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ON A FUNCTION ASSOCIATED WITH THE GENERALIZED EULER CONSTANT FOR AN ARITHMETIC PROGRESSION

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Abstract. The function $\mathfrak{L}(q)=\sum_{p|q}\frac{\log p}{p-1}$ appears in a work by D. H. Lehmer, where he studies a generalization of the Euler constant. In this short note, we obtain lower and upper bounds for $\mathfrak{L}(q)$.

1. Introduction

D. H. Lehmer [1] studied the generalized Euler constant $\gamma(a,q)$ for an arithmetic progression $a \pmod q$, defined by

$$\gamma(a,q)=\lim_{x\to\infty}\left(\sum_{\substack{j\leqslant x\\ n\equiv a\ [q]}}\frac{1}{j}-\frac{1}{q}\log x\right).$$

He proved that

$$\sum_{\substack{1\leqslant a\leqslant q\\ \gcd(a,q)=1}}\gamma(a,q)=\gamma\frac{\varphi(q)}{q}+\frac{1}{q}\log N(q),$$

for a number theoretic function $N(q)$, where γ is the Euler constant. More precisely, he showed that $N(q)$ is an integer, and its explicit value is given by

$$N(q)=\prod_{p|q}p^{\varphi(q)/(p-1)}.$$

This gives

$$\log N(q)=\varphi(q)\sum_{p|q}\frac{\log p}{p-1}:=\varphi(q)\mathfrak{L}(q),$$

say. In this note, we obtain some bounds for the function $\mathfrak{L}(q)$.

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2. Minimax of the function $\mathfrak{L}(q)$

Explicit Minimum. For $x \in (0,\infty) - \{1\}$ we let

$$\ell(x) = \frac{\log x}{x-1},$$

and we put $\ell(1) = 1$. Since ℓ is a decreasing function over $(0,\infty)$, we obtain

$$\min_{1<q\leqslant N} \mathfrak{L}(q) = \ell(p^\star) \quad \text{where} \quad p^\star = \max\{p : p \leqslant N\}.$$

More precisely, we have

$$\mathfrak{L}(q) \geqslant \ell(q) \qquad \text{for } q \geqslant 2.$$

Approximate Maximum. Also, we get the maximum value of $\mathfrak{L}(q)$ over $q \in [1,N]$, when we take q to be the largest $\prod_{p\leqslant x} p$ not larger than N. This happens for

$$x = x(N) := \max\{y : \theta(y) \leqslant \log N\}, \tag{2.1}$$

where $\theta(x) = \sum_{p\leqslant x} \log x$. Thus, we obtain

$$\max_{1<q\leqslant N} \mathfrak{L}(q) = \sum_{p\leqslant x(N)} \ell(p).$$

Since $\ell \in C^1(\mathbb{R}^+)$, using Stieljes integral and integrating by parts, we have

$$\begin{aligned}\sum_{p\leqslant x} \ell(p) &= \int_{2^-}^{x} \frac{\ell(x)}{\log x} \mathrm{d}\theta(x) = \frac{\ell(x)\theta(x)}{\log x} + \int_2^x \theta(t) \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{-\ell(t)}{\log t}\right) \mathrm{d}t \\ &= \frac{\theta(x)}{x-1} + \int_2^x \frac{\theta(t)}{(t-1)^2} \mathrm{d}t = (1+o(1)) \log x.\end{aligned}$$

This gives $\max_{1<q\leqslant N} \mathfrak{L}(q) = (1+o(1)) \log x(N)$. But, the prime number theorem asserts that $x(N) = (1+o(1)) \log N$. Thus, we obtain

$$\max_{1<q\leqslant N} \mathfrak{L}(q) = (1+o(1)) \log\log N.$$

Explicit Maximum. Here we give an approach which leads to an explicit upper bound for $\max_{1<q\leqslant N} \mathfrak{L}(q)$, as more as has capacity to reprove obtained bound above. To do this, we write

$$\sum_{p\leqslant x} \ell(p) = \sum_{p\leqslant x} \frac{\log p}{p} + A - \sum_{p>x} \frac{\log p}{p(p-1)},$$

where A is an absolute constant given by

$$A = \sum_p \frac{\log p}{p(p-1)}.$$

We recall that

$$\sum_{p\leqslant x}\frac{\log p}{p}=\log x+E+O(\exp(-c\sqrt{x}))$$

is valid for absolute constants $c>0$, and E given by

$$E=-\gamma-\sum_{n=2}^{\infty}\sum_{p}\frac{\log p}{p^n}.$$

More precisely, it is known [2] that

$$\sum_{p\leqslant x}\frac{\log p}{p}<\log x+E+\frac{1}{\log x}\qquad\text{for } x\geqslant 32.$$

Thus, we obtain

$$\sum_{p\leqslant x}\ell(p)<\log x+A+E+\frac{1}{\log x}\qquad\text{for } x\geqslant 32,$$

or, we obtain

$$\max_{1<q\leqslant N}\mathfrak{L}(q)<\log x(N)+A+E+\frac{1}{\log x(N)}\qquad\text{for } x(N)\geqslant 32.$$

It is known [2] that $t(1-\frac{1}{\log t})<\theta(t)$ is valid for $t\geqslant 41$. Thus, considering the relation (2.1) we have $x(N)<x^\star$, in which

$$x^\star\left(1-\frac{1}{\log x^\star}\right)=\log N.$$

It is clear that $x^\star>\log N$, so we put $x^\star=\log N+\varepsilon_N$ for some function $\varepsilon_N>0$, with the property $\varepsilon_N=o(\log N)$. We obtain

$$1+\frac{\log N}{\varepsilon_N}-\log\log N=\log(1+\frac{\varepsilon_N}{\log N}).$$

Exact solution of this equation is not accessible, but if we put $\varepsilon_N:=\frac{\log N}{\log\log N}+r_N$ then the left hand side of it becomes $1-\frac{r_N(\log\log N)^2}{\log N}+O(\frac{r_N^2(\log\log N)^3}{\log^2 N})$, and the right hand side of it becomes $\frac{1}{\log\log N}+\frac{r_N}{\log N}+O((\frac{1}{\log\log N}+\frac{r_N}{\log N})^2)$. Taking $r_N=\frac{\log N}{(\log\log N)^2}$, both of above terms become $O(\frac{1}{\log\log N})$. Thus, we obtain

$$x^\star=\log N\left(1+\frac{1}{\log\log N}+\frac{(1+o(1))}{(\log\log N)^2}\right).$$

Therefore, for N large enough, we obtain

$$\max_{1<q\leqslant N}\mathfrak{L}(q)<\log\log N+A+E+O\left(\frac{1}{\log\log N}\right).$$

Here, we note that $A+E\approx-0.58$, and above result asserts that we must have $\max_{1<q\leqslant N}\mathfrak{L}(q)<\log\log N+C$ for a constant C. Below figure shows that $C=0.5$ works as well at least for $7\leqslant q\leqslant 250$.

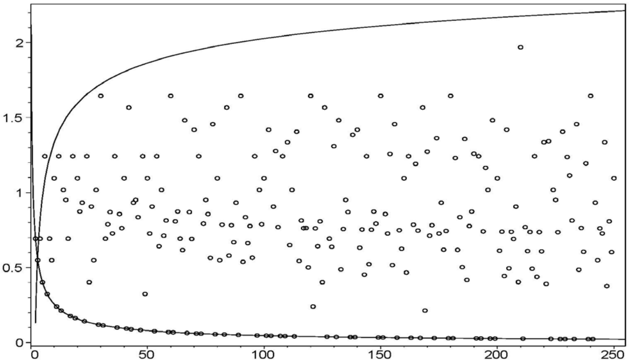

Figure 1: *Graphs of points* $(q,\mathfrak{L}(q))$ *for* $1 \leqslant q \leqslant 250$*, the function* $\frac{\log x}{x-1}$*, and the function* $\log\log x + 0.5$.

REFERENCES

[1] D. H. LEHMER, *Euler constants for arithmetic progressions*, Acta Arith. **27** (1975), 125–142.

[2] J. BARKLEY ROSSER & L. SCHOENFELD, *Approximate Formulas for Some Functions of Prime Numbers*, Illinois J. Math. **6** (1962), 64–94.

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