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AN INEQUALITY BETWEEN THE INTEGRAL NORM AND EUCLIDEAN NORM OF A SYMMETRIC BILINEAR FORM

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Abstract. In this note, we establish an inequality between the integral norm and Euclidean norm of a symmetric bilinear form f on a Euclidean space E^n, i.e.,

$$\frac{1}{vol\, S^{n-1}}\int_{S^{n-1}} f^2(\theta,\theta)d\theta \leqslant \frac{1}{n}|f|^2_{Euc}$$

where the equality holds if and only if the eigenvalues of f are all the same.

1. Introduction

For a linear map $L: V \to W$ between linear spaces, the Euclidean norm is given by

$$|L| = \sqrt{\mathrm{tr}(L^* \circ L)} = \sqrt{\mathrm{tr}(L \circ L^*)}$$

where $L^*: W \to V$ is the adjoint. (see [2, p. 54], for example)

In particular, for a symmetric bilinear form f on a Euclidean space E^n, $f: E^n \times E^n \to R$, there is a linear map $\tilde{f}: E^n \to E^n$ associated with f and the inner product in E^n, i.e.,

$$f(x,y) = \langle \tilde{f}(x), y\rangle.$$

We define the Euclidean norm of f by

$$|f|_{Euc} := |\tilde{f}|.$$

Below we choose an orthonormal base $\{e_i\}_{i=1}^n$ for E^n such that the matrix for $\tilde{f}$ is diagonal, and denote the eigenvalues of $\tilde{f}$ by $\lambda_i := f(e_i,e_i), \quad i = 1,\cdots n$, thus,

$$|f|_{Euc} = \sqrt{\sum_{i=1}^n (f(e_i,e_i))^2} = \sqrt{\sum_{i=1}^n \lambda_i^2}.$$

For a bilinear form f on a Euclidean space E^n, we define the integral norm of f by

$$|f|_{int} := \sqrt{\int_{S^{n-1}} f^2(\theta,\theta)d\theta},$$

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here and thereafter S^{n-1} is the unit sphere in E^n.

Similarly, for a linear function f on E^n, $f : E^n \to R$, one may also define the the integral norm of f by

$$|f|_{int} := \sqrt{\int_{S^{n-1}} f^2(\theta)d\theta}.$$

It is well known that for a bilinear form f on a Euclidean space E^n,

$$\frac{1}{n}\,\mathrm{tr} f = \frac{1}{vol\, S^{n-1}} \int_{S^{n-1}} f(\theta,\theta)d\theta.$$

How about the the integral norm and Euclidean norm of f?

THEOREM. *For a symmetric bilinear form f on a Euclidean space E^n, $f : E^n \times E^n \to R$, the integral norm and Euclidean norm of f satisfy*

$$\frac{1}{vol\, S^{n-1}} \int_{S^{n-1}} f^2(\theta,\theta)d\theta \leqslant \frac{1}{n}|f|^2_{Euc}$$

where the equality holds if and only if the eigenvalues of f are all the same.

The proof is based on the spherical coordinates in Riemann integrals.

REMARK. The motivation is to generalize the Bochner formula

$$\frac{1}{2}\triangle|\nabla f|^2 = |\nabla^2 f|^2 + < \nabla f, \nabla\triangle f > + Ric(\nabla f, \nabla f)$$

from Riemannian geometry to Alexandrov geometry (see [1, 2] for Riemannian geometry and Alexandrov geometry in details, respectively). On Alexandrov spaces, the tangent cone is not always a Euclidean space but a norm space or even a metric cone, we find that the integral norm is an appropriate alternative for the Euclidean norm $|\nabla^2 f|$ involved in the Bochner formula [3]. Generally, having this inequality, it is inspirational to generalize formulas associated with the Euclidean norm of a symmetric bilinear form on a Euclidean space to the corresponding ones associated with the the integral norm of a linear function on a norm space or a positively homogeneous function on a metric cone.

2. Proof

$$\begin{aligned}
\int_{S^{n-1}} f^2(\theta,\theta)d\theta &= \int_{S^{n-1}} f^2(\sum_{i=1}^n x_i e_i, \sum_{j=1}^n x_j e_j)d\theta \\
&= \int_{S^{n-1}} (\sum_{i,j=1}^n x_i x_j f(e_i,e_j))^2 d\theta \\
&= \int_{S^{n-1}} (\sum_{i,j=1}^n x_i^2 f(e_i,e_i))^2 d\theta
\end{aligned}$$

$$
\begin{aligned}
&= \int_{S^{n-1}} \sum_{i,j=1}^{n} x_i^2 x_j^2 f(e_i,e_i) f(e_j,e_j) d\theta \\
&=: \int_{S^{n-1}} \sum_{i,j=1}^{n} x_i^2 x_j^2 \lambda_i \lambda_j d\theta \\
&= \int_{S^{n-1}} (\sum_{i=1}^{n} \lambda_i^2 x_i^4 + 2 \sum_{1\leqslant i<j\leqslant n} \lambda_i \lambda_j x_i^2 x_j^2) d\theta,
\end{aligned}
$$

In calculation below we choose $n-$ spherical coordinates,

$$
\begin{cases}
x_1 = \cos\varphi_1, \\
x_2 = \sin\varphi_1 \cos\varphi_2, \\
x_3 = \sin\varphi_1 \sin\varphi_2 \cos\varphi_3, \\
\cdots\cdots\cdots\cdots \\
x_{n-1} = \sin\varphi_1 \sin\varphi_2 \cdots \sin\varphi_{n-2} \cos\varphi_{n-1}, \\
x_n = \sin\varphi_1 \sin\varphi_2 \cdots \sin\varphi_{n-2} \sin\varphi_{n-1},
\end{cases}
$$

$$
J = \sin^{n-2}\varphi_1 \cdot \sin^{n-3}\varphi_2 \cdots \sin^2\varphi_{n-3} \cdot \sin\varphi_{n-2},
$$

where $0 \leqslant \varphi_1, \varphi_2, \cdots, \varphi_{n-2} \leqslant \pi$, $0 \leqslant \varphi_{n-1} \leqslant 2\pi$.

And use

$$
\int_0^{\pi} \sin^n \theta d\theta = \begin{cases} 2 \cdot \frac{(2m-1)!!}{(2m)!!} \cdot \frac{\pi}{2}, & n = 2m, \\ 2 \cdot \frac{(2m)!!}{(2m+1)!!}, & n = 2m+1, \end{cases}
$$

$$
vol\, S^{n-1} = \begin{cases} \frac{2\pi^m}{(m-1)!}, & n = 2m, \\ \frac{2(2\pi)^m}{(2m-1)!!}, & n = 2m+1. \end{cases}
$$

And *suppose in calculation below the* n, i, j *are even, it is similar for other cases,* i.e., the n, i, j are even or odd respectively, and actually the final result is the same for all cases.

For n, i even and $1 \leqslant i \leqslant n-2$,

$$
\begin{aligned}
&\int_{S^{n-1}} x_i^4 d\theta \\
&= \int_0^{2\pi} d\varphi_{n-1} \cdot \int_{[0,\pi]^{n-2}} (\sin\varphi_1 \cdot \sin\varphi_2 \cdots \sin\varphi_{i-1} \cdot \sin\varphi_i)^4 \\
&\quad \cdot \sin^{n-2}\varphi_1 \cdot \sin^{n-3}\varphi_2 \cdots \sin^2\varphi_{n-3} \cdot \sin\varphi_{n-2} d\varphi_1 \cdots d\varphi_{n-2} \\
&= \int_0^{2\pi} d\varphi_{n-1} \cdot \int_{[0,\pi]^{n-2}} \sin^{n+2}\varphi_1 \cdot \sin^{n+1}\varphi_2 \cdots \sin^{n-i+4}\varphi_{i-1} \cdot \sin^{n-i-1}\varphi_i \cdots \\
&\quad \sin^2\varphi_{n-3} \cdot \sin\varphi_{n-2} \cdot \cos^4\varphi_i d\varphi_1 \cdots d\varphi_{n-2} \\
&= \int_0^{2\pi} d\varphi_{n-1} \cdot \int_0^{\pi} \sin^{n+2}\varphi_1 d\varphi_1 \cdot \int_0^{\pi} \sin^{n+1}\varphi_2 d\varphi_2 \cdots \int_0^{\pi} \sin^{n-i+4}\varphi_{i-1} d\varphi_{i-1}
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \int_0^\pi \sin^{n-i-1}\varphi_i \cdot \cos^4\varphi_i d\varphi_i \cdot \int_0^\pi \sin^{n-i-2}\varphi_{i+1} d\varphi_{i+1} \cdot \int_0^\pi \sin^{n-i-3}\varphi_{i+2} d\varphi_{i+2} \cdots \\
&\int_0^\pi \sin^2\varphi_{n-3} d\varphi_{n-3} \cdot \int_0^\pi \sin\varphi_{n-2} d\varphi_{n-2} \\
=&2\pi \cdot \left(2\cdot \frac{(n+2-1)!!}{(n+2)!!}\cdot\frac{\pi}{2}\right)\cdot\left(2\cdot\frac{(n+1-1)!!}{(n+1)!!}\right)\cdot\left(2\cdot\frac{(n-1)!!}{n!!}\cdot\frac{\pi}{2}\right)\cdots \\
&\left(2\cdot\frac{(n-i+4-1)!!}{(n-i+4)!!}\cdot\frac{\pi}{2}\right)\cdot 2\left[\frac{(n-i-2)!!}{(n-i-1)!!}-2\frac{(n-i)!!}{(n-i+1)!!}+\frac{(n-i+2)!!}{(n-i+3)!!}\right] \\
&\cdot\left(2\cdot\frac{(n-i-3)!!}{(n-i-2)!!}\cdot\frac{\pi}{2}\right)\cdot\left(2\cdot\frac{(n-i-4)!!}{(n-i-3)!!}\right)\cdots\left(2\cdot\frac{1!!}{2!!}\cdot\frac{\pi}{2}\right)\cdot\left(2\cdot\frac{0!!}{1!!}\right) \\
=&2\pi\cdot 2^{i-1}\cdot\left(\frac{\pi}{2}\right)^{\frac{i}{2}}\cdot\frac{(n-i+3)!!}{(n+2)!!}\cdot 2\cdot\frac{(n-i-2)!!}{(n-i-1)!!} \\
&\cdot\left[1-2\frac{(n-i)}{(n-i+1)}+\frac{(n-i+2)(n-i)}{(n-i+3)(n-i+1)}\right]\cdot 2^{n-2-i}\cdot\left(\frac{\pi}{2}\right)^{\frac{n-2-i}{2}}\cdot\frac{0!!}{(n-i-2)!!} \\
=&2\pi\cdot 2^{n-2}\cdot\left(\frac{\pi}{2}\right)^{\frac{n-2}{2}}\cdot\frac{(n-i+3)!!}{(n+2)!!}\cdot\frac{(n-i-2)!!}{(n-i-1)!!}\cdot\frac{3}{(n-i+3)(n-i+1)}\cdot\frac{0!!}{(n-i-2)!!} \\
=&2\pi\cdot 2^{n-2}\cdot\left(\frac{\pi}{2}\right)^{\frac{n-2}{2}}\cdot\frac{3}{(n+2)!!}.
\end{aligned}
$$

More easily,

$$
\int_{S^{n-1}} x_{n-1}^4 d\theta = 2\pi\cdot 2^{n-2}\cdot\left(\frac{\pi}{2}\right)^{\frac{n-2}{2}}\cdot\frac{3}{(n+2)!!},
$$

$$
\int_{S^{n-1}} x_n^4 d\theta = 2\pi\cdot 2^{n-2}\cdot\left(\frac{\pi}{2}\right)^{\frac{n-2}{2}}\cdot\frac{3}{(n+2)!!}.
$$

For n,i,j even and $1\leqslant i<j\leqslant n-2$,

$$
\begin{aligned}
&\int_{S^{n-1}} x_i^2 x_j^2 d\theta \\
=&\int_0^{2\pi} d\varphi_{n-1}\cdot\int_{[0,\pi]^{n-2}}(\sin\varphi_1\cdots\sin\varphi_{i-1}\cdot\cos\varphi_i)^2\cdot(\sin\varphi_1\cdots\sin\varphi_{j-1}\cdot\cos\varphi_j)^2 \\
&\cdot\sin^{n+2}\varphi_1\cdot\sin^{n-3}\varphi_2\cdots\sin\varphi_{n-2}d\varphi_1\cdots d\varphi_{n-2} \\
=&\int_0^{2\pi} d\varphi_{n-1}\cdot\int_{[0,\pi]^{n-2}}\sin^4\varphi_1\cdot\sin^4\varphi_2\cdots\sin^4\varphi_{i-1}\cdot\cos^2\varphi_i\cdot\sin^2\varphi_i \\
&\cdot\sin^2\varphi_{i+1}\cdots\sin^2\varphi_{j-1}\cdot\cos^2\varphi_j\cdot\sin^{n-2}\varphi_1\cdot\sin^{n-3}\varphi_2\cdots\sin\varphi_{n-2}d\varphi_1\cdots d\varphi_{n-2} \\
=&\int_0^{2\pi} d\varphi_{n-1}\cdot\int_{[0,\pi]^{n-2}}\sin^{n+2}\varphi_1\cdots\sin^{n-i+4}\varphi_{i-1}\cdot(\sin^{n-i+1}\varphi_i\cdot\cos^2\varphi_i)\cdot\sin^{n-i}\varphi_{i+1}\cdots \\
&\sin^{n-j+2}\varphi_{j-1}\cdot(\cos^2\varphi_j\cdot\sin^{n-j-1}\varphi_j)\cdot\sin^{n-j-2}\varphi_{j+1}\cdots\sin\varphi_{n-2}d\varphi_1\cdots d\varphi_{n-2} \\
=&\int_0^{2\pi} d\varphi_{n-1}\cdot\int_0^\pi\sin^{n+2}\varphi_1 d\varphi_1\cdot\int_0^\pi\sin^{n+1}\varphi_2 d\varphi_2\cdots\int_0^\pi\sin^{n-i+4}\varphi_{i-1}d\varphi_{i-1}
\end{aligned}
$$

$$
\begin{aligned}
&\cdot\int_0^{\pi}(\sin^{n-i+1}\varphi_i-\sin^{n-i+3}\varphi_i)d\varphi_i\cdot\int_0^{\pi}\sin^{n-i}\varphi_{i+1}d\varphi_{i+1}\cdots\int_0^{\pi}\sin^{n-j+2}\varphi_{j-1}d\varphi_{j-1}\\
&\cdot\int_0^{\pi}(\sin^{n-j-1}\varphi_j-\sin^{n-j+1}\varphi_j)d\varphi_j\cdot\int_0^{\pi}\sin^{n-j-2}\varphi_{j+1}d\varphi_{j+1}\cdots\int_0^{\pi}\sin\varphi_{n-2}d\varphi_{n-2}\\
=&2\pi\cdot(2\cdot\frac{(n+1)!!}{(n+2)!!}\cdot\frac{\pi}{2})\cdot(2\cdot\frac{(n)!!}{(n+1)!!})\cdots\left(2\cdot\frac{(n-i+3)!!}{(n-i+4)!!}\cdot\frac{\pi}{2}\right)\\
&\cdot 2\cdot\left[\frac{(n-i)!!}{(n-i+1)!!}-\frac{(n-i+2)!!}{(n-i+3)!!}\right]\cdot\left(2\cdot\frac{(n-i-1)!!}{(n-i)!!}\cdot\frac{\pi}{2}\right)\cdots\left(2\cdot\frac{(n-j+1)!!}{(n-j+2)!!}\cdot\frac{\pi}{2}\right)\\
&\cdot 2\cdot\left[\frac{(n-j-2)!!}{(n-j-1)!!}-\frac{(n-j)!!}{(n-j+1)!!}\right]\cdot\left(2\cdot\frac{(n-j-3)!!}{(n-j-2)!!}\cdot\frac{\pi}{2}\right)\cdots\left(2\cdot\frac{0!!}{1!!}\right)\\
=&2\pi\cdot 2^{n-2}\cdot\left(\frac{\pi}{2}\right)^{\frac{n-2}{2}}\cdot\frac{(n-i+3)!!}{(n+2)!!}\cdot\left[\frac{(n-i)!!}{(n-i+1)!!}\cdot\frac{1}{n-i+3}\right]\cdot\frac{(n-j+1)!!}{(n-i)!!}\\
&\cdot\left[\frac{(n-j-2)!!}{(n-j-1)!!}\cdot\frac{1}{n-j+1}\right]\cdot\frac{0!!}{(n-j-2)!!}\\
=&2\pi\cdot 2^{n-2}\cdot\left(\frac{\pi}{2}\right)^{\frac{n-2}{2}}\cdot\frac{1}{(n+2)!!}.
\end{aligned}
$$

More easily,

$$\int_{S^{n-1}}x_i^2x_{n-1}^2d\theta=2\pi\cdot 2^{n-2}\cdot\left(\frac{\pi}{2}\right)^{\frac{n-2}{2}}\cdot\frac{1}{(n+2)!!},\qquad 1\leqslant i\leqslant n-2,$$

$$\int_{S^{n-1}}x_i^2x_n^2d\theta=2\pi\cdot 2^{n-2}\cdot\left(\frac{\pi}{2}\right)^{\frac{n-2}{2}}\cdot\frac{1}{(n+2)!!},\qquad 1\leqslant i\leqslant n-2,$$

$$\int_{S^{n-1}}x_{n-1}^2x_n^2d\theta=2\pi\cdot 2^{n-2}\cdot\left(\frac{\pi}{2}\right)^{\frac{n-2}{2}}\cdot\frac{1}{(n+2)!!}.$$

Thus, for n even,

$$
\begin{aligned}
&\frac{1}{vol\,S^{n-1}}\int_{S^{n-1}}f^2(\theta,\theta)d\theta\\
=&\frac{(\frac{n}{2}-1)!}{2\pi^{\frac{n}{2}}}\int_{S^{n-1}}(\sum_{i=1}^{n}\lambda_i^2x_i^4+2\sum_{1\leqslant i<j\leqslant n}\lambda_i\lambda_jx_i^2x_j^2)d\theta\\
=&\frac{(\frac{n}{2}-1)!}{2\pi^{\frac{n}{2}}}\cdot 2\pi\cdot 2^{n-2}\cdot\left(\frac{\pi}{2}\right)^{\frac{n-2}{2}}\cdot\frac{1}{(n+2)!!}\cdot[\sum_{i=1}^{n}3\lambda_i^2+2\sum_{1\leqslant i<j\leqslant n}\lambda_i\lambda_j]\\
=&\frac{1}{n(n+2)}\cdot[\sum_{i=1}^{n}3\lambda_i^2+2\sum_{1\leqslant i<j\leqslant n}\lambda_i\lambda_j],
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{vol\, S^{n-1}} \int_{S^{n-1}} f^2(\theta,\theta)d\theta - \frac{1}{n}|f|^2_{Euc} \\
&= \frac{1}{n(n+2)} \cdot [\sum_{i=1}^{n} 3\lambda_i^2 + 2 \sum_{1 \leqslant i<j \leqslant n} \lambda_i\lambda_j] - \frac{1}{n}\sum_{i=1}^{n}\lambda_i^2 \\
&= -\frac{1}{n(n+2)} \cdot [(n-1)\sum_{i=1}^{n}\lambda_i^2 - 2\sum_{1 \leqslant i<j \leqslant n} \lambda_i\lambda_j] \\
&= -\frac{1}{n(n+2)} \cdot \sum_{1 \leqslant i<j \leqslant n} (\lambda_i - \lambda_j)^2 \\
&\leqslant 0
\end{aligned}
$$

where the equality holds if and only if $\lambda_1 = \cdots = \lambda_n$.

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