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SOME NEW PACHPATTE TYPE INEQUALITIES ON TIME SCALES AND THEIR APPLICATIONS

LIWEI DU AND RUN XU

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Abstract. The aim of this paper is to establish some new Pachpatte type inequalities on time scales, which provide explicit bounds on unknown functions. Our results extend some known Pachpatte inequalities on times scales, unify and extend some continuous inequalities and their corresponding discrete analogues. The inequalities given here can be used as tools in the qualitative theory of certain differential equations and dynamic equations. Some examples are given in the end of this paper.

1. Introduction

The theory of time scales was introduced by Hilger [2] in 1988 in order to contain both difference and differential calculus in a consistent way. Since then many authors have expounded on various aspects of the theory of dynamic equations and dynamic inequalities on time scales. For example, we refer the reader to the papers [2–7], the monographs [8,9] and the references cited therein. At the same time, the paper [1] includes a study of the theory of Pachpatte type inequalities on time scales, which extend some unknown dynamic inequalities on time scales, unify and extend some continuous inequalities and their corresponding discrete analogues.

On the basis of [1], we continue to investigate some new Pachpatte type inequalities on time scales, which extend some inequalities established by Wei Nian Li [1]. We validate Pachpatte type inequalities by virtue of Lemma 1 mainly. The obtained inequalities can be used as important tools in the study of dynamic equations on time scales.

This paper is organized as follows: In the second section, we give some basic definitions and some preliminary lemmas and theorems with respect to the calculus on time scales, which can also be found in [1]. In Section 3 we deal with our Pachpatte type inequalities on time scales. In the fourth section, we give some applications of our main results.

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2. Some preliminaries on time scales

In what follows, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = [0,+\infty)$, $\mathbb{Z}$ denotes the set of integers, $\mathbb{N}_0$ denotes the set of nonnegative integers, $\mathbb{C}$ denotes the set of complex numbers, and $C(M,S)$ denotes the class of all continuous functions defined on set M with range in the set S. We use the usual conventions that empty sums and products are taken to be 0 and 1, respectively. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. The forward jump operator σ on $\mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \in \mathbb{T}, \ for\ all\ \ t \in \mathbb{T}.$$

In this definition we put $\inf\emptyset = \sup\mathbb{T}$, where $\emptyset$ is the empty set. If $\sigma(t) > t$, then we say that t is right-scattered. If $\sigma(t) = t$ and t$<\sup\mathbb{T}$, then we say that t is right-dense. The graininess $\mu : \mathbb{T} \to [0,\infty)$ is defined by $\mu(t) := \sigma(t) - t$. The set $\mathbb{T}^k$ is derived from $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$; otherwise, $\mathbb{T}^k = \mathbb{T}$.

We say that $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous provided f is continuous at each right-dense point of $\mathbb{T}$ and has a finite left-sided limit at each left-dense point of $\mathbb{T}$. As usual, C_{rd} denotes the set of rd-continuous functions. We say that $p : \mathbb{T} \to \mathbb{R}$ is regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$. We denote by $\mathscr{R}$ the set of all regressive and rd-continuous functions. We define the set of all positively regressive functions by $\mathscr{R}^+ = \{p \in \mathscr{R} : 1 + \mu(t)p(t) > 0, t \in \mathbb{T}\}$. Obviously, if $p \in C_{rd}$ and $p(t) \geqslant 0$ for $t \in \mathbb{T}$, then $p \in \mathscr{R}^+$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, we define $f^{\triangle}(t)$ as follows (provided it exists):

$$f^{\triangle}(t) := \lim_{s \to t} \frac{f^{\sigma}(t) - f(s)}{\sigma(t) - s},$$

we call $f^{\triangle}(t)$ the *delta derivative* of f at t.

We first introduce the following lemmas, which are very useful in our main results.

LEMMA 2.1. [7] *Assume that* $a \geqslant 0$, $p \geqslant q > 0$, *then*

$$a^{\frac{q}{p}} \leqslant \frac{q}{p} K^{\frac{q-p}{p}} a + \frac{p-q}{p} K^{\frac{q}{p}}, \qquad for\ any\ K > 0.$$

THEOREM 2.1. [4] *If* $p \in \mathscr{R}$ *and fix* $t_0 \in \mathbb{T}$, *then the exponential function* $e_p(\cdot, t_0)$ *is for the unique solution of the initial value problem*

$$x^{\triangle} = p(t)x, \quad x(t_0) = 1 \quad on \ \ \mathbb{T}.$$

THEOREM 2.2. [4] *Let* $t_0 \in \mathbb{T}^k$ *and* $w : \mathbb{T} \times \mathbb{T}^k \to \mathbb{R}$ *be continuous at* (t,t), *where* $t \geqslant t_0$. *Assume that* $w^{\triangle}(t,\cdot)$ *is rd-continuous on* $[t_0, \sigma(t)]$. *If for any* $\varepsilon > 0$, *there exists a neighborhood* U *of* t, *independent of* $\tau \in [t_0, \sigma(t)]$, *such that*

$$|w(\sigma(t),\tau) - w(s,\tau) - w^{\triangle}(t,\tau)(\sigma(t) - s)| \leqslant \varepsilon |\sigma(t) - s|, \ \ s \in U,$$

where $w^{\triangle}$ denotes the derivative of w with respect to the first variable, then

$$g(t) := \int_{t_0}^{t} w(t,\tau)\triangle\tau$$

implies

$$g^{\triangle}(t) := \int_{t_0}^{t} w^{\triangle}(t,\tau)\triangle\tau + w(\sigma(t),t).$$

The following theorem is a foundational result in dynamic inequalities.

THEOREM 2.3. (Comparison Theorem [4]) *Suppose $u,b \in C_{rd}$, $a \in \mathscr{R}^{+}$, then*

$$u^{\triangle}(t) \leqslant a(t)u(t) + b(t), \;\; t \geqslant t_0, \; t \in \mathbb{T}^k$$

implies

$$u(t) \leqslant u(t_0)e_a(t,t_0) + \int_{t_0}^{t} b(\tau)e_a(t,\sigma(\tau))\triangle\tau, \;\; t \geqslant t_0, \; t \in \mathbb{T}^k.$$

2. Main results

In this section, we deal with Pachpatte type inequalities on time scales. Throughout this paper, we always assume that $m \geqslant 1$, m,n,l are real constants, $m \geqslant n \geqslant l > 0$, and $t \geqslant t_0$, $t \in \mathbb{T}^k$.

THEOREM 3.1. *Assume that $u,f,p \in C_{rd}, u(t), f(t)$ and $p(t)$ are nonnegative, and u_0 is a nonnegative constant. If $\omega(t,s)$ is as defined in Theorem 2.2 such that $\omega(t,s) \geqslant 0$ and $\omega^{\Delta}(t,s) \geqslant 0$ for $t,s \in \mathbb{T}$ with $s \leqslant t$, then*

$$\begin{aligned} u^m(t) \leqslant\; & u_0 + \int_{t_0}^{t} [f(\tau)u^m(\tau) + p(\tau)]\Delta\tau \\ & + \int_{t_0}^{t} f(\tau)(\int_{t_0}^{\tau} \omega(\tau,s)u^n(s)\Delta s)\Delta\tau, \;\; t \in \mathbb{T}^k \end{aligned} \tag{3.1}$$

implies

$$\begin{aligned} u(t) \leqslant \Big\{ & u_0 + \int_{t_0}^{t} \Big[B(\tau) + f(\tau)\Big(u_0 e_{f+A}(\tau,t_0) \\ & + \int_{t_0}^{\tau} e_{f+A}(\tau,\sigma(s))B(s)\Delta s\Big)\Big]\Delta\tau\Big\}^{\frac{1}{m}}, \; K>0, \; t \in \mathbb{T}^k, \end{aligned} \tag{3.2}$$

where

$$A(t) = \frac{n}{m}K^{\frac{n-m}{m}}\left(\omega(\sigma(t),t) + \int_{t_0}^{t} \omega^{\triangle}(t,s)\Delta s\right), \tag{3.3}$$

$$B(t) = p(t) + \frac{m-n}{m}K^{\frac{n}{m}} f(t)\int_{t_0}^{t} \omega(t,s)\Delta s, \;\; t \in \mathbb{T}^k. \tag{3.4}$$

Proof. Define a function $z(t)$ by the right hand of (3.1), then $z(t_0) = u_0$, and

$$z^{\triangle}(t) = f(t)u^m(t) + p(t) + f(t)\int_{t_0}^{t} \omega(t,s)u^n(s)\Delta s$$
$$\leqslant f(t)z(t) + p(t) + f(t)\int_{t_0}^{t} \omega(t,s)z(s)^{\frac{n}{m}}\Delta s.$$

Using Lemma 2.1, from the above inequality, for any $K > 0$, we obtain

$$\begin{aligned} z^{\triangle}(t) &\leqslant p(t) + f(t)z(t) + f(t)\int_{t_0}^{t} \omega(t,s)\left[\frac{n}{m}K^{\frac{n-m}{m}}z(s) + \frac{m-n}{m}K^{\frac{n}{m}}\right]\Delta s \\ &= p(t) + \frac{m-n}{m}K^{\frac{n}{m}}f(t)\int_{t_0}^{t}\omega(t,s)\Delta s + f(t)\left[z(t) + \frac{n}{m}K^{\frac{n-m}{m}}\int_{t_0}^{t}\omega(t,s)z(s)\Delta s\right] \\ &= B(t) + f(t)\left[z(t) + \frac{n}{m}K^{\frac{n-m}{m}}\int_{t_0}^{t}\omega(t,s)z(s)\Delta s\right],\ t \in \mathbb{T}^k, \end{aligned} \tag{3.5}$$

where $B(t)$ is as defined in (3.4).

Let

$$v(t) = z(t) + \frac{n}{m}K^{\frac{n-m}{m}}\int_{t_0}^{t}\omega(t,s)z(s)\Delta s. \tag{3.6}$$

Obviously, $v(t_0) = z(t_0) = u_0$, $z(t) \leqslant v(t)$ and $z^{\Delta}(t) \leqslant B(t) + f(t)v(t)$. Using Theorem 2.2, we have

$$\begin{aligned} v^{\triangle}(t) &= z^{\triangle}(t) + \frac{n}{m}K^{\frac{n-m}{m}}\left(\int_{t_0}^{t}\omega^{\triangle}(t,s)z(s)\Delta s + \omega(\sigma(t),t)z(t)\right) \\ &\leqslant B(t) + \left[f(t) + \frac{n}{m}K^{\frac{n-m}{m}}\left(\int_{t_0}^{t}\omega^{\triangle}(t,s)\Delta s + \omega(\sigma(t),t)\right)\right]v(t) \\ &= B(t) + (f(t) + A(t))\,v(t),\ t \in \mathbb{T}^k, \end{aligned} \tag{3.7}$$

where $A(t)$ is as defined in (3.3). It is easy to see that $(f + A) \in \mathscr{R}^+$. Therefore, using Theorem 2.3, from the above inequality, we have

$$v(t) \leqslant u_0 e_{f+A}(t,t_0) + \int_{t_0}^{t} e_{f+A}(t,\sigma(s))B(s)\Delta s,\ t \in \mathbb{T}^k. \tag{3.8}$$

Combining (3.5), (3.6), (3.8), we obtain

$$z^{\triangle}(t) \leqslant B(t) + f(t)\left[u_0 e_{f+A}(t,t_0) + \int_{t_0}^{t} e_{f+A}(t,\sigma(s))B(s)\Delta s\right],\ t \in \mathbb{T}^k. \tag{3.9}$$

Setting $t = \tau$ in (3.9), integrating from t_0 to t, and noting $z(t_0) = u_0$ and $u(t) \leqslant z(t)$, we easily obtain the desired inequality (3.2). The proof is complete. $\square$

REMARK 3.1. If $m = n = 1$ in Theorem 3.1, then the inequality given in (3.2) reduces to the inequality in [1, Theorem 3.1].

REMARK 3.2. If $m=n=1$, $p(t)=0$, and $\omega(t,s)=\omega(s)$ in Theorem 3.1, then the inequality given in (3.2) reduces to the inequality in [5, Theorem 1].

REMARK 3.3. The result of Theorem 3.1 holds for an arbitrary time scale. Let $\omega(t,s)=\omega(s)$ and $m=n=1$ in Theorem 3.1. If $\mathbb{T}=\mathbb{R}$, then the inequality established in Theorem 3.1 reduces to the inequality established by Pachpatte in [13, Theorem 1.7.2(i)]. If $\mathbb{T}=\mathbb{Z}$, then from Theorem 3.1, we easily obtain Theorem 1.8.7 in [14].

THEOREM 3.2. *Assume that* $u,f,p\in C_{rd}$, $u(t),f(t)$ *and* $p(t)$ *are nonnegative, and* u_0 *is a nonnegative constant. If* $\omega(t,s)$ *is as defined in Theorem 2.2 such that* $\omega(t,s)\geqslant 0$ *and* $\omega^{\Delta}(t,s)\geqslant 0$ *for* $t,s\in\mathbb{T}$ *with* $s\leqslant t$, *then*

$$u^m(t)\leqslant u_0+\int_{t_0}^{t}f(\tau)u^m(\tau)\Delta\tau+\int_{t_0}^{t}f(\tau)\left[\int_{t_0}^{\tau}(\omega(\tau,s)u^n(s)+p(s))\Delta s\right]\Delta\tau,\quad K>0,\ t\in\mathbb{T}^k \tag{3.10}$$

implies

$$u(t)\leqslant\left\{u_0+\int_{t_0}^{t}f(\tau)\left[u_0e_{f+A}(\tau,t_0)+\int_{t_0}^{\tau}e_{f+A}(\tau,\sigma(s))B_1(s)\Delta s\right]\Delta\tau\right\}^{\frac{1}{m}},\quad K>0,\ t\in\mathbb{T}^k, \tag{3.11}$$

where A(t) is as defined in (3.3),

$$B_1(t)=p(t)+\frac{m-n}{m}K^{\frac{n}{m}}\left[\omega(\sigma(t),t)+\int_{t_0}^{t}\omega^{\Delta}(t,s)\Delta s\right],\ t\in\mathbb{T}^k. \tag{3.12}$$

Proof. Define a function $z(t)$ by the right hand side of (3.10). Then $z(t_0)=u_0$, $u^m(t)\leqslant z(t)$, and

$$\begin{aligned} z^{\triangle}(t)&=f(t)u^m(t)+f(t)\int_{t_0}^{t}\left[\omega(t,s)(u^m(s))^{\frac{n}{m}}+p(s)\right]\Delta s\\ &\leqslant f(t)z(t)+f(t)\int_{t_0}^{t}\left[\omega(t,s)(z(s))^{\frac{n}{m}}+p(s)\right]\Delta s. \end{aligned} \tag{3.13}$$

Using Lemma 1, from the above inequality, we have

$$z^{\triangle}(t)\leqslant f(t)\left\{z(t)+\int_{t_0}^{t}\left[\left(\frac{n}{m}K^{\frac{n-m}{m}}z(s)+\frac{m-n}{m}K^{\frac{n}{m}}\right)\omega(t,s)+p(s)\right]\Delta s\right\}. \tag{3.14}$$

Let

$$m(t)=z(t)+\int_{t_0}^{t}\left[\left(\frac{n}{m}K^{\frac{n-m}{m}}z(s)+\frac{m-n}{m}K^{\frac{n}{m}}\right)\omega(t,s)+p(s)\right]\Delta s, \tag{3.15}$$

it is easy to see that $m(t_0)=z(t_0)=u_0$, $z(t)\leqslant m(t)$, $z^{\triangle}(t)\leqslant f(t)m(t)$, and

$$\begin{aligned}
m^{\triangle}(t) &= z^{\triangle}(t)+\left(\frac{n}{m}K^{\frac{n-m}{m}}z(t)+\frac{m-n}{m}K^{\frac{n}{m}}\right)\omega(\sigma(t),t)\\
&\quad+\int_{t_0}^{t}\left(\frac{n}{m}K^{\frac{n-m}{m}}z(s)+\frac{m-n}{m}K^{\frac{n}{m}}\right)\omega^{\triangle}(t,s)\Delta s+p(t)\\
&\leqslant\left\{f(t)+\frac{n}{m}K^{\frac{n-m}{m}}\left[\omega(\sigma(t),t)+\int_{t_0}^{t}\omega^{\triangle}(t,s)\Delta s\right]\right\}m(t)\\
&\quad+p(t)+\frac{m-n}{m}K^{\frac{n}{m}}\left[\omega(\sigma(t),t)+\int_{t_0}^{t}\omega^{\triangle}(t,s)\Delta s\right]\\
&=(f(t)+A(t))m(t)+B_1(t),\ \ t\in\mathbb{T}^k, \qquad (3.16)
\end{aligned}$$

where $A(t)$ is as defined in (3.3), $B_1(t)$ is as defined in (3.13). Using Theorem 2.3, from (3.16), we obtain

$$m(t)\leqslant u_0e_{f+A}(t,t_0)+\int_{t_0}^{t}e_{f+A}(t,\sigma(s))B_1(s)\Delta s,\ \ t\in\mathbb{T}^k. \qquad (3.17)$$

Therefore,

$$z^{\triangle}(t)\leqslant f(t)\left[u_0e_{f+A}(t,t_0)+\int_{t_0}^{t}e_{f+A}(t,\sigma(s))B_1(s)\Delta s\right],\ t\in\mathbb{T}^k. \qquad (3.18)$$

Setting $t=\tau$ in (3.18), integrating it from t_0 to t, and noting $z(t_0)=u_0$ and $u(t)\leqslant z(t)$, we easily obtain the desired inequality (3.11). The proof of Theorem 3.2 is complete. □

REMARK 3.5. If $m=n=1$ in Theorem 3.2, then the inequality given in (3.11) reduces to the inequality in [1, Theorem 3.2].

REMARK 3.6. By taking $m=n=1$, $\omega(t,s)=\omega(s)$ from Theorem 3.2, we easily obtain Theorem 4.8(ii)[11].

REMARK 3.7. Letting $m=n=1$, $\omega(t,s)=\omega(s)$ in Theorem 3.2, we can obtain the inequality established in [13, Theorem 1.7.2(ii)] if $\mathbb{T}=\mathbb{R}$, and the inequality established in [14, Theorem 1.4.6(ii)] if $\mathbb{T}=\mathbb{Z}$.

THEOREM 3.3. *Assume that $u,f,g\in C_{rd}$, $u(t)$, $f(t)$ and $g(t)$ are nonnegative, and u_0 is a nonnegative constant. If $\omega(t,s)$ is as defined in Theorem 2.2 such that $\omega(t,s)\geqslant 0$ and $\omega^{\Delta}(t,s)\geqslant 0$ for $t,s\in\mathbb{T}$ with $s\leqslant t$, then*

$$\begin{aligned}
u^m(t)\leqslant u_0&+\int_{t_0}^{t}f(\tau)u^m(\tau)\Delta\tau+\int_{t_0}^{t}g(\tau)\\
&\cdot\left[u^n(\tau)+\int_{t_0}^{\tau}\omega(\tau,s)u^l(s)\Delta s\right]\Delta\tau,\ t\in\mathbb{T}^k \qquad (3.19)
\end{aligned}$$

implies

$$u(t) \leqslant \left\{u_0\left[e_f(t,t_0)+\frac{n}{m}K^{\frac{n-m}{m}}\int_{t_0}^t e_f(t,\sigma(\tau))g(\tau)e_{f+g+A_1}(\tau,t_0)\Delta\tau\right]\right.$$
$$\left.+\int_{t_0}^t e_f(t,\sigma(\tau))C(\tau)\Delta\tau\right\}^{\frac{1}{m}},\ K>0,\ t\in\mathbb{T}^k, \tag{3.20}$$

where

$$A_1(t) = \frac{l}{n}K^{\frac{l-n}{m}}\left[w(\sigma(t),t)+\int_{t_0}^t \omega^{\triangle}(t,s)\Delta s\right], \tag{3.21}$$

$$B_2(t) = \frac{m-n}{m}K^{\frac{n}{m}}+\frac{m-l}{m}K^{\frac{l}{m}}\int_{t_0}^t \omega^{\triangle}(t,s)\Delta s, \tag{3.22}$$

$$C(t) = g(t)\left[B_2(t)+\frac{n}{m}K^{\frac{n-m}{m}}\int_{t_0}^t e_{f+g+A_1}(t,\sigma(s))g(s)B_2(s)\Delta s\right]. \tag{3.23}$$

Proof. Define a function $z(t)$ by the right hand of (3.20). Then $z(t_0)=u_0$, $u^m(t)\leqslant z(t)$, and

$$z^{\triangle}(t) = f(t)u^m(t)+g(t)\left[u^n(t)+\int_{t_0}^t \omega(t,s)u^l(s)\Delta s\right]$$
$$\leqslant f(t)z(t)+g(t)\left[(z(t)^{\frac{n}{m}}+\int_{t_0}^t \omega(t,s)(z(s))^{\frac{l}{m}}\Delta s\right]. \tag{3.24}$$

Using Lemma 1, from the above inequality, we have

$$z^{\triangle}(t) \leqslant f(t)z(t)+g(t)\left[\frac{m-n}{m}K^{\frac{n}{m}}+\frac{m-l}{m}K^{\frac{l}{m}}\int_{t_0}^t \omega(t,s)\Delta s\right]$$
$$+g(t)\left[\frac{n}{m}K^{\frac{n-m}{m}}z(t)+\frac{l}{m}K^{\frac{l-m}{m}}\int_{t_0}^t \omega(t,s)z(s)\Delta s\right],\ t\in\mathbb{T}^k. \tag{3.25}$$

Let

$$v(t) = \frac{n}{m}K^{\frac{n-m}{m}}z(t)+\frac{l}{m}K^{\frac{l-m}{m}}\int_{t_0}^t \omega(t,s)z(s)\Delta s,\ t\in\mathbb{T}^k, \tag{3.26}$$

then $v(t_0)=\frac{n}{m}K^{\frac{n-m}{m}}u_0$, $\frac{n}{m}K^{\frac{n-m}{m}}z(t)\leqslant v(t)$, and

$$v^{\triangle}(t) \leqslant [f(t)+g(t)]v(t)+\frac{l}{m}K^{\frac{l-m}{m}}\left[\omega(\sigma(t),t)+\int_{t_0}^t \omega^{\triangle}(t,s)\Delta s\right]v(t)$$
$$+\frac{n}{m}K^{\frac{n-m}{m}}g(t)B_2(t)$$
$$= [f(t)+g(t)+A_1(t)]v(t)+\frac{n}{m}K^{\frac{n-m}{m}}g(t)B_2(t),\ t\in\mathbb{T}^k, \tag{3.27}$$

where $A_1(t)$, $B_2(t)$ are as defined in (3.21), (3.22). It is easy to see that $(f+g+A_1) \in \mathscr{R}^+$. Therefore, by using Theorem 2.3, from (3.27), we easily have

$$\begin{aligned} v(t) &\leqslant v(t_0)e_{f+g+A_1}(t,t_0) + \frac{n}{m}K^{\frac{n-m}{m}} \int_{t_0}^{t} e_{f+g+A_1}(t,\sigma(s))g(s)B_2(s)\Delta s \\ &= \frac{n}{m}K^{\frac{n-m}{m}} \left[u_0 e_{f+g+A_1}(t,t_0) + \int_{t_0}^{t} e_{f+g+A_1}(t,\sigma(s))g(s)B_2(s)\Delta s \right]. \end{aligned} \tag{3.28}$$

Combining (3.25), (3.26) and (3.28), we obtain

$$\begin{aligned} z^{\triangle}(t) &\leqslant f(t)z(t) + \frac{n}{m}K^{\frac{n-m}{m}} u_0 g(t)e_{f+g+A_1}(t,t_0) \\ &\quad + g(t)\left[B_2(t) + \frac{n}{m}K^{\frac{n-m}{m}} \int_{t_0}^{t} e_{f+g+A_1}(t,\sigma(s))g(s)B_2(s)\Delta s \right] \\ &= f(t)z(t) + \frac{n}{m}K^{\frac{n-m}{m}} u_0 g(t)e_{f+g+A_1}(t,t_0) + C(t), \end{aligned} \tag{3.29}$$

where C(t) is as defined in (3.23). Which implies

$$\begin{aligned} z(t) &\leqslant u_0 \left[e_f(t,t_0) + \frac{n}{m}K^{\frac{n-m}{m}} \int_{t_0}^{t} e_f(t,\sigma(\tau)g(\tau)e_{f+g+A_1}(\tau,t_0)\Delta\tau \right] \\ &\quad + \int_{t_0}^{t} e_f(t,\sigma(\tau)C(\tau)\Delta\tau,\ t \in \mathbb{T}^k. \end{aligned} \tag{3.30}$$

It is obvious that the desired inequality (3.20) follows from $u^m(t) \leqslant z(t)$ and (3.30). The proof of Theorem 3.3 is complete. □

REMARK 3.8. If $m = n = l = 1$ in Theorem 3.3, then the inequality given in (3.20) reduces to the inequality in [1, Theorem 3.3].

REMARK 3.9. Let $m = n = l = 1$, $\omega(t,s) = \omega(s)$ in Theorem 3.3. Then we observe that Theorem 1.7.2(iii) in [13] is a peculiar case of Theorem 3.3 if $\mathbb{T} = \mathbb{R}$, and Theorem 1.4.6(iii) in [14] is also a peculiar case of Theorem 3.3 if $\mathbb{T} = \mathbb{Z}$.

4. Some applications

In this section, we present some immediate applications of our results.

EXAMPLE 4.1. Consider the dynamic equation

$$[u^m(t)]^{\triangle} = F\left(t, u^m(t), \int_{t_0}^{t} H(t,s,u^n(s))\triangle s\right),\quad u^m(t_0) = C,\ \ t \in \mathbb{T}^k, \tag{4.1}$$

where C is a constant, $F : \mathbb{T}^k \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and $H : \mathbb{T}^k \times \mathbb{T}^k \times \mathbb{R} \to \mathbb{R}$ is also a continuous function.

Assume that

$$|F(t,U,V)| \leqslant f(t)(|U| + |V|) + p(t), \tag{4.2}$$

$$|H(t,s,U)| \leqslant \omega(t,s)|U|, \quad t,s \in \mathbb{T}^k. \tag{4.3}$$

If $u(t)$ is a solution of (4.1), then

$$|u^m(t)| \leqslant |C| + \int_{t_0}^{t} \left\{ B(\tau) + f(\tau)\left[|C|e_{f+A}(\tau,t_0) + \int_{t_0}^{\tau} e_{f+A}(\tau,\sigma(s))B(s)\Delta s \right] \right\} \Delta\tau, \tag{4.4}$$

where $p, f \in C_{rd}, f(t), p(t)$ are nonnegative, $\omega(t,s)$ is as defined in Theorem 2.2 such that $\omega(t,s) \geqslant 0$ and $\omega^{\Delta}(t,s) \geqslant 0$ for $t,s \in \mathbb{T}$ with $s \leqslant t$, and A, B are as defined in (3.3), (3.4).

Proof. Clearly, the solution $u(t)$ of (4.1) satisfies the following equivalent equation

$$u^m(t) = C + \int_{t_0}^{t} F\left(\tau, u^m(\tau), \int_{t_0}^{\tau} H(\tau,s,u^n(s))\Delta s\right)\Delta\tau, \; t \in \mathbb{T}^k. \tag{4.5}$$

It follows from (4.2), (4.3), (4.5) that

$$\begin{aligned}
|u^m(t)| &\leqslant |C| + \int_{t_0}^{t} \left| F\left(\tau, u^m(\tau), \int_{t_0}^{\tau} H(\tau,s,u^n(s))\Delta s\right)\right| \Delta\tau \\
&\leqslant |C| + \int_{t_0}^{t} f(\tau)\left[|u^m(\tau)| + \int_{t_0}^{\tau} |H(\tau,s,u^n(s))|\Delta s\right]\Delta\tau + \int_{t_0}^{t} p(\tau)\Delta\tau \\
&\leqslant |C| + \int_{t_0}^{t} [f(\tau)|u(\tau)|^m + p(\tau)]\Delta\tau \\
&\quad + \int_{t_0}^{t} f(\tau)\left(\int_{t_0}^{\tau} \omega(\tau,s)|u^n(s)|\Delta s\right)\Delta\tau, \; t \in \mathbb{T}^k.
\end{aligned} \tag{4.6}$$

Using Theorem 3.1, the desired inequality (4.4) is obtained from (4.6). The proof of Example 4.1 is complete. □

EXAMPLE 4.2. Assume that

$$|F(t,U_1,U_2) - F(t,V_1,V_2)| \leqslant f(t)\left(|U_1 - V_1| + |U_2 - V_2|\right), \tag{4.7}$$

$$|H(t,s,U_1) - H(t,s,U_2)| \leqslant \omega(t,s)|U_1 - U_2|, \; t,s \in \mathbb{T}^k, \tag{4.8}$$

where f, ω are as defined in Example 4.1. Then (31) has at most one solution.

Proof. Let $u_1(t), u_2(t)$ be two solutions of (4.1). Then we have

$$\begin{aligned}
u_1^m(t) - u_2^m(t) = \int_{t_0}^{t} \Bigg[& F\left(\tau, u_1^m(\tau), \int_{t_0}^{\tau} H(\tau,s,u_1^n(\tau))\Delta s\right) \\
& - F\left(\tau, u_2^m(\tau), \int_{t_0}^{\tau} H(\tau,s,u_2^n(\tau))\Delta s\right)\Bigg]\Delta\tau, \; t \in \mathbb{T}^k.
\end{aligned} \tag{4.9}$$

It follows from (4.7)-(4.9) that

$$\begin{aligned}|u_1^m(t)-u_2^m(t)| &\leqslant \int_{t_0}^{t} f(\tau)\,[|u_1^m(\tau)-u_2^m(\tau)| \\ &\quad + \int_{t_0}^{\tau} |H(\tau,s,u_1^n(s))-H(\tau,s,u_2^n(s))|\Delta s\Big]\,\Delta\tau \\ &\leqslant \int_{t_0}^{t} f(\tau)\left[|u_1^m(\tau)-u_2^m(\tau)|+\int_{t_0}^{\tau}\omega(\tau,s)|u_1^n(s)-u_2^n(s)|\Delta s\right]\Delta\tau, \\ &\quad t\in\mathbb{T}^k. \end{aligned} \tag{4.10}$$

By Theorem 3.1, we have $|u_1(t)-u_2(t)|\equiv 0,\ t\in\mathbb{T}^k$. Therefore, $u_1(t)=u_2(t)$, then (4.1) has the most one solution. The results are obtained. $\square$

EXAMPLE 4.3. Consider the dynamic equation

$$[u^m(t)]^{\triangle}=F\left(t,u^m(t),\int_{t_0}^{t}H(t,s,u^n(s))\triangle s\right),\ \ u^m(t_0)=C,\ \ t\in\mathbb{T}^k, \tag{4.11}$$

where C is a constant, $F:\mathbb{T}^k\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function, and $H:\mathbb{T}^k\times\mathbb{T}^k\times\mathbb{R}\to\mathbb{R}$ is also a continuous function.

Assume that

$$|F(t,U,V)|\leqslant f(t)(|U|+|V|), \tag{4.12}$$

$$|H(t,s,U)|\leqslant \omega(t,s)|U|+p(s),\quad t,s\in\mathbb{T}^k. \tag{4.13}$$

If $u(t)$ is a solution of (4.1), then

$$|u^m(t)|\leqslant |C|+\int_{t_0}^{t} f(\tau)\left[|C|e_{f+A}(\tau,t_0)+\int_{t_0}^{\tau}e_{f+A}(\tau,\sigma(s))B_1(s)\Delta s\right]\Delta\tau, \tag{4.14}$$

where $p,f\in C_{rd}, f(t),p(t)$ are nonnegative, $\omega(t,s)$ is as defined in Theorem 2.2 such that $\omega(t,s)\geqslant 0$ and $\omega^{\Delta}(t,s)\geqslant 0$ for $t,s\in\mathbb{T}$ with $s\leqslant t$, and A,B_1 are as defined in (3.3), (3.12).

Proof. Clearly, the solution $u(t)$ of (4.11) satisfies the following equivalent equation

$$u^m(t)=C+\int_{t_0}^{t}F\left(\tau,u^m(\tau),\int_{t_0}^{\tau}H(\tau,s,u^n(s))\Delta s\right)\Delta\tau,\ t\in\mathbb{T}^k. \tag{4.15}$$

It follows from (4.12), (4.13), (4.15) that

$$\begin{aligned}|u^m(t)| &\leqslant |C|+\int_{t_0}^{t}\left|F\left(\tau,u^m(\tau),\int_{t_0}^{\tau}H(\tau,s,u^n(s))\Delta s\right)\right|\Delta\tau \\ &\leqslant |C|+\int_{t_0}^{t}f(\tau)\left[|u^m(\tau)|+\int_{t_0}^{\tau}|H(\tau,s,u^n(s))|\,\Delta s\right]\Delta\tau \\ &\leqslant |C|+\int_{t_0}^{t}f(\tau)|u^m(\tau)|\Delta\tau \\ &\quad +\int_{t_0}^{t}f(\tau)\left[\int_{t_0}^{\tau}(\omega(\tau,s)|u^n(s)|+p(s))\,\Delta s\right]\Delta\tau,\ t\in\mathbb{T}^k. \end{aligned} \tag{4.16}$$

Using Theorem 3.2, the desired inequality (4.14) is obtained from (4.16). The proof of Example 4.3 is complete. □

EXAMPLE 4.4. Consider the dynamic equation

$$[u^m(t)]^{\triangle} = F\left(t, u^m(t), u^n(t), \int_{t_0}^{t} H(t,s,u^l(s))\triangle s\right),\ u^m(t_0) = C,\ t \in \mathbb{T}^k, \quad (4.17)$$

where C is a constant, $F : \mathbb{T}^k \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and $H : \mathbb{T}^k \times \mathbb{T}^k \times \mathbb{R} \to \mathbb{R}$ is also a continuous function.

Assume that

$$|F(t,U,V,W)| \leqslant f(t)|U| + g(t)(|V| + |W|), \quad (4.18)$$

$$|H(t,s,U)| \leqslant \omega(t,s)|U|, \quad t,s \in \mathbb{T}^k. \quad (4.19)$$

If $u(t)$ is a solution of (4.17), then

$$\begin{aligned} |u^m(t)| \leqslant |C| &\left[e_f(t,t_0) + \frac{n}{m} K^{\frac{n-m}{m}} \int_{t_0}^{t} e_f(t,\sigma(\tau)) g(\tau) e_{f+g+A_1}(\tau,t_0)\Delta\tau \right] \\ &+ \int_{t_0}^{t} e_f(t,\sigma(\tau)) C(\tau)\Delta\tau,\ K > 0,\ t \in \mathbb{T}^k, \end{aligned} \quad (4.20)$$

where $p, f \in C_{rd}, f(t), p(t)$ are nonnegative, $\omega(t,s)$ is as defined in Theorem 2.2 such that $\omega(t,s) \geqslant 0$ and $\omega^{\Delta}(t,s) \geqslant 0$ for $t,s \in \mathbb{T}$ with $s \leqslant t$, and A_1, B_2, C is as defined in (3.22), (3.23), (3.24).

Proof. Clearly, the solution $u(t)$ of (4.17) satisfies the following equivalent equation

$$u^m(t) = C + \int_{t_0}^{t} F\left(\tau, u^m(\tau), u^n(\tau), \int_{t_0}^{\tau} H(\tau,s,u^l(s))\Delta s\right)\Delta\tau,\ t \in \mathbb{T}^k. \quad (4.21)$$

It follows from (4.18), (4.19), (4.21) that

$$\begin{aligned} |u^m(t)| &\leqslant |C| + \int_{t_0}^{t} \left| F\left(\tau, u^m(\tau), u^n(\tau), \int_{t_0}^{\tau} H(\tau,s,u^l(s))\Delta s\right) \right| \Delta\tau \\ &\leqslant |C| + \int_{t_0}^{t} \left[f(\tau)|u^m(\tau)| + g(\tau)|u^n(\tau)| + g(\tau)\int_{t_0}^{\tau} |H(\tau,s,u^l(s))|\Delta s \right] \Delta\tau \\ &\leqslant |C| + \int_{t_0}^{t} f(\tau)|u^m(\tau)| \\ &\quad + \int_{t_0}^{t} g(\tau)\left(|u^n(\tau)| + \int_{t_0}^{\tau} \omega(t,s)|u^l(s)|\Delta s\right)\Delta\tau,\ t \in \mathbb{T}^k. \end{aligned} \quad (4.22)$$

Using Theorem 3.3, the desired inequality (4.20) is obtained from (4.22). The proof of Example 4.4 is complete. □

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Liwei Du
Department of Mathematics
Qufu Normal University
Qufu, 273165, Shandong
People's Republic of China

Run Xu
Department of Mathematics
Qufu Normal University
Qufu, 273165, Shandong
People's Republic of China
e-mail: xurun2005@163.com