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ON DIAZ–METCALF AND KLAMKIN–MCLENAGHAN TYPE OPERATOR INEQUALITIES

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Abstract. In this note, a result of M. S. Moslehian, R. Nakamoto and Y. Seo [Electron. J. Linear Algebra, 22 (2011) 179–190] on Diaz-Metcalf and Klamkin-McLenaghan type inequalities for positive definite operators is extended to operators having accretive transforms.

1. Introduction and summary

Throughout for a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$, we denote by $\mathbb{B}(H)$ the algebra of all bounded linear operators on H with the identity operator I on H.

For positive definite operators A and B on H, the *geometric mean* of A and B is defined by

$$A\sharp B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2} \tag{1}$$

(see [7]).

Recently, M. S. Moslehian, R. Nakamoto and Y. Seo [10, Theorem 2.1, part (i)] proved, among other results, the following:

Let H and K be complex Hilbert spaces. Let $A,B \in \mathbb{B}(H)$ be positive definite operators and $\Phi : \mathbb{B}(H) \to \mathbb{B}(K)$ be a positive linear map. If

$$m^2A \leqslant B \leqslant M^2A \quad \text{for some positive real numbers } m < M, \tag{2}$$

then the following inequalities hold:

(i) *operator Diaz-Metcalf (D-M) type inequality*:

$$Mm\,\Phi(A) + \Phi(B) \leqslant (M+m)\,\Phi(A\sharp B), \tag{3}$$

(ii) *operator Klamkin-McLenaghan (K-L) type inequality*:

$$\Phi(A\sharp B)^{-1/2}\Phi(B)\Phi(A\sharp B)^{-1/2} - \Phi(A\sharp B)^{1/2}\Phi(A)^{-1}\Phi(A\sharp B)^{1/2} \leqslant (\sqrt{M}-\sqrt{m})^2 I. \tag{4}$$

(Here and in the sequel, $\Phi(C)^p$ means $(\Phi(C))^p$ for an operator C and exponent p.)

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Also, M. S. Moslehian, R. Nakamoto and Y. Seo [10, Theorem 2.1, part (ii)] showed that if

$$m_1^2 I \leqslant A \leqslant M_1^2 I \quad \text{and} \quad m_2^2 I \leqslant B \leqslant M_2^2 I$$

for some positive numbers $m_1 < M_1$ and $m_2 < M_2$, then the following inequalities hold

(iii) *operator Diaz-Metcalf (D-M) type inequality*:

$$\frac{M_2 m_2}{M_1 m_1}\Phi(A) + \Phi(B) \leqslant \left(\frac{M_2}{m_1} + \frac{m_2}{M_1}\right) \Phi(A\sharp B), \tag{5}$$

(iv) *operator Shisha-Mond (S-M) type inequality*:

$$\Phi(A\sharp B)^{-1/2}\Phi(B)\Phi(A\sharp B)^{-1/2} - \Phi(A\sharp B)^{1/2}\Phi(A)^{-1}\Phi(A\sharp B)^{1/2} \leqslant \left(\sqrt{\frac{M_2}{m_1}} - \sqrt{\frac{m_2}{M_1}}\right)^2 I. \tag{6}$$

In light of (1) and (2), the positive operator $Z = (A^{-1/2}BA^{-1/2})^{1/2}$ with $mI \leqslant Z \leqslant MI$ for positive scalars m, M, plays a key role in inequalities (3)–(4).

In this paper, our purpose is to extend (3)–(4) by using some other Z's.

In Section 2, we generalize (3)–(4) by employing an operator $Z \in \mathbb{B}(H)$ such that $Z^*Z = A^{-1/2}BA^{-1/2}$ and

$$\operatorname{Re}(Z - mI)^*(MI - Z) \geqslant 0 \tag{7}$$

for some complex scalars $m, M \in \mathbb{C}$. The condition (7) says that the transform $(Z - mI)^*(MI - Z)$ of Z is *accretive* [4, 5]. (See [2, 3, 4, 5, 6, 9, 10, 11, 12, 13, 14] for applications of (7) in deriving inequalities.) In Theorem 2.1 we show that the accretivity is equivalent to an operator D-M type inequality and to a preliminary version of K-L inequality. In our approach we use complex scalars m and M and arbitrary (not necessarily Hermitian) operators Z. A special case of Theorem 2.1 for positive scalars m and M is demonstrated in Corollary 2.2.

In Section 3, we present further generalizations of (3)–(4). Here we replace the map $A^{-1/2}(\cdot)A^{-1/2}$ by an arbitrary invertible strictly positive map Ψ having strictly positive inverse Ψ^{-1}. Moreover, in place of (7) we apply the condition

$$\operatorname{Re}(Z - mW)^*(MW - Z) \geqslant 0 \tag{8}$$

with $\Psi(A) = W^*W$ and $\Psi(B) = Z^*Z$ (see Theorem 3.1).

Finally, in Corollary 3.3 we give a specialization of Theorem 3.1 for positive scalars m and M.

2. Accretive operators and D-M and K-L type inequalities

For operators $X,Y \in \mathbb{B}(H)$, we write $Y \leqslant X$ (resp., $Y < X$) if $X - Y$ is positive semidefinite (resp., positive definite).

An operator $C \in \mathbb{B}(H)$ is said to be *accretive* (resp., *strictly accretive*) if $\mathrm{Re}\,(C) \geqslant 0$ (resp., $\mathrm{Re}\,(C) > 0$), where the symbol $\mathrm{Re}\,(C)$ stands for $\frac{1}{2}(C + C^*)$, and $C^* \in \mathbb{B}(H)$ is the adjoint of C in the sense that $\langle Cx, y\rangle = \langle x, C^*y\rangle$ for all $x,y \in H$ [4, p. 2753].

For an operator $Z \in \mathbb{B}(H)$ and scalars $m,M \in \mathbb{C}$, we denote

$$C_{m,M}(Z) = (Z - mI)^*(MI - Z) \tag{9}$$

(see [4, p. 2752]). It follows that

$$C_{m,M}(Z) \text{ is accretive } \text{ iff } \mathrm{Re}\,\langle h, (Z - mI)^*(MI - Z)h\rangle \geqslant 0 \text{ for all } h \in H \tag{10}$$

(see [11, 12]). Hereafter $\mathrm{Re}\, z = \frac{1}{2}(z + \overline{z})$ stands for the realis of a complex number z.

For scalars $m,M \in \mathbb{C}$ such that $\mathrm{Re}\,(M + m) \neq 0$, we define

$$\alpha_{m,M} = \frac{\overline{M + m}}{|\mathrm{Re}\,(M + m)|}. \tag{11}$$

For positive definite operators $A,B \in \mathbb{B}(H)$ and an operator $Z \in \mathbb{B}(H)$ satisfying

$$Z^*Z = A^{-1/2}BA^{-1/2}, \tag{12}$$

we denote

$$A\sharp_Z B = A^{1/2}ZA^{1/2} \tag{13}$$

(see [14, p. 3]).

We now discuss properties of the binary operation $\sharp_Z$. It is easily seen that $A\sharp_Z B = A\sharp B$ whenever $Z > 0$. Thus $\sharp_Z$ reduces to the geometric mean $\sharp$ for positive definite Z.

It is evident that $A\sharp_Z B$ strongly depends on Z satisfying (12). For instance, observe that (12) is fulfilled for $Z = U(A^{-1/2}BA^{-1/2})^{1/2}$ with unitary $U \in \mathbb{B}(H)$. In consequence,

$$A\sharp_Z B = A^{1/2}U(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}. \tag{14}$$

Clearly, if A and U commute then $A\sharp_Z B = UA\sharp B$.

If $B = A$ or $B = A^{-1}$ then (14) gives $A\sharp_Z A = A^{1/2}UA^{1/2}$ and $A\sharp_Z A^{-1} = A^{1/2}UA^{-1/2}$, respectively.

So, in general, $A\sharp_Z B$ is not positive definite for $A,B > 0$. Therefore $A\sharp_Z B$ is not a "mean" in the usual meaning.

It is known that

$$A\sharp B \leqslant \frac{M + m}{2\sqrt{Mm}}\mathrm{Re}\,(A\sharp_Z B)$$

whenever $0 < m \leqslant M$ and $C_{m,M}(Z)$ is accretive (see [14, Theorem 2.1]).

We return to definitions. For $\alpha \in \mathbb{C}$, we introduce

$$G_{\alpha,Z}(A,B) = \mathrm{Re}\,(\alpha A\sharp_Z B). \tag{15}$$

It follows that $G_{\alpha_{m,M},Z}(A,B) = A\sharp B$ whenever $Z > 0$ with (12) and $M+m$ is positive.

Let H and K be complex Hilbert spaces. A linear map $\Phi : \mathbb{B}(H) \to \mathbb{B}(K)$ is said to be *positive* if $0 \leqslant \Phi(X)$ whenever $0 \leqslant X \in \mathbb{B}(H)$. If in addition $0 < \Phi(X)$ for $0 < X \in \mathbb{B}(H)$, then Φ is said to be *strictly positive*.

In the forthcoming theorem we show the equivalency between the accretivity of the transform $C_{m,M}(Z)$ satisfying (12) and inequalities of Diaz-Metcalf and pre- Klamkin-McLenaghan's type. This theorem is motivated by [6, Theorem 2], [10, Theorem 2.1], [12, Theorem 1.1, Proposition 2.1] and [14, Theorem 2.1].

THEOREM 2.1. *Let H be a complex Hilbert space. Let $A,B \in \mathbb{B}(H)$ be positive definite operators and $m,M \in \mathbb{C}$ be scalars with $\operatorname{Re}(\overline{m}M) > 0$ and $\operatorname{Re}(M+m) \neq 0$. Let $Z \in \mathbb{B}(H)$ satisfy $Z^*Z = A^{-1/2}BA^{-1/2}$.*

The following three statements **(i)–(iii)** *are equivalent:*

(i) *The operator $C_{m,M}(Z)$ is accretive.*

(ii) *For any strictly positive linear map $\Phi : \mathbb{B}(H) \to \mathbb{B}(K)$ with a complex Hilbert space K, the following Diaz-Metcalf type inequality holds*

$$\operatorname{Re}(\overline{m}M)\,\Phi(A) + \Phi(B) \leqslant \Phi\left(\operatorname{Re}(\overline{M+m}A\sharp_Z B)\right). \tag{16}$$

(iii) *For any strictly positive linear map $\Phi : \mathbb{B}(H) \to \mathbb{B}(K)$ with a complex Hilbert space K, and for the operator*

$$G = \operatorname{Re}(\alpha_{m,M}A\sharp_Z B) \quad \textit{with} \quad \alpha_{m,M} = \frac{\overline{M+m}}{|\operatorname{Re}(M+m)|}, \tag{17}$$

the operator $\Phi(G) \in \mathbb{B}(K)$ is positive definite, and the following pre-Klamkin-McLenaghan type inequality holds

$$\begin{aligned} &\Phi(G)^{-1/2}\Phi(B)\Phi(G)^{-1/2} - \Phi(G)^{1/2}\Phi(A)^{-1}\Phi(G)^{1/2} \\ &\leqslant \left(|\operatorname{Re}(M+m)| - 2\sqrt{\operatorname{Re}(\overline{m}M)}\right)I - \left(\sqrt{\operatorname{Re}(\overline{m}M)}X^{1/2} - X^{-1/2}\right)^2, \end{aligned} \tag{18}$$

where $X = \Phi(G)^{-1/2}\Phi(A)\Phi(G)^{-1/2}$.

In consequence, each of the equivalent statements **(i)–(iii)** *implies the following Klamkin-McLenaghan type inequality.*

(iv) *For any strictly positive linear map $\Phi : \mathbb{B}(H) \to \mathbb{B}(K)$ with a complex Hilbert space K,*

$$\begin{aligned} &\Phi(G)^{-1/2}\Phi(B)\Phi(G)^{-1/2} - \Phi(G)^{1/2}\Phi(A)^{-1}\Phi(G)^{1/2} \\ &\leqslant \left(|\operatorname{Re}(M+m)| - 2\sqrt{\operatorname{Re}(\overline{m}M)}\right)I, \end{aligned} \tag{19}$$

where G is defined by (17).

Proof. **(i)** $\Rightarrow$ **(ii)**. In a similar manner as in the proof of [14, Theorem 2.1], one can get the following Diaz-Metcalf type inequality

$$\mathrm{Re}\,(\overline{m}M)\,A+B\leqslant \mathrm{Re}\,(\overline{M+m}A\sharp_Z B). \tag{20}$$

Now, it is obvious that (20) gives (16), as required.

(ii) $\Rightarrow$ **(iii)**. Applying (16) for $K=H$ and $\Phi=$ the identity on H, we obtain (20). By pre- and post-multiplying both sides of inequality (20) by $A^{-1/2}$, we get

$$\mathrm{Re}\,(\overline{m}M)\,I+Z^*Z\leqslant \mathrm{Re}\,(\overline{M+m}Z). \tag{21}$$

However $\mathrm{Re}\,(\overline{m}M)>0$, so we may easily deduce

$$0<\mathrm{Re}\,(\overline{m}M)\,I+Z^*Z.$$

This and (21) directly imply that the operator $\overline{M+m}Z$ is strictly accretive, i.e.,

$$0<\mathrm{Re}\,(\overline{M+m}Z). \tag{22}$$

Fix any complex Hilbert space K and any strictly positive linear map $\Phi:\mathbb{B}(H)\to\mathbb{B}(K)$. From (22) we get

$$0<\Phi(A^{1/2}\mathrm{Re}\,(\overline{M+m}Z)A^{1/2})=\Phi(\mathrm{Re}\,(\overline{M+m}A^{1/2}ZA^{1/2})).$$

Thus we have

$$0<\Phi(\mathrm{Re}\,(\overline{M+m}A\sharp_Z B))=|\mathrm{Re}\,(M+m)|\Phi(G). \tag{23}$$

Because $0<|\mathrm{Re}\,(M+m)|$, we obtain

$$0<\Phi(G), \tag{24}$$

completing proof of the first part of **(iii)**.

It follows from (16) and (23) that

$$\Phi(B)\leqslant|\mathrm{Re}\,(M+m)|\Phi(G)-\mathrm{Re}\,(\overline{m}M)\,\Phi(A). \tag{25}$$

By pre- and post-multiplying both sides of the inequality (25) by $\Phi(G)^{-1/2}$, we get

$$\begin{aligned}&\Phi(G)^{-1/2}\Phi(B)\Phi(G)^{-1/2}\\ &\leqslant|\mathrm{Re}\,(M+m)|I-\mathrm{Re}\,(\overline{m}M)\Phi(G)^{-1/2}\Phi(A)\Phi(G)^{-1/2}.\end{aligned} \tag{26}$$

Denoting

$$L=\Phi(G)^{-1/2}\Phi(B)\Phi(G)^{-1/2}-\Phi(G)^{1/2}\Phi(A)^{-1}\Phi(G)^{1/2}, \tag{27}$$

we may rewrite (26) in the form

$$L\leqslant|\mathrm{Re}\,(M+m)|I-\mathrm{Re}\,(\overline{m}M)X-X^{-1}, \tag{28}$$

where $X = \Phi(G)^{-1/2}\Phi(A)\Phi(G)^{-1/2}$.

We find that

$$\operatorname{Re}(\overline{m}M)X + X^{-1} = \left(\sqrt{\operatorname{Re}(\overline{m}M)}X^{1/2} - X^{-1/2}\right)^2 + 2\sqrt{\operatorname{Re}(\overline{m}M)}I.$$

Combining this and (28) gives

$$L \leqslant \left(|\operatorname{Re}(M+m)| - 2\sqrt{\operatorname{Re}(\overline{m}M)}\right)I - \left(\sqrt{\operatorname{Re}(\overline{m}M)}X^{1/2} - X^{-1/2}\right)^2. \tag{29}$$

By making use of (27) and (29) we obtain (18).

This completes the proof of the implication **(ii)** $\Rightarrow$ **(iii)**.

As the proof of implications **(i)** $\Rightarrow$ **(ii)** $\Rightarrow$ **(iii)** can be easily reversed, we obtain the validity of implications **(iii)** $\Rightarrow$ **(ii)** $\Rightarrow$ **(i)**.

The implication **(iii)** $\Rightarrow$ **(iv)** is obvious. □

The next result is a direct consequence of Theorem 2.1 for positive scalars m and M.

COROLLARY 2.2. *Under the hypotheses of Theorem 2.1, if in addition m and M are positive then the following three statements* **(i')–(iii')** *are equivalent:*

(i') *The operator $C_{m,M}(Z)$ is accretive.*

(ii') *For any strictly positive linear map $\Phi : \mathbb{B}(H) \to \mathbb{B}(K)$ with a complex Hilbert space K, the following Diaz-Metcalf type inequality holds*

$$mM\,\Phi(A) + \Phi(B) \leqslant (M+m)\,\Phi(\operatorname{Re}(A\sharp_Z B)). \tag{30}$$

(iii') *For any strictly positive linear map $\Phi : \mathbb{B}(H) \to \mathbb{B}(K)$ with a complex Hilbert space K, the operator $\Phi(G) \in \mathbb{B}(K)$ is positive definite, and the following pre-Klamkin-McLenaghan type inequality holds*

$$\begin{aligned}&\Phi(G)^{-1/2}\Phi(B)\Phi(G)^{-1/2} - \Phi(G)^{1/2}\Phi(A)^{-1}\Phi(G)^{1/2}\\ &\leqslant \left(\sqrt{M} - \sqrt{m}\right)^2 I - \left(\sqrt{mM}X^{1/2} - X^{-1/2}\right)^2,\end{aligned} \tag{31}$$

where $G = \operatorname{Re}(A\sharp_Z B)$ and $X = \Phi(G)^{-1/2}\Phi(A)\Phi(G)^{-1/2}$.

In consequence, each of the equivalent statements **(i')–(iii')** *implies the following Klamkin-McLenaghan type inequality.*

(iv') *For any strictly positive linear map $\Phi : \mathbb{B}(H) \to \mathbb{B}(K)$ with a complex Hilbert space K,*

$$\begin{aligned}&\Phi(G)^{-1/2}\Phi(B)\Phi(G)^{-1/2} - \Phi(G)^{1/2}\Phi(A)^{-1}\Phi(G)^{1/2}\\ &\leqslant \left(\sqrt{M} - \sqrt{m}\right)^2 I,\end{aligned} \tag{32}$$

where $G = \operatorname{Re}(A\sharp_Z B)$.

3. Further generalizations of D-M and K-L inequalities

For operators $Z, W \in \mathbb{B}(H)$ and scalars $m, M \in \mathbb{C}$, we denote

$$C_{m,M}(Z,W) = (Z - mW)^*(MW - Z). \tag{33}$$

THEOREM 3.1. *Let H be a complex Hilbert space. Let $A, B \in \mathbb{B}(H)$ be positive definite operators and $m, M \in \mathbb{C}$ be scalars with $\mathrm{Re}(\overline{m}M) > 0$ and $\mathrm{Re}(M+m) \neq 0$. Assume $\Psi : \mathbb{B}(H) \to \mathbb{B}(H)$ is an invertible linear operator such that Ψ and Ψ^{-1} are strictly positive linear maps.*

If $W, Z \in \mathbb{B}(H)$ are linear operators satisfying $\Psi(A) = W^*W$ and $\Psi(B) = Z^*Z$, then the following three statements **(i)–(iii)** *are equivalent:*

(i) *The operator $C_{m,M}(Z,W)$ is accretive.*

(ii) *For any strictly positive linear map $\Phi : \mathbb{B}(H) \to \mathbb{B}(K)$ with a complex Hilbert space K, the following Diaz-Metcalf type inequality holds*

$$\mathrm{Re}(\overline{m}M)\Phi(A) + \Phi(B) \leqslant \Phi\left(\Psi^{-1}(\mathrm{Re}(\overline{M+m}W^*Z))\right). \tag{34}$$

(iii) *For any strictly positive linear map $\Phi : \mathbb{B}(H) \to \mathbb{B}(K)$ with a complex Hilbert space K, and for the operator*

$$G = \Psi^{-1}\mathrm{Re}(\alpha_{m,M} W^*Z) \quad \text{with} \quad \alpha_{m,M} = \frac{\overline{M+m}}{|\mathrm{Re}(M+m)|}, \tag{35}$$

the operator $\Phi(G) \in \mathbb{B}(K)$ is positive definite, and the following pre-Klamkin-McLenaghan type inequality holds

$$\begin{aligned} &\Phi(G)^{-1/2}\Phi(B)\Phi(G)^{-1/2} - \Phi(G)^{1/2}\Phi(A)^{-1}\Phi(G)^{1/2} \\ &\leqslant \left(|\mathrm{Re}(M+m)| - 2\sqrt{\mathrm{Re}(\overline{m}M)}\right)I - \left(\sqrt{\mathrm{Re}(\overline{m}M)}X^{1/2} - X^{-1/2}\right)^2, \end{aligned} \tag{36}$$

where $X = \Phi(G)^{-1/2}\Phi(A)\Phi(G)^{-1/2}$.

In consequence, each of the equivalent statements **(i)–(iii)** *implies the following Klamkin-McLenaghan type inequality.*

(iv) *For any strictly positive linear map $\Phi : \mathbb{B}(H) \to \mathbb{B}(K)$ with a complex Hilbert space K,*

$$\begin{aligned} &\Phi(G)^{-1/2}\Phi(B)\Phi(G)^{-1/2} - \Phi(G)^{1/2}\Phi(A)^{-1}\Phi(G)^{1/2} \\ &\leqslant \left(|\mathrm{Re}(M+m)| - 2\sqrt{\mathrm{Re}(\overline{m}M)}\right)I, \end{aligned} \tag{37}$$

where G is defined by (35).

Proof. Equivalences **(i)** $\Leftrightarrow$ **(ii)** $\Leftrightarrow$ **(iii)** can be easily proved by an analogous method as in the proof of Theorem 2.1. $\square$

REMARK 3.2. The case of Theorem 3.1 for $\Psi = A^{-1/2}(\cdot)A^{-1/2}$ and $W = I$ (the identity operator on H) leads to Theorem 2.1. In this context, inequalities (34), (36) and (37) reduce to (16), (18) and (19), respectively, because $\Psi^{-1} = A^{1/2}(\cdot)A^{1/2}$ and

$$G = \Psi^{-1}\mathrm{Re}\,(\alpha_{m,M}\,W^*Z) = A^{1/2}\mathrm{Re}\,(\alpha_{m,M}Z)A^{1/2} = \mathrm{Re}\,(\alpha_{m,M}A\sharp_Z B).$$

By making use of Theorem 3.1 for positive scalars m and M we obtain

COROLLARY 3.3. *Under the hypotheses of Theorem 3.1, if in addition m and M are positive then the following three statements* **(i')–(iii')** *are equivalent:*

(i') *The operator $C_{m,M}(Z,W)$ is accretive.*

(ii') *For any strictly positive linear map $\Phi : \mathbb{B}(H) \to \mathbb{B}(K)$ with a complex Hilbert space K, the following Diaz-Metcalf type inequality holds*

$$mM\,\Phi(A) + \Phi(B) \leqslant (M+m)\Phi\left(\Psi^{-1}\mathrm{Re}\,(W^*Z)\right). \tag{38}$$

(iii') *For any strictly positive linear map $\Phi : \mathbb{B}(H) \to \mathbb{B}(K)$ with a complex Hilbert space K, the operator $\Phi(G) \in \mathbb{B}(K)$ is positive definite, and the following pre-Klamkin-McLenaghan type inequality holds*

$$\begin{aligned} &\Phi(G)^{-1/2}\Phi(B)\Phi(G)^{-1/2} - \Phi(G)^{1/2}\Phi(A)^{-1}\Phi(G)^{1/2} \\ &\leqslant \left(\sqrt{M}-\sqrt{m}\right)^2 I - \left(\sqrt{mM}X^{1/2} - X^{-1/2}\right)^2, \end{aligned} \tag{39}$$

where $G = \Psi^{-1}\mathrm{Re}\,(W^*Z)$ and $X = \Phi(G)^{-1/2}\Phi(A)\Phi(G)^{-1/2}$.

In consequence, each of the equivalent statements **(i')-(iii')** *implies the following Klamkin-McLenaghan type inequality.*

(iv') *For any strictly positive linear map $\Phi : \mathbb{B}(H) \to \mathbb{B}(K)$ with a complex Hilbert space K,*

$$\begin{aligned} &\Phi(G)^{-1/2}\Phi(B)\Phi(G)^{-1/2} - \Phi(G)^{1/2}\Phi(A)^{-1}\Phi(G)^{1/2} \\ &\leqslant \left(\sqrt{M}-\sqrt{m}\right)^2 I, \end{aligned} \tag{40}$$

where $G = \Psi^{-1}\mathrm{Re}\,(W^*Z)$.

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