integrable functions.

We can now state our main result.

THEOREM 1.1. *Suppose that* $0\leqslant\beta<1<\alpha\leqslant n\leqslant\delta+\beta$, $\delta\leqslant n$ *and* $\lambda(n-\beta)=\delta(n-\alpha)$. *Then there exists a constant* $C>0$ *such that for any function* f *belonging to* $W^{1,1}_{loc}$ *and satisfying*

$$\limsup_{r\to 0} f_{Q(x,r)} = f(x)\,, \qquad x\in\mathbb{R}^n$$

and

$$\lim_{r\to\infty} |f_{Q(x,r)}| = 0\,, \qquad x\in\mathbb{R}^n$$

we have

$$\|f\|_{L_*^{\frac{\delta}{n-\beta}\frac{\alpha-\beta}{\alpha-1},\lambda}(H^\delta)} \leqslant C\|\nabla f\|_{L^{1,n-\alpha}(dx)}.$$

As an immediate consequence we have the following weak Sobolev inequality.

COROLLARY 1.2. *Suppose that* $1<\alpha\leqslant n$. *Then there exists a constant* $C>0$ *such that for any function* f *belonging to* $W^{1,1}$ *we have*

$$\|f\|_{L_*^{\frac{\alpha}{\alpha-1},n-\alpha}(dx)} \leqslant C\|\nabla f\|_{L^{1,n-\alpha}(dx)}. \tag{1.2}$$

Inequality (1.2) generalizes the inequality

$$\|f\|_{L_*^{\frac{n}{n-1}}(dx)} \leqslant C\|\nabla f\|_{L^1(dx)}, \tag{1.3}$$

which is a weak form of the Sobolev classical inequality

$$\|f\|_{L^{\frac{n}{n-1}}(dx)} \leqslant C\|\nabla f\|_{L^1(dx)}, \tag{1.4}$$

(see [8]).

The remainder of this paper is organized as follows: in section 2 we establish a boundedness property for the Riez potential in Morrey type spaces. In section 3, we prove technical lemmas used in the proof of the principal result. In section 4, we prove Theorem 1.1 and Corollary 1.2 .

2. Weak Inequality For Riesz Potential

Let f be a locally integrable function on $\mathbb{R}^n$. The Riesz potential of f of order α $(0 \leqslant \alpha < n)$ is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{\alpha-n}} dy, \ x \in \mathbb{R}^n.$$

It is well known that I_α is related to the fractional maximal operator M_α defined by

$$M_\alpha f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |f(y)| dy, \ x \in \mathbb{R}^n$$

where the supremum is taken over all cubes Q containing x.

The theory of boundedness of fractional maximal operator and Riesz potential from one Morrey-type space constructed on the base of Lebesgue measure to another one is well studied. See [3], [4], [5] and [6].

In this section, boundedness property for the Riesz potential (Theorem 2.4) in Morrey-type space constructed on the base of Hausdorff content is obtained. This result is interesting in the sense that there exists a set $E \subset \mathbb{R}^n$ such that $|E| = 0$ and $H^\delta(E) > 0$ for $\delta < n$.

An instance of this relation is the following form of Welland's inequality.

LEMMA 2.1. *Assume $0 \leqslant \beta < \gamma < \alpha \leqslant n$. Then there exists a constant $C > 0$, such that, for any positive function f, we have:*

$$I_\gamma f(x) \leqslant C \left(M_\beta f(x)\right)^{\frac{\alpha-\gamma}{\alpha-\beta}} \left(M_\alpha f(x)\right)^{\frac{\gamma-\beta}{\alpha-\beta}}, \quad x \in \mathbb{R}^n.$$

The proof is the same as that of inequality (2.3) in [13].

Now, we recall a boundedness property for the fractional maximal operator in Morrey type spaces (see [7]).

PROPOSITION 2.2. *Assume that* $0 \leqslant \beta < n$, $0 \leqslant \lambda \leqslant n$, $0 < \delta \leqslant n$, $\delta \geqslant \lambda$, $\delta \geqslant n-\beta$ *and* $\mu = \frac{\lambda}{\delta}(n-\beta)$. *Then there exists a constant* $C > 0$, *such that for any locally integrable function* f, *we have*

$$\|M_\beta f\|_{L_*^{\frac{\delta}{n-\beta},\lambda}(H^\delta)} \leqslant C\|f\|_{L^{1,\mu}(dx)}.$$

From the above results, we shall deduce a norm estimate for the Riez potential. Before this, we establish the following lemma which will be useful in the of proof of our result.

LEMMA 2.3. *Let* $p,q > 0$, $0 \leqslant \lambda \leqslant n$ *and* $0 < \delta \leqslant n$. *Then for any locally integrable function* f, *we have*

$$\|f^p\|_{L_*^{q,\lambda}(H^\delta)} = \|f\|^p_{L_*^{pq,\lambda}(H^\delta)}.$$

Proof. We have

$$\begin{aligned}
\|f^p\|_{L_*^{q,\lambda}(H^\delta)} &= \sup_{\substack{Q\\ t>0}} t\left(H^\delta\{x\in Q, |f(x)|^p > t\}\right)^{\frac{1}{q}} l(Q)^{\frac{-\lambda}{q}}\\
&= \sup_{\substack{Q\\ t>0}} t\left(H^\delta\left\{x\in Q, |f(x)| > t^{\frac{1}{p}}\right\}\right)^{\frac{1}{q}} l(Q)^{\frac{-\lambda}{q}}\\
&= \sup_{\substack{Q\\ u>0}} u^p\left(H^\delta\{x\in Q, |f(x)| > u\}\right)^{\frac{1}{q}} l(Q)^{\frac{-\lambda}{q}}\\
&= \left(\sup_{\substack{Q\\ u>0}} u\left(H^\delta\{x\in Q, |f(x)| > u\}\right)^{\frac{1}{pq}} l(Q)^{\frac{-\lambda}{pq}}\right)^p\\
&= \|f\|^p_{L_*^{pq,\lambda}(H^\delta)}. \qquad \square
\end{aligned}$$

THEOREM 2.4. *If* $0 \leqslant \beta < \gamma < \alpha \leqslant n \leqslant \delta+\beta$, $\delta \leqslant n$ *and* $\lambda(n-\beta) = \delta(n-\alpha)$, *then there exists a constant* $C > 0$ *such that for any function* $f \in L^{1,n-\alpha}(dx)$, *we have*

$$\|I_\gamma f\|_{L_*^{\frac{\delta}{n-\beta}\frac{\alpha-\beta}{\alpha-\gamma},\lambda}(H^\delta)} \leqslant C\|f\|_{L^{1,n-\alpha}(dx)}.$$

Proof. We have

$$M_\alpha f(x) = \sup_{Q\ni x} \frac{1}{|Q|^{1-\frac{\alpha}{n}}}\int_Q |f(y)|dy \leqslant \|f\|_{L^{1,n-\alpha}(dx)}, x \in \mathbb{R}^n.$$

It follows from Lemma 2.1 that

$$I_\gamma|f|(x) \leqslant C_1\|f\|^{\frac{\gamma-\beta}{\alpha-\beta}}_{L^{1,n-\alpha}(dx)}\left(M_\beta f(x)\right)^{\frac{\alpha-\gamma}{\alpha-\beta}}, x \in \mathbb{R}^n.$$

So

$$\|I_\gamma |f|\|_{L_*^{\frac{\delta}{n-\beta}\frac{\alpha-\beta}{\alpha-\gamma},\lambda}(H^\delta)} \leqslant C_1 \|f\|_{L^{1,n-\alpha}(dx)}^{\frac{\gamma-\beta}{\alpha-\beta}} \| \left(M_\beta f\right)^{\frac{\alpha-\gamma}{\alpha-\beta}} \|_{L_*^{\frac{\delta}{n-\beta}\frac{\alpha-\beta}{\alpha-\gamma},\lambda}(H^\delta)}.$$

Hence, by Lemma 2.3 we have

$$\|I_\gamma |f|\|_{L_*^{\frac{\delta}{n-\beta}\frac{\alpha-\beta}{\alpha-\gamma},\lambda}(H^\delta)} \leqslant C_1 \|f\|_{L^{1,n-\alpha}(dx)}^{\frac{\gamma-\beta}{\alpha-\beta}} \|M_\beta f\|_{L_*^{\frac{\delta}{n-\beta},\lambda}(H^\delta)}^{\frac{\alpha-\gamma}{\alpha-\beta}}.$$

It is clear that the assumptions of Proposition 2.2 hold: $\delta \geqslant n-\beta$, $\frac{\lambda}{\delta} = \frac{n-\alpha}{n-\beta} < 1$ and $\frac{\lambda}{\delta}(n-\beta) = n-\alpha$. So by virtue of Proposition 2.2 we have

$$\|M_\beta f\|_{L_*^{\frac{\delta}{n-\beta},\lambda}(H^\delta)} \leqslant C_2 \|f\|_{L^{1,n-\alpha}(dx)}.$$

Therefore,

$$\|I_\gamma |f|\|_{L_*^{\frac{\delta}{n-\beta}\frac{\alpha-\beta}{\alpha-\gamma},\lambda}(H^\delta)} \leqslant C_2 C_1 \|f\|_{L^{1,n-\alpha}(dx)}^{\frac{\gamma-\beta}{\alpha-\beta}+\frac{\alpha-\gamma}{\alpha-\beta}} = C\|f\|_{L^{1,n-\alpha}(dx)} < \infty.$$

The claim follows. $\square$

3. Technical lemmas

The propositions of this section have been inspired by Chapter 4 in [10]. For a locally integrable function f we set

$$A(f) = \sup_{\substack{x\in\mathbb{R}^n \\ t>0 \\ r>0}} tH^\delta\left(\left\{y \in Q(x,r),\, |f - f_{Q(x,r)}|(y) > t\right\}\right)^{\frac{1}{p}} l(Q(x,r))^{-\frac{\lambda}{p}}.$$

LEMMA 3.1. *Assume that* $0 < \delta \leqslant n$, $0 \leqslant \lambda \leqslant \delta$, $1 \leqslant p$ *and* $0 < \sigma < \rho < +\infty$. *Then for any locally integrable function* f *and for all* $x \in \mathbb{R}^n$, *we have*

$$|f_{Q(x,\rho)} - f_{Q(x,\sigma)}| \leqslant 2\frac{\rho^{\frac{\lambda}{p}} + \sigma^{\frac{\lambda}{p}}}{\sigma^{\frac{\delta}{p}}} A(f).$$

Proof. We have

$$|f_{Q(x,\rho)} - f_{Q(x,\sigma)}|\chi_{Q(x,\sigma)} \leqslant |f_{Q(x,\rho)} - f|\chi_{Q(x,\rho)} + |f - f_{Q(x,\sigma)}|\chi_{Q(x,\sigma)}.$$

For $t > 0$, we have

$$\begin{aligned}\left\{y : |f_{Q(x,\rho)} - f_{Q(x,\sigma)}|\chi_{Q(x,\sigma)}(y) > 2t\right\} \quad \subset \quad & \left\{y : |f_{Q(x,\rho)} - f|\chi_{Q(x,\rho)}(y) > t\right\} \\ & \cup \left\{y : |f - f_{Q(x,\sigma)}|\chi_{Q(x,\sigma)}(y) > t\right\}.\end{aligned}$$

As $\frac{1}{p} \leqslant 1$, we obtain for any $t > 0$

$$\begin{aligned} & tH^{\delta}\left(\left\{y : |f_{Q(x,\rho)} - f_{Q(x,\sigma)}|\chi_{Q(x,\sigma)}(y) > 2t\right\}\right)^{\frac{1}{p}} \\ \leqslant\ & tH^{\delta}\left(\left\{y : |f_{Q(x,\rho)} - f|\chi_{Q(x,\rho)}(y) > t\right\}\right)^{\frac{1}{p}} \\ & + tH^{\delta}\left(\left\{y : |f - f_{Q(x,\sigma)}|\chi_{Q(x,\sigma)}(y) > t\right\}\right)^{\frac{1}{p}} \\ \leqslant\ & \rho^{\frac{\lambda}{p}} tH^{\delta}\left(\left\{y : |f_{Q(x,\rho)} - f|\chi_{Q(x,\rho)}(y) > t\right\}\right)^{\frac{1}{p}} \rho^{\frac{-\lambda}{p}} \\ & + \sigma^{\frac{\lambda}{p}} tH^{\delta}\left(\left\{y : |f - f_{Q(x,\sigma)}|\chi_{Q(x,\sigma)}(y) > t\right\}\right)^{\frac{1}{p}} \sigma^{\frac{-\lambda}{p}}. \end{aligned}$$

From above inequality, we obtain for any $u > 1$

$$\frac{|f_{Q(x,\rho)} - f_{Q(x,\sigma)}|}{2u} H^{\delta}\left(Q(x,r)\right)|^{\frac{1}{p}} \leqslant \left(\rho^{\frac{\lambda}{p}} + \sigma^{\frac{\lambda}{p}}\right) A(f).$$

The claim follows. $\square$

LEMMA 3.2. *Assume that* $0 < \delta \leqslant n$, $1 \leqslant p$ *and* $0 \leqslant \lambda \leqslant \delta$. *Then for any locally integrable function* f, *all non negative integer* k *and all* $x \in \mathbb{R}^n$, *we have*

$$|f_{Q(x,\rho)} - f_{Q(x,2^{-k}\rho)}| \leqslant 2^{\frac{p+\delta-\lambda}{p}} \left(1 + 2^{\frac{\lambda}{p}}\right) \frac{2^{k\frac{\delta-\lambda}{p}} - 1}{2^{\frac{\delta-\lambda}{p}} - 1} \rho^{-\frac{\delta-\lambda}{p}} A(f).$$

Proof. Let m be a non negative integer. By Lemma 3.1, we have

$$|f_{Q(x,2^{-m}\rho)} - f_{Q(x,2^{-m-1}\rho)}| \leqslant 2\left(1 + 2^{\frac{\lambda}{p}}\right) 2^{m\frac{\delta-\lambda}{p}} 2^{\frac{\delta-\lambda}{p}} \rho^{-\frac{\delta-\lambda}{p}} A(f).$$

Since for any non negative integer k we have

$$|f_{Q(x,\rho)} - f_{Q(x,2^{-k}\rho)}| \leqslant \sum_{m=0}^{k-1} |f_{Q(x,2^{-m}\rho)} - f_{Q(x,2^{-m-1}\rho)}|,$$

it follows that

$$\begin{aligned} |f_{Q(x,\rho)} - f_{Q(x,2^{-k}\rho)}| & \leqslant 2\left(1 + 2^{\frac{\lambda}{p}}\right) 2^{\frac{\delta-\lambda}{p}} \rho^{-\frac{\delta-\lambda}{p}} \left(\sum_{m=0}^{k-1} 2^{m\frac{\delta-\lambda}{p}}\right) A(f) \\ & \leqslant 2^{\frac{p+\delta-\lambda}{p}} \left(1 + 2^{\frac{\lambda}{p}}\right) \frac{2^{k\frac{\delta-\lambda}{p}} - 1}{2^{\frac{\delta-\lambda}{p}} - 1} \rho^{-\frac{\delta-\lambda}{p}} A(f). \qquad \square \end{aligned}$$

LEMMA 3.3. *Suppose that* $1 \leqslant p$, $0 < \rho < \sigma < +\infty$, $0 \leqslant \lambda \leqslant \delta$, $0 < \delta \leqslant n$ *and* k *is the unique integer satisfying* $2^{-k-1}\sigma \leqslant \rho < 2^{-k}\sigma$. *Then for any locally integrable function* f *and all* $x \in \mathbb{R}^n$, *we have*

$$|f_{Q(x,\rho)}| \leqslant |f_{Q(x,\sigma)}| + 2\left(1 + 2^{\frac{\lambda}{p}}\right)\left(1 + 2^{\frac{\delta-\lambda}{p}} \frac{1 - 2^{-k\frac{\delta-\lambda}{p}}}{2^{\frac{\delta-\lambda}{p}} - 1}\right) \rho^{-\frac{\delta-\lambda}{p}} A(f).$$

Proof. We have

$$|f_{Q(x,\rho)}| \leqslant |f_{Q(x,\sigma)}| + |f_{Q(x,2^{-k}\sigma)} - f_{Q(x,\sigma)}| + |f_{Q(x,2^{-k}\sigma)} - f_{Q(x,\rho)}|.$$

By Lemma 3.1,

$$\begin{aligned} |f_{Q(x,2^{-k}\sigma)} - f_{Q(x,\rho)}| &\leqslant 2\left(\left(2^{-k}\sigma\rho^{-1}\right)^{\frac{\lambda}{p}} + 1\right)\rho^{-\frac{\delta-\lambda}{p}} A(f) \\ &\leqslant 2\left(1+2^{\frac{\lambda}{p}}\right)\rho^{-\frac{\delta-\lambda}{p}} A(f). \end{aligned} \tag{3.1}$$

Lemma 3.2 yields

$$\begin{aligned} |f_{Q(x,2^{-k}\sigma)} - f_{Q(x,\sigma)}| &\leqslant 2^{\frac{p+\delta-\lambda}{p}}\left(1+2^{\frac{\lambda}{p}}\right)\frac{2^{k\frac{\delta-\lambda}{p}}-1}{2^{\frac{\delta-\lambda}{p}}-1}\sigma^{-\frac{\delta-\lambda}{p}} A(f) \\ &\leqslant 2^{\frac{p+\delta-\lambda}{p}}\left(1+2^{\frac{\lambda}{p}}\right)\frac{1-2^{-k\frac{\delta-\lambda}{p}}}{2^{\frac{\delta-\lambda}{p}}-1}\rho^{-\frac{\delta-\lambda}{p}} A(f). \end{aligned} \tag{3.2}$$

We deduce from (3.1) and (3.2) the claim. □

LEMMA 3.4. *Suppose that $1 \leqslant p$, $0 < \delta \leqslant n$ and $0 \leqslant \lambda \leqslant \delta$. Then there exists a constant $C > 0$ such that for any locally integrable function f satisfying*

$$\lim_{r\to+\infty} |f|_{Q(x,r)} = 0, \quad x \in \mathbb{R}^n$$

we have

$$\|f\|_{L_*^{p,\lambda}(H^\delta)} \leqslant CA(f).$$

Proof. Let (t,ρ,x) be any element of $(0,+\infty)\times(0,+\infty)\times\mathbb{R}^n$.
As

$$|f\chi_{Q(x,\rho)}| \leqslant |\left(f - f_{Q(x,\rho)}\right)\chi_{Q(x,\rho)}| + |f_{Q(x,\rho)}\chi_{Q(x,\rho)}|.$$

Hence for any $t > 0$, we have

$$\begin{aligned} tH^\delta\left(\{y : |f\chi_{Q(x,\rho)}|(y) > 2t\}\right)^{\frac{1}{p}} \leqslant{} & \rho^{\frac{\lambda}{p}} tH^\delta\left(\{y : |f - f_{Q(x,\rho)}|\chi_{Q(x,\rho)}(y) > t\}\right)^{\frac{1}{p}}\rho^{-\frac{\lambda}{p}} \\ & + tH^\delta\left(\{y : |f_{Q(x,\rho)}|\chi_{Q(x,\rho)}(y) > t\}\right)^{\frac{1}{p}}. \end{aligned}$$

So,

$$tH^\delta\left(\{y, |f\chi_{Q(x,\rho)}|(y) > 2t\}\right)^{\frac{1}{p}}\rho^{-\frac{\lambda}{p}} \leqslant A(f) + \rho^{\frac{\delta-\lambda}{p}}|f_{Q(x,\rho)}|.$$

Let $\sigma > \rho$ and k the unique integer satisfying $2^{-k-1}\sigma \leqslant \rho < 2^{-k}\sigma$. By Lemma 3.3,

$$\begin{aligned} & tH^\delta\left(\{y : |f\chi_{Q(x,\rho)}|(y) > 2t\}\right)^{\frac{1}{p}}\rho^{-\frac{\lambda}{p}} \\ & \leqslant A(f) + \rho^{\frac{\delta-\lambda}{p}}|f\chi_{Q(x,\sigma)}| + \rho^{\frac{\delta-\lambda}{p}}\cdot 2\left(1+2^{\frac{\lambda}{p}}\right)\left(1+2^{\frac{\delta-\lambda}{p}}\frac{1-2^{-k\frac{\delta-\lambda}{p}}}{2^{\frac{\delta-\lambda}{p}}-1}\right)\rho^{-\frac{\delta-\lambda}{p}}A(f). \end{aligned}$$

As $\lim\limits_{k\to+\infty} 2^{-k\frac{\delta-\lambda}{p}} = 0 = \lim\limits_{\sigma\to+\infty} |f\chi_{Q(x,\sigma)}|$, we get

$$tH^{\delta}\left(\left\{y,\, |f\chi_{Q(x,\rho)}|(y) > 2t\right\}\right)^{\frac{1}{p}}\rho^{-\frac{\lambda}{p}} \leqslant \left(1+2\left(1+2^{\frac{\lambda}{p}}\right)\left(1+\frac{2^{\frac{\delta-\lambda}{p}}}{2^{\frac{\delta-\lambda}{p}}-1}\right)\right)A(f).$$

The claim follows. $\square$

4. Proofs of Theorem 1.1 and Corollary 1.2

We recall that:

– a point $x\in\mathbb{R}^n$ is called Lebesgue point of an locally integrable function f if

$$\lim_{r\to0}|Q(x,r)|^{-1}\int_{Q(x,r)}|f(y)-f(x)|dy = 0.$$

– a function f belongs $\mathscr{C}^k$, $(k=0,1,\dots,\infty)$ when f is k-times continuously differentiable.

Now, we establish the following particular case of Theorem 1.1.

PROPOSITION 4.1. *Suppose that* $0\leqslant\beta<1<\alpha\leqslant n\leqslant\delta+\beta$, $\delta\leqslant n$ *and* $\lambda(n-\beta)=\delta(n-\alpha)$. *Then there exists a constant* $C>0$ *such that for any function* f *belonging to* C^1 *and satisfying*

$$\lim_{r\to+\infty}|f_{Q(x,r)}| = 0,$$

for all $x\in\mathbb{R}^n$, *we have*

$$\|f\|_{L_*^{\frac{\delta}{n-\beta}\frac{\alpha-\beta}{\alpha-1},\lambda}(H^{\delta})} \leqslant C\|\nabla f\|_{L^{1,n-\alpha}(dx)}.$$

Proof. Let f satisfying the assumptions of the proposition.

It is well known (see Lemma 7.16 in [9]) that there exists a real number C_1 depending only on n such that for all cubes Q and $x\in Q$ we have

$$|f(x)-f_Q| \leqslant C_1 I_1(|\nabla f|\chi_Q)(x).$$

Using lemma 3.4 and the above inequality we obtain

$$\begin{aligned}
\|f\|_{L_*^{\frac{\delta}{n-\beta}\frac{\alpha-\beta}{\alpha-1},\lambda}(H^{\delta})} &\leqslant C_2\sup_Q tH^{\delta}\left(\{x\in Q,\, |f-f_Q|(x)>t\}\right)^{\frac{\alpha-1}{\alpha-\beta}\frac{n-\beta}{\delta}} l(Q)^{-\frac{\lambda}{\delta}(n-\beta)\frac{\alpha-1}{\alpha-\beta}}\\
&\leqslant C_1C_2\sup_Q tH^{\delta}\left(\{x\in Q,\, I_1\left(|\nabla f|\right)(y)>t\}\right)^{\frac{\alpha-1}{\alpha-\beta}\frac{n-\beta}{\delta}} l(Q)^{-\frac{\lambda}{\delta}(n-\beta)\frac{\alpha-1}{\alpha-\beta}}\\
&= C_1C_2\|I_1\left(|\nabla f|\right)\|_{L_*^{\frac{\delta}{n-\beta}\frac{\alpha-\beta}{\alpha-1},\lambda}(H^{\delta})}.
\end{aligned}$$

It follows from Theorem 2.4 that

$$\|f\|_{L_*^{\frac{\delta}{n-\beta}\frac{\alpha-\beta}{\alpha-1},\lambda}(H^{\delta})} \leqslant C\|\nabla f\|_{L^{1,n-\alpha}(dx)}. \qquad\square$$

Let us recall the following proposition used in the proof of Theorem 1.1.

PROPOSITION 4.2. (see Theorem 7.8 in [11]) *Let f in $W^{1,1}_{loc}$. Then there exists a constant C such that*

$$\int_Q |f - f_Q| dy \leqslant Cl(Q) \int_Q |\nabla f| dy,$$

for any cube Q of $\mathbb{R}^n$.

Proof of Theorem 1.1. Let f satisfying the assumptions of the theorem.

We can assume that $\|\nabla f\|_{L^{1,n-\alpha}(dx)} < +\infty$ otherwise the claim of Theorem 1.1 is trivial.

a) Let φ be a non negative element of $\mathscr{C}^\infty$ with support $supp\varphi$ is included in the unit ball $B(0,1)$ and such that $\int_{\mathbb{R}^n} \varphi(x)dx = 1$.

For any real number $\varepsilon > 0$, we write $f_\varepsilon = f * \varphi_\varepsilon$ with

$$\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(\varepsilon^{-1}x).$$

Then $f_\varepsilon \in \mathrm{C}^\infty$ and for all $x \in \mathbb{R}^n$ and $\rho > 0$, we have

$$\begin{aligned}
|Q(x,\rho)|^{-1} \int_{Q(x,\rho)} |f_\varepsilon|(y)dy &\leqslant |Q(x,\rho)|^{-1} \int_{Q(x,\rho)} \int_{supp\varphi_\varepsilon} |f_\varepsilon(y-z)|\varphi_\varepsilon(z)dzdy \\
&\leqslant \int_{supp\varphi_\varepsilon} \left(|Q(x,\rho)|^{-1} \int_{Q(x,\rho)-z} |f(y)|dy \right) \varphi_\varepsilon(z)dz \\
&\leqslant \left(\frac{\rho_\varepsilon}{\rho}\right)^n \int_{\mathbb{R}^n} \left(|Q(x,\rho_\varepsilon)|^{-1} \int_{Q(x,\rho_\varepsilon)} |f(y)|dy \right) \varphi_\varepsilon(z)dz \\
&\leqslant \left(\frac{\rho_\varepsilon}{\rho}\right)^n |Q(x,\rho_\varepsilon)|^{-1} \int_{Q(x,\rho_\varepsilon)} |f(y)|dy,
\end{aligned}$$

where $\rho_\varepsilon = \rho + 2\varepsilon$.

Therefore for all $x \in \mathbb{R}^n$, $\lim_{\rho \to +\infty} |f_\varepsilon|_{Q(x,\rho)} = 0$.

Thus from Proposition 4.1, we get

$$\|f_\varepsilon\|_{L_*^{\frac{\delta}{n-\beta}\frac{\alpha-\beta}{\alpha-1},\lambda}(H^\delta)} \leqslant C_1 \|\nabla f_\varepsilon\|_{L^{1,n-\alpha}(dx)}. \tag{4.1}$$

In addition, for all cubes Q, we have

$$
\begin{aligned}
\int_Q |\nabla f_\varepsilon|(x)dx &= \int_Q |\nabla (f * \varphi_\varepsilon)|(x)dx \\
&= \int_Q |\nabla f * \varphi_\varepsilon|(x)dx \\
&\leqslant \int_Q \int_{\mathbb{R}^n} |\nabla f(x-y)|\varphi_\varepsilon(y)dy(x)dx \\
&= \int_{\mathbb{R}^n} \left(\int_Q |\nabla f(x-y)|dx\right) \varphi_\varepsilon(y)dy \\
&= \int_{\mathbb{R}^n} \left(\int_{Q-y} |\nabla f(z)|dz\right) \varphi_\varepsilon(y)dy \\
&\leqslant \|\nabla f\|_{L^{1,n-\alpha}(dx)} l(Q)^{n-\alpha} \int_{\mathbb{R}^n} \varphi_\varepsilon(y)dy
\end{aligned}
$$

$$
\frac{1}{l(Q)^{n-\alpha}} \int_Q |\nabla f_\varepsilon|(x)dx \leqslant \|\nabla f\|_{L^{1,n-\alpha}(dx)}.
$$

Hence

$$
\|\nabla f_\varepsilon\|_{L^{1,n-\alpha}(dx)} \leqslant \|\nabla f\|_{L^{1,n-\alpha}(dx)}.
$$

So, by (4.1) we obtain

$$
\|f_\varepsilon\|_{L_*^{\frac{\delta}{n-\beta}\frac{\alpha-\beta}{\alpha-1},\lambda}(H^\delta)} \leqslant C_1 \|\nabla f\|_{L^{1,n-\alpha}(dx)}. \tag{4.2}
$$

b) Consider σ, ρ with $0<\sigma<\rho<+\infty$ and k the unique integer satisfying $2^{-k-1}\rho \leqslant \sigma < 2^k\rho$.

(i) For all $x\in\mathbb{R}^n$, we have

$$
|f_{Q(x,\rho)} - f_{Q(x,\sigma)}| \leqslant \sum_{i=0}^{k} |f_{Q(x,2^{-i}\rho)} - f_{Q(x,2^{-i-1}\rho)}| + |f_{Q(x,2^{-k}\rho)} - f_{Q(x,\sigma)}|.
$$

For all i such that $0\leqslant i\leqslant k-1$,

$$
\begin{aligned}
|f_{Q(x,2^{-i}\rho)} - f_{Q(x,2^{-i-1}\rho)}| &= \left| |Q(x,2^{-i-1}\rho)|^{-1} \int_{Q(x,2^{-i-1}\rho)} \left(f_{Q(x,2^{-i}\rho)} - f(y)\right) dy \right| \\
&\leqslant |Q(x,2^{-i-1}\rho)|^{-1} \int_{Q(x,2^{-i-1}\rho)} |f_{Q(x,2^{-i}\rho)} - f(y)|dy \\
&\leqslant 2^{-n}|Q(x,2^{-i}\rho)|^{-1} \int_{Q(x,2^{-i}\rho)} |f_{Q(x,2^{-i}\rho)} - f(y)|dy
\end{aligned}
$$

and

$$
\begin{aligned}
|f_{Q(x,\sigma)} - f_{Q(x,2^{-k}\rho)}| &= \left| |Q(x,\sigma)|^{-1} \int_{Q(x,\sigma)} \left(f(y) - f_{Q(x,2^{-k}\rho)}\right) dy \right| \\
&\leqslant 2^n |Q(x,2^{-k}\rho)|^{-1} \int_{Q(x,2^{-k}\rho)} |f(y) - f_{Q(x,2^{-i}\rho)}|dy.
\end{aligned}
$$

Therefore,

$$|f_{Q(x,\rho)} - f_{Q(x,\sigma)}| \leqslant 2^n \sum_{i=0}^{k} |Q(x,2^{-i}\rho)|^{-1} \int_{Q(x,2^{-i}\rho)} |f_{Q(x,2^{-i}\rho)} - f(y)|dy.$$

By Proposition 4.2, we have

$$\begin{aligned} |f_{Q(x,\rho)} - f_{Q(x,\sigma)}| &\leqslant 2^n \sum_{i=0}^{k} |Q(x,2^{-i}\rho)|^{\frac{1}{n}-1} \int_{Q(x,2^{-i}\rho)} |\nabla f|(y)dy \\ &\leqslant 2^n C_1 \sum_{i=0}^{k} |Q(x,2^{-i}\rho)|^{\frac{1}{n}-\frac{\beta}{n}} M_\beta(|\nabla f|)(x). \end{aligned}$$

So

$$|f_{Q(x,\rho)} - f_{Q(x,\sigma)}| \leqslant 2^n \frac{C_1}{1-2^{-1+\beta}} M_\beta(|\nabla f|)(x)\rho^{1-\beta}. \tag{4.3}$$

(ii) Now consider $N = \left\{x \in \mathbb{R}^n, M_\beta(|\nabla f|)(x) = +\infty\right\}$, Q a cube of $\mathbb{R}^n$, $N_Q = \left\{x \in Q, M_\beta(|\nabla f|)(x) = +\infty\right\}$, η a positive real number and $N_Q^\eta = \{x \in Q, M_\beta(|\nabla f|)(x) > \eta\}$.

By Proposition 2.2, we have

$$\begin{aligned} \eta H^\delta(N_Q^\eta)^{\frac{n-\beta}{\delta}} l(Q)^{-\frac{\lambda}{\delta}(n-\beta)} &\leqslant C_2 \|\nabla f\|_{L^{1,n-\alpha}(dx)} \\ H^\delta(N_Q^\eta) &\leqslant C_2^{\frac{\delta}{n-\beta}} l(Q)^\lambda \frac{1}{\eta^{\frac{\delta}{n-\beta}}} \|\nabla f\|_{L^{1,n-\alpha}(dx)}^{\frac{\delta}{n-\beta}}. \end{aligned}$$

As $N_Q \subset N_Q^\eta$, we get

$$H^\delta(N_Q) \leqslant C_2^{\frac{\delta}{n-\beta}} l(Q)^\lambda \frac{1}{\eta^{\frac{\delta}{n-\beta}}} \|\nabla f\|_{L^{1,n-\alpha}(dx)}^{\frac{\delta}{n-\beta}}.$$

Letting $\eta \to \infty$, we obtain $H^\delta(N_Q) = 0$.

Since $\mathbb{R}^n$ can be written as a countable union of cubes, we get finally

$$H^\delta(N) = 0. \tag{4.4}$$

(iii) Fix $x \in \mathbb{R}^n \setminus N$.

Since by hypothesis $\limsup_{\sigma\to 0} f_{Q(x,\sigma)} = f(x)$, inequality (4.3) leads to

$$|f_{Q(x,\rho)} - f(x)| \leqslant 2^n \frac{C_1}{1-2^{-1+\beta}} M_\beta(|\nabla f|)(x)\rho^{1-\beta}.$$

In addition, by Proposition 4.2, we have

$$\begin{aligned} |Q(x,\rho)|^{-1} \int_{Q(x,\rho)} |f(y) - f_{Q(x,\rho)}|dy &\leqslant C_3 \rho^{-n+1} \int_{Q(x,\rho)} |\nabla f(y)|dy \\ &\leqslant C_3 M_\beta(|\nabla f|)(x)\rho^{1-\beta}. \end{aligned}$$

So

$$|Q(x,\rho)|^{-1}\int_{Q(x,\rho)}|f(y)-f(x)|dy \leqslant |Q(x,\rho)|^{-1}\int_{Q(x,\rho)}|f(y)-f_{Q(x,\rho)}|dy + |f_{Q(x,\rho)}-f(x)| \leqslant C_4\left(1+\frac{2^n}{1-2^{-1+\beta}}\right)M_\beta(|\nabla f|)(x)\rho^{1-\beta}.$$

Thus for any $x\in\mathbb{R}^n\setminus N$,

$$\lim_{\rho\to 0}|Q(x,\rho)|^{-1}\int_{Q(x,\rho)}|f(y)-f(x)|dy=0,$$

i.e any $x\in\mathbb{R}^n\setminus N$ is a Lebesgue point for f.

c) Let Q be a cube of $\mathbb{R}^n$ and $t>0$. We set

$$\begin{aligned} G &= \{y\in Q,\ |f(y)|>t\}\\ G_k &= \left\{y\in Q(x,r),\ |f_{\frac{1}{k}}(y)|>t\right\},\quad k\in\mathbb{N}^*\\ \Gamma_j &= \bigcap_{k\geqslant j} G_k,\quad j\in\mathbb{N}^*. \end{aligned}$$

(i) Let $y\in G$ and $y\notin N$.

We have $|f(y)|>t$ and $\lim_{\eta\to+\infty} f_{\frac{1}{\eta}}(y)=f(y)$ (since y is a Lebesgue point). So, there exists $j>0$ such that $f_{\frac{1}{k}}(y)>t$ for all $k\geqslant j$; that is $y\in\bigcup_j\Gamma_j$.

Therefore

$$G\setminus N\subset\bigcup_j\Gamma_j.$$

(ii) By equality (4.4) and the above inclusion, we have

$$H^\delta(G)=H^\delta(G\setminus N)\leqslant H^\delta(\bigcup_j\Gamma_j).$$

Using Properties P1 and P2, we have

$$\begin{aligned} H^\delta(G) &\leqslant AH^\delta_\triangle(\bigcup_j\Gamma_j)=A\lim_{j\to+\infty}H^\delta_\triangle(\Gamma_j)\\ &\leqslant A\lim_{k\to+\infty}H^\delta_\triangle(G_k)\leqslant AB\liminf_{k\to+\infty}H^\delta(G_k). \end{aligned}$$

Therefore,

$$\begin{aligned} tH^\delta(G)^{\frac{n-\beta}{\delta}\frac{\alpha-1}{\alpha-\beta}}l(Q)^{-\lambda\frac{n-\beta}{\delta}\frac{\alpha-1}{\alpha-\beta}} &\leqslant (AB)^{\frac{n-\beta}{\delta}\frac{\alpha-1}{\alpha-\beta}}\liminf_{k\to+\infty}H^\delta(G_k)^{\frac{n-\beta}{\delta}\frac{\alpha-1}{\alpha-\beta}}l(Q)^{-\lambda\frac{n-\beta}{\delta}\frac{\alpha-1}{\alpha-\beta}}\\ &\leqslant (AB)^{\frac{n-\beta}{\delta}\frac{\alpha-1}{\alpha-\beta}}\liminf_{k\to+\infty}\|f_\varepsilon\|_{L_*^{\frac{\delta}{n-\beta}\frac{\alpha-\beta}{\alpha-1},\lambda}(H^\delta)}. \end{aligned}$$

So by inequality (4.2), we have

$$H^\delta(G)^{\frac{n-\beta}{\delta}\frac{\alpha-1}{\alpha-\beta}}l(Q)^{-\lambda\frac{n-\beta}{\delta}\frac{\alpha-1}{\alpha-\beta}}\leqslant (AB)^{\frac{n-\beta}{\delta}\frac{\alpha-1}{\alpha-\beta}}C_1\|\nabla f\|_{L_*^{\frac{\delta}{n-\beta}\frac{\alpha-\beta}{\alpha-1},\lambda}(H^\delta)}.$$

Thus
$$\|f\|_{L_*^{\frac{\delta}{n-\beta}\frac{\alpha-\beta}{\alpha-1},\lambda}(H^\delta)} \leqslant C\|\nabla f\|_{L^{1,n-\alpha}(dx)}. \qquad \square$$

Proof of Corollary 1.2. Since $f \in W^{1,1}$, then we choose Borel representatives $\tilde{f}$ defined at every point x by
$$\tilde{f}(x) := \limsup_{r\to 0} f_{Q(x,r)}.$$

By the Lebesgue differentiation theorem, [12], $f = \tilde{f}$ a.e.

Applying Theorem 1.1 for $\beta = 0$, $\delta = n$ and $\tilde{f}$, we obtain the result because $f = \tilde{f}$ a.e. $\square$

REFERENCES

[1] D. R. ADAMS, *A note on Choquet integrals with respect to Hausdorff capacity*, in Functions spaces and Applications, Lund 1986, Lecture Notes in Math.1302, Springer–Verlag, 1988, pp. 115–184.

[2] D. R. ADAMS, *Choquet integrals in potential theory*, Publ. Mat., 42 no. 1(1998) 3–66.

[3] D. R. ADAMS, *A note on Riez Potentials*, Duke Mat. J. Vol 42. no. 4 (1978) 765–778.

[4] V. I. BURENKOV, H. V. GULIYEV, *Necessary and sufficient conditions for boundedness of the maximal operator in the local Morrey-type spaces*, Studia Mathematica, 163 (2) (2004), 157–176.

[5] V. I. BURENKOV, H. V. GULIYEV, V. S. GULIYEV, *Necessary and sufficient conditions for boundedness of the Riesz potential in the local Morrey-type spaces*, Doklady Ross. Akad. Nauk, (412) (2007), 1–5.

[6] F. CHIARENZA, M. FRASCA, *Morrey spaces and Hardy-Littlewood maximal operator*, Rend. Math. 7 (1987), 273–279.

[7] M. ESSOH, I. FOFANA, K. KONIN, *Inégalités de type faible pour l'opérateur maximal fractionnaire dans les espaces de Morrey par rapport à la capacité de Hausdorff*, Italian Journal of pure and applied mathematic, no. 28 (2011), 81–92.

[8] L. C. EVANS AND R. F. GARIEPY, *Measure theory and fine properties of functions*, CCR Press, 1992.

[9] D. GILBARG AND N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag 1983.

[10] A. KUFNER, O. JOHN, F. FUCIK, *Functions spaces*, Noordhoff Int. Pub. 1977 Prague.

[11] J. MALÝ, *Lectures on change of variables in integral*, Graduate School in Helsinki, 2001.

[12] E. M. STEIN, *Singular Integral and Differentiability Properties of Functions*, Princeton, New Jersey, 1975.

[13] G. V. WELLAND, *Weighted Norm Inequalities for Fractionnal Integrals*, Proc. Am. Math. Soc. Vol 51, no. 1 Aug. 1975.

[14] W. P. ZIEMER, *Weakly Differentiable Functions*, Springer-Verlag 1989.

(Received December 13, 2010)

Modeste Essoh
UFR de Mathématiques et Informatique
Université de Cocody, Abidjan
22 BP 582 ABJ 22 Côte d'Ivoire
e-mail: birilowe@yahoo.fr

Ibrahim Fofana
UFR de Mathématiques et Informatique
Université de Cocody, Abidjan
22 BP 582 ABJ 22 Côte d'Ivoire
e-mail: fofana_ib_math_ab@yahoo.fr