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INEQUALITIES FOR UNITARILY INVARIANT NORMS

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Abstract. This paper aims to discuss some inequalities for unitarily invariant norms. We obtain several inequalities for unitarily invariant norms.

1. Introduction

Let $M_{m,n}$ be the space of $m\times n$ complex matrices and $M_n = M_{n,n}$. Suppose that the eigenvalues of A are $\lambda_1(A),\cdots,\lambda_n(A)$ and $|\lambda_1(A)|\geqslant\cdots\geqslant|\lambda_n(A)|$. Let $A,B\in M_n$ be positive semidefinite, the order relation $A\geqslant B$ means, as usual, that $A-B$ is positive semidefinite. Let $\|\cdot\|$ denote any unitarily invariant norm on M_n. So, $\|UAV\| = \|A\|$ for all $A\in M_n$ and for all unitary matrices $U,V\in M_n$. For $k=1,\cdots,n$, the *Ky Fan k-norm* $\|\cdot\|_{(k)}$ is defined as

$$\|A\|_{(k)} = \sum_{j=1}^{k} s_j(A).$$

For $1\leqslant p<\infty$, the *Schatten p-norm* $\|\cdot\|_p$ is defined as

$$\|A\|_p = \left(\sum_{j=1}^{n} s_j^p(A)\right)^{1/p} = \left(tr|A|^p\right)^{1/p},$$

where tr is the usual trace functional and $s_1(A)\geqslant\cdots\geqslant s_n(A)$ are the singular values of A, that is, the eigenvalues of the positive semidefinite matrix $|A| = (AA^*)^{1/2}$, arranged in decreasing order and repeated according to multiplicity. It is known that these norms are unitarily invariant, and it is evident that each unitarily invariant norm is a symmetric gauge function of singular values [5]. For more information on unitarily invariant norms the reader is referred to [5]. For $A=[a_{ij}]\in M_n$, the norm

$$\|A\|_2 = \left(\sum_{j=1}^{n} s_j^2(A)\right)^{1/2} = \left(tr|A|^2\right)^{1/2} = \left(\sum_{i,j=1}^{n} |a_{ij}|^2\right)^{1/2}$$

is also called the Hilbert-Schmidt norm or Frobenius norm. Obviously, the Hilbert-Schmidt norm or Frobenius norm is in the class of *Schatten* norms.

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2. Two inequalities for the Hilbert-Schmidt norm

Bhatia and Davis proved in [1] that if $A,B,X \in M_n$ such that A and B are positive semidefinite and if $0 \leqslant v \leqslant 1$, then

$$2\left\|A^{1/2}XB^{1/2}\right\| \leqslant \left\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\right\| \leqslant \|AX+XB\|.$$

Recently, Kittaneh and Manasrah [2] obtained an improvement of the Heinz inequality for the Hilbert-Schmidt norm which is stated as follows:

$$\left\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\right\|_2+2r_0\left((\|AX\|_2)^{1/2}-(\|XB\|_2)^{1/2}\right)^2 \leqslant \|AX+XB\|_2,$$

where $r_0=\min\{v,1-v\}$.

Meanwhile, Kittaneh and Manasrah [2] obtained another improvement of the Heinz inequality for the Hilbert-Schmidt norm which is stated as follows:

$$\left\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\right\|_2^2+2r_0\|AX-XB\|_2^2 \leqslant \|AX+XB\|_2^2,$$

where $r_0=\min\{v,1-v\}$.

In this section, we present two upper bounds for $\|AX+XB\|_2^2$. To do this, we need the following lemmas.

LEMMA 2.1. *If $a,b,s \in R$ and $s \neq 0$, then*

$$\frac{(s+1)^2}{s^2+1}(a+b)^2 \leqslant (a+sb)^2+\left(a+\frac{b}{s}\right)^2. \tag{2.1}$$

Proof. Let

$$K=\left(s^2+1\right)\left\{(a+sb)^2+\left(a+\frac{b}{s}\right)^2\right\}-(s+1)^2(a+b)^2.$$

Then, we have

$$\begin{aligned} K &= (s-1)^2a^2+2\left(s^3-s^2-1+\tfrac{1}{s}\right)ab+\left(s^2-\tfrac{1}{s}\right)^2b^2 \\ &= \left\{(s-1)a+\left(s^2-\tfrac{1}{s}\right)b\right\}^2 \geqslant 0. \end{aligned}$$

This completes the proof. □

LEMMA 2.2. *If $a,b,s,t \in R$, then*

$$\frac{1}{2}(s+t)^2(a+b)^2 \leqslant (sa+tb)^2+(ta+sb)^2. \tag{2.2}$$

Proof. Let

$$K=2(sa+tb)^2+2(ta+sb)^2-(s+t)^2(a+b)^2.$$

Then, we have

$$K=(s-t)^2(a-b)^2 \geqslant 0.$$

This completes the proof. □

THEOREM 2.1. *Let $A,B,X \in M_n$ such that A and B are positive semidefinite. If $s \in R$ and $s \neq 0,-1$, then*

$$\|AX+XB\|_2^2 \leqslant \frac{s^2+1}{(s+1)^2}\left\{\|AX+sXB\|_2^2+\left\|AX+\frac{1}{s}XB\right\|_2^2\right\}.$$

Proof. Since every positive semidefinite matrix is unitarily diagonalizable, it follows that there exist unitary matrices $U,V \in M_n$ such that $A=U\Lambda_1U^*$ and $B=V\Lambda_2V^*$, where

$$\Lambda_1 = diag(\lambda_1,\cdots,\lambda_n), \Lambda_2 = diag(\mu_1,\cdots,\mu_n), \lambda_i,\mu_i \geqslant 0, i=1,\cdots,n.$$

Let

$$Y = U^*XV = [y_{ij}].$$

Then

$$AX+XB = U[(\lambda_i+\mu_j)y_{ij}]V^*,$$

$$AX+sXB = U[(\lambda_i+s\mu_j)y_{ij}]V^*,$$

and

$$AX+\frac{1}{s}XB = U\left[\left(\lambda_i+\frac{1}{s}\mu_j\right)y_{ij}\right]V^*.$$

By the inequality (2.1), we have

$$\begin{aligned}\|AX+XB\|_2^2 &= \sum_{i,j=1}^{n}(\lambda_i+\mu_j)^2|y_{ij}|^2\\ &\leqslant \frac{s^2+1}{(s+1)^2}\left\{\sum_{i,j=1}^{n}(\lambda_i+s\mu_j)^2|y_{ij}|^2+\sum_{i,j=1}^{n}\left(\lambda_i+\frac{1}{s}\mu_j\right)^2|y_{ij}|^2\right\}\\ &= \frac{s^2+1}{(s+1)^2}\left\{\|AX+sXB\|_2^2+\left\|AX+\frac{1}{s}XB\right\|_2^2\right\}\end{aligned}$$

This completes the proof. □

THEOREM 2.2. *Let $A,B,X \in M_n$ such that A and B are positive semidefinite. If $s,t \in R$ and $s+t \neq 0$, then*

$$\|AX+XB\|_2^2 \leqslant \frac{2}{(s+t)^2}\left\{\|sAX+tXB\|_2^2+\|tAX+sXB\|_2^2\right\}.$$

Proof. Let U,V and Y have the same meaning as in the proof of Theorem 2.1. Then

$$sAX+tXB = U[(s\lambda_i+t\mu_j)y_{ij}]V^*, \quad tAX+sXB = U[(t\lambda_i+s\mu_j)y_{ij}]V^*.$$

By the inequality (2.2), we have

$$\begin{aligned}\|AX+XB\|_2^2 &= \sum_{i,j=1}^{n} (\lambda_i+\mu_j)^2 \left|y_{ij}\right|^2 \\ &\leqslant \frac{2}{(s+t)^2}\left\{\sum_{i,j=1}^{n} (s\lambda_i+t\mu_j)^2\left|y_{ij}\right|^2+\sum_{i,j=1}^{n} (t\lambda_i+s\mu_j)^2\left|y_{ij}\right|^2\right\} \\ &= \frac{2}{(s+t)^2}\left\{\|sAX+tXB\|_2^2+\|tAX+sXB\|_2^2\right\}.\end{aligned}$$

This completes the proof. □

3. Two inequalities for unitarily invariant norms

Bhatia and Kittaneh proved in [3] that if $A,B \in M_n$ are positive semidefinite, then

$$\left\|A^{3/2}B^{1/2}+A^{1/2}B^{3/2}\right\| \leqslant \frac{1}{2}\left\|(A+B)^2\right\|. \tag{3.1}$$

By the matrix arithmetic-geometric mean inequality [5, p. 263], we have

$$\left\|A^{1/2}B^{1/2}\right\| \leqslant \frac{1}{2}\|A+B\|. \tag{3.2}$$

It follows from the triangle inequality, (3.1) and (3.2) that

$$\left\|A^{3/2}B^{1/2}+A^{1/2}B^{3/2}+A^{1/2}B^{1/2}\right\| \leqslant \frac{1}{2}\left\|(A+B)^2\right\|+\frac{1}{2}\|A+B\|. \tag{3.3}$$

THEOREM 3.1. *Let $A,B \in M_n$ be positive semidefinite. Then*

$$\left\|A^{3/2}B^{1/2}+A^{1/2}B^{3/2}+A^{1/2}B^{1/2}\right\| \leqslant \frac{1}{2}\left\|(A+B)^2+A+B\right\|.$$

Proof. Let $X \in M_{2n}$ be defined by

$$X=\begin{bmatrix} A^{1/2} & 0 \\ B^{1/2} & 0 \end{bmatrix}.$$

Then

$$XX^*=\begin{bmatrix} A & A^{1/2}B^{1/2} \\ B^{1/2}A^{1/2} & B \end{bmatrix} \geqslant 0, X^*X=\begin{bmatrix} A+B & 0 \\ 0 & 0 \end{bmatrix} \geqslant 0.$$

Meanwhile, we have

$$(XX^*)^2=\begin{bmatrix} * & A^{3/2}B^{1/2}+A^{1/2}B^{3/2} \\ B^{1/2}A^{3/2}+B^{3/2}A^{1/2} & * \end{bmatrix}$$

and

$$(X^*X)^2=\begin{bmatrix} (A+B)^2 & 0 \\ 0 & 0 \end{bmatrix}.$$

So

$$XX^* + (XX^*)^2 = \begin{bmatrix} * & A^{3/2}B^{1/2} + A^{1/2}B^{3/2} + A^{1/2}B^{1/2} \\ B^{1/2}A^{3/2} + B^{3/2}A^{1/2} + B^{1/2}A^{1/2} & * \end{bmatrix} \geqslant 0.$$

Then, by Theorem 1 of [4], we have

$$s_j\left(A^{3/2}B^{1/2} + A^{1/2}B^{3/2} + A^{1/2}B^{1/2}\right) \leqslant \frac{1}{2}\lambda_j\left(XX^* + (XX^*)^2\right). \tag{3.4}$$

Note that there exists unitary matrix U such that $XX^* = UX^*XU^*$. So, we have

$$XX^* + (XX^*)^2 = U\left(X^*X + (X^*X)^2\right)U^*$$

and hence $XX^* + (XX^*)^2$ is unitarily equivalent to $X^*X + (X^*X)^2$. By (3.4), we have

$$s_j\left(A^{3/2}B^{1/2} + A^{1/2}B^{3/2} + A^{1/2}B^{1/2}\right) \leqslant \frac{1}{2}s_j\left((A+B)^2 + A + B\right).$$

By Fan's dominance principle [5, p. 93], we have

$$\left\|A^{3/2}B^{1/2} + A^{1/2}B^{3/2} + A^{1/2}B^{1/2}\right\| \leqslant \frac{1}{2}\left\|(A+B)^2 + A + B\right\|.$$

This completes the proof. □

Obviously, Theorem 3.1 is a refinement of the inequality (3.3).

THEOREM 3.2. *Let $A, B \in M_n$. Then*

$$\|A(A^*A + B^*B)B^*\| \leqslant \frac{1}{2}\left\|(AA^* + BB^*)^2\right\|.$$

Proof. Let $X \in M_{2n}$ be defined by

$$X = \begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix}.$$

Then

$$XX^* = \begin{bmatrix} AA^* & AB^* \\ BA^* & BB^* \end{bmatrix}, X^*X = \begin{bmatrix} AA^* + BB^* & 0 \\ 0 & 0 \end{bmatrix}.$$

Meanwhile, we have

$$(XX^*)^2 = \begin{bmatrix} * & A(A^*A + B^*B)B^* \\ B(A^*A + B^*B)A^* & * \end{bmatrix} \geqslant 0.$$

Then, by Theorem 1 of [4], we have

$$s_j(A(A^*A + B^*B)B^*) \leqslant \frac{1}{2}\lambda_j(XX^*)^2. \tag{3.5}$$

Since $(XX^*)^2$ is unitarily equivalent to $(X^*X)^2$, (3.5) is the same as follows:

$$s_j\left(A\left(A^*A+B^*B\right)B^*\right) \leqslant \frac{1}{2}s_j\left(AA^*+BB^*\right)^2.$$

By Fan's dominance principle [5, p. 93], we have

$$\|A\left(A^*A+B^*B\right)B^*\| \leqslant \frac{1}{2}\left\|\left(AA^*+BB^*\right)^2\right\|.$$

This completes the proof. □

If $A,B \in M_n$ are positive semidefinite, then by Theorem 3.2, we have the inequality (3.1).

4. Hölder type inequalities for unitarily invariant norms

Throughout this section we assume that $p,q>0$ and $1/p+1/q=1$. The classical Young's inequality for $a,b \geqslant 0$,

$$ab \leqslant \frac{a^p}{p}+\frac{b^q}{q},$$

was refined by Kittaneh and Manasrah [2] as follows:

$$ab+\frac{1}{s_0}\left(a^{p/2}-b^{q/2}\right)^2 \leqslant \frac{a^p}{p}+\frac{b^q}{q}, \tag{4.1}$$

where $s_0=\max\{p,q\}$.

Hölder's inequality for $A,B \in M_n$ and any unitarily invariant norm is given by

$$\|AB\| \leqslant \||A|^p\|^{1/p}\,\||B|^q\|^{1/q}$$

(see [5, p. 95]).

Let $A,B,X \in M_n$ with A,B positive semidefinite. The following more general Hölder type inequality valid for any $r>0$ was proved in [6, Theorem 3]:

$$\||AXB|^r\| \leqslant \||A^pX|^r\|^{1/p}\,\||XB^q|^r\|^{1/q}. \tag{4.2}$$

It was shown in [7] that under these assumptions on A,B,X,

$$\|AXB\| \leqslant \frac{1}{p}\|A^pX\|+\frac{1}{q}\|XB^q\|.$$

Kittaneh and Manasrah [2] improved this inequality to

$$\|AXB\|+\frac{1}{s_0}\left(\|A^pX\|^{1/2}-\|XB^q\|^{1/2}\right)^2 \leqslant \frac{1}{p}\|A^pX\|+\frac{1}{q}\|XB^q\|, \tag{4.3}$$

where $s_0=\max\{p,q\}$.

In view of the inequalities (4.1) and (4.2), we can generalize the inequality (4.3).

THEOREM 4.1. *Let $A,B,X \in M_n$ such that A and B are positive semidefinite. If $p,q,r > 0$ and $1/p + 1/q = 1$, then for every unitarily invariant norm*

$$|||AXB|^r\| + \frac{1}{s_0}\left(\||A^pX|^r\|^{1/2} - \||XB^q|^r\|^{1/2}\right)^2 \leqslant \frac{1}{p}\||A^pX|^r\| + \frac{1}{q}\||XB^q|^r\|,$$

where $s_0 = \max\{p,q\}$.

Proof. It follows from (4.1) and (4.2) that

$$\begin{aligned}
&\||AXB|^r\| + \frac{1}{s_0}\left(\||A^pX|^r\|^{1/2} - \||XB^q|^r\|^{1/2}\right)^2 \\
&\quad \leqslant \||A^pX|^r\|^{1/p}\||XB^q|^r\|^{1/q} + \frac{1}{s_0}\left(\||A^pX|^r\|^{1/2} - \||XB^q|^r\|^{1/2}\right)^2 \\
&\quad \leqslant \frac{1}{p}\||A^pX|^r\| + \frac{1}{q}\||XB^q|^r\|.
\end{aligned}$$

This completes the proof. □

COROLLARY 4.1. *Let $A,B \in M_n$. If $p,q > 0$ and $1/p + 1/q = 1$, then for every unitarily invariant norm*

$$\|AB\| + \frac{1}{s_0}\left(\||A|^p\|^{1/2} - \||B|^q\|^{1/2}\right)^2 \leqslant \frac{1}{p}\||A|^p\| + \frac{1}{q}\||B|^q\|, \tag{4.4}$$

where $s_0 = \max\{p,q\}$.

For all $A,B,C,D \in M_n$ and every unitarily invariant norm, Hiai and Zhan [8] obtained the following inequality

$$2^{|1/p-1/2|}\|C^*A + D^*B\| \leqslant \||A|^p + |B|^p\|^{1/p}\||C|^q + |D|^q\|^{1/q}$$

They also showed that, for $r \geqslant 1$,

$$2^{1/r-1}\|C^*A + D^*B\|_r \leqslant \||A|^p + |B|^p\|_r^{1/p}\||C|^q + |D|^q\|_r^{1/q}$$

Here, we give some similar inequalities.

THEOREM 4.2. *Let $A,B,C,D \in M_n$. If $p,q > 0$ and $1/p + 1/q = 1$, then for every unitarily invariant norm*

$$\|AC + BD\| \leqslant \frac{a}{p} + \frac{b}{q} - \frac{1}{s_0}\left(\sqrt{a} - \sqrt{b}\right)^2,$$

where

$$a = \left\|\left(|A|^2 + |B|^2\right)^{p/2}\right\|, \quad b = \left\|(|C|^q + |D|^q)^{q/2}\right\|, \quad s_0 = \max\{p,q\}.$$

Proof. Let $X,Y \in M_{2n}$ be defined by

$$X = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \text{ and } Y = \begin{bmatrix} C & 0 \\ D & 0 \end{bmatrix}.$$

Note that

$$\|AA^*\| = \|A^*A\| \text{ and } XY = \begin{bmatrix} AC+BD & 0 \\ 0 & 0 \end{bmatrix}.$$

It follows from (4.4) that

$$\begin{aligned} \|AC+BD\| &= \|XY\| \\ &\leqslant \frac{1}{p}\,\||X|^p\| + \frac{1}{q}\,\||Y|^q\| - \frac{1}{s_0}\left(\||X|^p\|^{1/2} - \||Y|^q\|^{1/2}\right)^2 \\ &= \frac{a}{p} + \frac{b}{q} - \frac{1}{s_0}\left(\sqrt{a} - \sqrt{b}\right)^2. \end{aligned}$$

This completes the proof. □

REMARK 4.1. For any matrix X and $r > 0$, $\||X^*|^r\| = \||X|^r\|$ [8, p. 161]. So, we have

$$\|AC^* + BD^*\| \leqslant \frac{a}{p} + \frac{b}{q} - \frac{1}{s_0}\left(\sqrt{a} - \sqrt{b}\right)^2,$$

$$\|A^*C + B^*D\| \leqslant \frac{a}{p} + \frac{b}{q} - \frac{1}{s_0}\left(\sqrt{a} - \sqrt{b}\right)^2,$$

$$\|A^*C^* + B^*D^*\| \leqslant \frac{a}{p} + \frac{b}{q} - \frac{1}{s_0}\left(\sqrt{a} - \sqrt{b}\right)^2.$$

THEOREM 4.3. *Let* $A,B,C,D \in M_n$*,* $p,q > 0$ *and* $1/p + 1/q = 1$*. Then for each unitarily invariant norm*

$$\|AC+BD\| \leqslant \frac{a_1}{p} + \frac{a_2}{q} - \frac{1}{s_0}\left(a_3^2 + a_4^2\right),$$

where,

$$a_1 = \||A|^p\| + \||B|^p\|, \quad a_2 = \||C|^q\| + \||D|^q\|,$$

$$a_3 = \||A|^p\|^{1/2} - \||C|^q\|^{1/2}, a_4 = \||B|^p\|^{1/2} - \||D|^q\|^{1/2}, s_0 = \max\{p,q\}.$$

Proof. Note that

$$\|AC+BD\| \leqslant \|AC\| + \|BD\|.$$

Using (4.4), we have

$$\|AC\| + \|BD\| \leqslant \frac{a_1}{p} + \frac{a_2}{q} - \frac{1}{s_0}\left(a_3^2 + a_4^2\right).$$

This completes the proof. □

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