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INEQUALITIES INVOLVING MULTIVARIATE CONVEX FUNCTIONS IV

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Abstract. This paper deals with the inequalities involving logarithmically convex functions of several variables. The results here provide generalizations of inequalities for univariate functions obtained by Dragomir and Dragomir and Mond.

1. Introduction

Convex functions of one or several variables play an important role in many areas of pure and applied mathematics (see, e.g., [11]). In this paper we shall establish several new inequalities involving logarithmically convex functions of several variables.

Let $I = [a,b]$ be a proper subinterval of the number line. A function $g : I \to \mathbb{R}$ is said to be convex if $g[(1-\lambda)x+\lambda y)] \leqslant (1-\lambda)g(x)+\lambda g(y)$ holds for all $x, y \in I$ and $0 \leqslant \lambda \leqslant 1$. An important inequality for univariate convex functions has been obtained by Hermite and Hadamard. It reads as follows

$$g\left(\frac{x+y}{2}\right) \leqslant \frac{1}{x-y}\int_y^x g(t)\,dt \leqslant \frac{g(x)+g(y)}{2} \tag{1.1}$$

(see, e.g., [11, p. 137]). Many generalizations and refinements of this result have been obtained in recent years. The interested reader is referred to the monograph [5].

In order to present one of these results let us introduce more notation. By

$$E_n = \left\{u = (u_0,\ldots,u_n) : u_i \geqslant 0 \;\; (0 \leqslant i \leqslant n),\;\; u_0+\ldots+u_n = 1\right\},$$

$n \geqslant 1$, we will denote the Euclidean simplex. In what follows we will always choose $u_0 = 1-(u_1+\ldots+u_n)$. Further, let μ stand for a probability measure on E_n. In what follows the weights w_i $(0 \leqslant i \leqslant n)$ of the measure μ are the natural weights. They are defined as follows

$$w_i = \int_{E_n} u_i\,d\mu(u). \tag{1.2}$$

Clearly all the weights w_i are nonnegative and $w_0+\ldots+w_n = 1$.

An important subfamily of the class of convex functions, called the logarithmically convex (log-convex) functions, have been found of interest in the mathematical

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statistics, [11], theory of special functions [1], and theory of means [10], to name a few areas. Recall that a function $g : I \to (0,\infty)$ is log-convex if

$$g[(1-\lambda)x+\lambda y] \leqslant [g(x)]^{1-\lambda}[g(y)]^{\lambda}$$

holds for all $x, y \in I$ and $0 \leqslant \lambda \leqslant 1$. Since the log-convex functions are also convex, they satisfy the Hermite-Hadamard inequality (1.1).

Some refinements of the inequality (1.1) have been obtained by S. Dragomir and B. Mond in [4]. They have proven that any log-convex function g satisfies the inequalities

$$\begin{aligned} g\left(\frac{a+b}{2}\right) &\leqslant \exp\left[\frac{1}{b-a}\int_a^b \ln[g(x)]\,dx\right] \\ &\leqslant \frac{1}{b-a}\int_a^b \sqrt{g(x)g(a+b-x)}\,dx \\ &\leqslant \frac{1}{b-a}\int_a^b g(x)\,dx \leqslant L[g(a),g(b)] \leqslant \frac{g(a)+g(b)}{2}, \end{aligned} \tag{1.3}$$

where $L(p,q) = \frac{p-q}{\ln p - \ln q}$ $(p \neq q)$ and $L(p,p) = p$ is the logarithmic mean of $p > 0$ and $q > 0$.

Another refinement of the first inequality in (1.1) appears in [3, Theorem 1]. Let $A = (a+b)/2$. If the function g is log-convex and differentiable on $\operatorname{Int}(I)$, then

$$\begin{aligned} &\frac{1}{b-a}\int_a^b g(x)\,dx/g(A) \\ &\quad \geqslant L\left(\exp\left[\frac{g'(A)}{g(A)}\left(\frac{b-a}{2}\right)\right], \exp\left[-\frac{g'(A)}{g(A)}\left(\frac{b-a}{2}\right)\right]\right) \geqslant 1. \end{aligned} \tag{1.4}$$

This paper is the fourth in the series of papers devoted to the study of inequalities for the multivariate convex functions (see [9], [6], and [8]). The goal of the present paper is to obtain generalizations of the inequalities (1.3) and (1.4) for the log-convex functions of several variables. Notation and definitions are introduced in Section 2. The main results of this paper are presented in Section 3.

2. Notation and Definitions

Let U be an open subset of $\mathbb{R}^k$ $(k \geqslant 1)$ and let $x^0, \ldots, x^n \in U$ $(n \geqslant k)$. Further, let $X = [x^0, \ldots, x^n]$ be a k by $(n+1)$ matrix whose columns are the vectors $x^0, \ldots, x^n$ and let $\sigma = \operatorname{conv}(X)$ denote the convex hull of the columns of X, i.e.,

$$\sigma = \left\{ x \in \mathbb{R}^k : x = \sum_{i=0}^{n} u_i x^i, \quad (u_0, \ldots, u_n) \in E_n \right\}.$$

Clearly $\sum_{i=0}^{n} u_i x^i = Xu$ for $u = (u_0, \ldots, u_n) \in E_n$. In what follows we will always assume that the columns of X span a *proper* simplex in $\mathbb{R}^k$, i.e., that $\operatorname{vol}(\sigma) \neq 0$. When $k = n$, then $\operatorname{vol}(\sigma) = (\det A)/n!$, where $A = [x^1 - x^0, \ldots, x^n - x^0]$ is the n by n matrix.

The generalized simplex spline $M_\mu(\cdot|X)$ can be realized as the kernel of the distribution

$$\int_{E_n} f(Xu)\,d\mu(u), \quad f \in C_0^\infty(\mathbb{R}^k),$$

i.e., $M_\mu(\cdot|X)$ satisfies

$$\int_{E_n} f(Xu)\,d\mu(u) = \int_\sigma f(x) M_\mu(x|X)\,dx \tag{2.1}$$

(see [2]). Here $x \in \mathbb{R}^k$, $dx = dx_1 \cdots dx_k$. When $n = k$, then we have

$$M_\mu(x|X) = \begin{cases} \dfrac{\mu(A^{-1}(x-x^0))}{n!|\mathrm{vol}(\sigma)|}, & x \in \sigma \\ 0, & \text{otherwise} \end{cases} \tag{2.2}$$

(see [6, Lemma 3.1]). It is worth mentioning that for $n \geqslant k$, $\operatorname{supp} M_\mu(\cdot|X) = \sigma$ and $\int\limits_\sigma M_\mu(x|X)\,dx = 1$.

A function $f : U \to (0,\infty)$ is said to be log-convex if

$$f[(1-\lambda)x + \lambda y] \leqslant [f(x)]^{1-\lambda}[f(y)]^\lambda$$

holds for all $x,y \in U$ and $0 \leqslant \lambda \leqslant 1$. Any log-convex function f also satisfies the inequality

$$f(Xu) \leqslant \prod_{i=0}^n [f(x^i)]^{u_i}, \tag{2.3}$$

where $u = (u_0,\ldots,u_n) \in E_n$. This is a consequence of application of Jensen's inequality for multivariate functions (see [9]) to the function $\ln(f)$.

In what follows, the inner product of $x,y \in \mathbb{R}^{n+1}$ will be denoted by $x \cdot y$, i.e., $x \cdot y = x_0y_0 + \ldots + x_ny_n$. Also, we will use the weighted logarithmic mean $\mathcal{L}_\mu(a)$ of $a = (a_0,\ldots,a_n)$, $a_i > 0$, $0 \leqslant i \leqslant n$. Following [7] we define

$$\mathcal{L}_\mu(a) = \int_{E_n} \prod_{i=0}^n a_i^{u_i}\,d\mu(u) = \int_{E_n} \exp(u \cdot \ln(a))\,d\mu(u), \tag{2.4}$$

where $\ln a := (\ln a_0,\ldots,\ln a_n)$. It is known ([7]) that the logarithmic mean interpolates the inequality of the arithmetic and geometric means, i.e.,

$$\prod_{i=0}^n a_i^{w_i} \leqslant \mathcal{L}_\mu(a) \leqslant \sum_{i=0}^n w_i a_i, \tag{2.5}$$

where the w_i's are defined in (1.2). If $\mu(u) = n!$ – the Lebesgue measure on E_n, then

$$\mathcal{L}_\mu(a) = n![\ln a_0,\ldots,\ln a_n]e^t, \tag{2.6}$$

where $[\ln a_0,\dots,\ln a_n]e^t$ stands for the divided difference of order n of $\exp(t)$ (see [7, (4.21)]). If in addition the variables $a_0,\dots,a_n$ are pairwise distinct, i.e., if $a_i \neq a_j$ for all $i \neq j$, then (2.6) can be written as

$$\mathscr{L}_\mu(a) = n!\sum_{i=0}^{n}\left[a_i\Big/\prod_{\substack{j=0\\ j\neq i}}^{n}\ln(a_i/a_j)\right] \tag{2.7}$$

(see [7, p. 899]).

3. Refinements of the Hermite-Hadamard inequality for the multivariate log-convex functions

For later use we recall the following result [6, Theorem 4.2]. Let $g:\sigma\to\mathbb{R}$ be a convex function, $x^0,\dots,x^n\in U$, and let the weights w_i $(0\leqslant i\leqslant n)$ be the same as in (1.2). Then

$$g\left(\sum_{i=0}^{n} w_i x^i\right) \leqslant \int_{E_n} g(Xu)\,d\mu(u) \leqslant \sum_{i=0}^{n} w_i g(x^i) \tag{3.1}$$

It is worth mentioning that if $n=1$ and if $x^0 =: y$ and $x^1 =: x$, then (3.1) becomes (1.1).

Our first result reads as follows.

THEOREM 3.1. *Let $f:\sigma\to(0,\infty)$ $(\mathrm{vol}(\sigma)\neq 0)$ be a log-convex function. Then*

$$\begin{aligned} f\left(\sum_{i=0}^{n} w_i x^i\right) &\leqslant \exp\left(\int_\sigma [\ln f(x)] M_\mu(x|X)\,dx\right)\\ &\leqslant \int_\sigma f(x) M_\mu(x|X)\,dx\\ &\leqslant \mathscr{L}_\mu\big(f(x^0),\dots,f(x^n)\big) \leqslant \sum_{i=0}^{n} w_i f(x^i). \end{aligned} \tag{3.2}$$

Proof. In order to establish the first inequality in (3.2) we utilize the first inequality in (3.1) with g replaced by $\ln f$ and next employ (2.1) to obtain

$$\ln f\left(\sum_{i=0}^{n} w_i x^i\right) \leqslant \int_{E_n} \ln f(Xu)\,d\mu(u) = \int_\sigma [\ln f(x)] M_\mu(x|X)\,dx.$$

Application of Jensen's inequality for integrals to the second member of (3.2) gives

$$\exp\left(\int_\sigma [\ln f(x)] M_\mu(x|X)\,dx\right) \leqslant \int_\sigma f(x) M_\mu(x|X)\,dx.$$

The third inequality in (3.2) can be established as follows. First we apply (2.1) to the second member of (3.1), with g replaced by f, next we utilize logarithmic convexity of f (see (2.3)) followed by application of (2.4) to obtain

$$\int_\sigma f(x)M_\mu(x|X)\,dx = \int_{E_n} f(Xu)\,d\mu(u) \leqslant \int_{E_n}\prod_{i=0}^{n}[f(x^i)]^{u_i}\,d\mu(u)$$
$$= \mathscr{L}_\mu\big(f(x^0),\dots,f(x^n)\big).$$

The last inequality in (3.2) follows from the second one in (2.5).

The Hermite-Hadamard inequality for the multivariate convex functions is obtained in [9, Corollary 3.1]. Its refinements for the multivariate log-convex functions read as follow.

COROLLARY 3.2. *Let $k=n$. If $f:\sigma\to(0,\infty)$ is a logarithmically convex function, then*

$$\begin{aligned} f\left(\frac{x^0+\ldots+x^n}{n+1}\right) &\leqslant \exp\left(\frac{1}{|\mathrm{vol}\,(\sigma)|}\int_\sigma \ln f(x)\,dx\right)\\ &\leqslant \frac{1}{|\mathrm{vol}\,(\sigma)|}\int_\sigma f(x)\,dx \\ &\leqslant n![\ln f(x^0),\dots,\ln f(x^n)]e^t \leqslant \frac{1}{n+1}\sum_{i=0}^{n} f(x^i). \end{aligned} \tag{3.3}$$

Proof. Let $\mu(u)=n!$ $(u\in E_n)$. Then $M_\mu(x|X)=1/|\mathrm{vol}\,(\sigma)|$ for $x\in\sigma$ and $M_\mu(x|X)=0$ otherwise (see (2.2)). Also, $w_i=1/(n+1)$ for $0\leqslant i\leqslant n$. Making use of (3.2) and (2.6) we obtain the desired inequalities (3.3).

The inequalities (1.3), with the third member being omitted, follow from (3.3) by letting $n=1$, x^0 = a and $x^1=b$ $(a\neq b)$.

Before we will state and prove the next result, let us introduce more notation. For $y\in\sigma$ let $c=\nabla\ln f(y)$ stand for the gradient of $\ln f$. Also, let

$$z_i=(x^i-y)\cdot c, \tag{3.4}$$

$0\leqslant i\leqslant n$, let $z=(z_0,\dots,z_n)$, and let $\exp(z)=(\exp(z_0),\dots,\exp(z_n))$.

We have the following.

THEOREM 3.3. *Let $f:\sigma\to\mathbb{R}$ be a log-convex function. If f has continuous partial derivatives of order one on Int (σ), then*

$$\int_\sigma f(x)M_\mu(x|X)\,dx \geqslant f(y)\mathscr{L}_\mu(\exp(z)) \tag{3.5}$$

holds for any $y\in\sigma$.

Proof. The proof presented below bears some resemblance of that of Theorem 3.1 in [8]. Logarithmic convexity of $f(\cdot)$ implies the following inequality

$$\ln f(x)-\ln f(y)\geqslant (x-y)\cdot\nabla\ln f(y)$$

which is valid for all $x, y \in \sigma$. Hence

$$f(x) \geqslant f(y)\exp[(x-y)\cdot c].$$

Letting above $x = Xu$ and next integrating both sides against the probability measure μ we obtain

$$\int_{E_n} f(Xu)\,d\mu(u) \geqslant f(y)\int_{E_n} \exp[(Xu-y)\cdot c]\,d\mu(u).$$

Since

$$\begin{aligned}(Xu-y)\cdot c &= \left(\sum_{i=0}^{n} u_i x^i - \sum_{i=0}^{n} u_i y\right)\cdot c = \left(\sum_{i=0}^{n} u_i(x^i - y)\right)\cdot c \\ &= \sum_{i=0}^{n} u_i z_i = u\cdot z,\end{aligned}$$

the last inequality together with (2.4) implies

$$\int_{E_n} f(Xu)\,d\mu(u) \geqslant f(y)\int_{E_n} \exp(u\cdot z)\,d\mu(u) = f(y)\mathscr{L}_\mu(\exp(z)).$$

This in conjunction with (2.1) gives the assertion.

REMARK 3.4. If $y = \sum\limits_{i=0}^{n} w_i x^i$, then

$$\mathscr{L}_\mu(\exp(z)) \geqslant 1, \tag{3.6}$$

where the weights w_i $(0 \leqslant i \leqslant n)$ are defined in (1.2) and exp(z) is the same as in Theorem 3.3. Moreover, the number 1 is the largest lower bound in (3.6).

Proof. Application of the first inequality in (2.5) gives

$$\begin{aligned}\mathscr{L}_\mu(\exp(z)) \geqslant \exp(w\cdot z) &= \exp\left(\sum_{i=0}^{n} w_i(x^i - y)\cdot c)\right) \\ &= \exp\left(\left(\sum_{i=0}^{n} w_i x^i - y\right)\cdot c\right) \\ &= \exp(0\cdot c) = 1,\end{aligned}$$

where 0 stands for the origin in $\mathbb{R}^{n+1}$. The last statement of Remark 3.4 follows from Theorem 3.2 of [8].

COROLLARY 3.5. *Let $k = n$. Then under the assumptions of Theorem 3.3, the following inequality*

$$\frac{1}{|\mathrm{vol}\,(\sigma)|}\int_\sigma f(x)\,dx \geqslant f(y)(n![z_0,\ldots,z_n]e^t) \tag{3.7}$$

holds for all $y \in \sigma$.

Proof. We let $\mu(u) = n!$ $(u \in E_n)$ in (3.5) and next utilize (2.2) and (2.6) to obtain the desired result.

Let us note that the inequalities (1.4) follow from (3.7), (2.6) and (3.4) by letting $n = 1$, $x^0 = a$, $x^1 = b$ $(a \neq b)$, and $y = A = (a+b)/2$. Then (3.7) becomes

$$\frac{1}{b-a}\int_a^b f(x)dx \geqslant f(A)L(exp(z_0), exp(z_1)),$$

where $z_0 = (a-A)c$, $z_1 = (b-A)c$, and $c = f'(A)/f(A)$. Using (3.6) we see that the right side of the last inequality is always greater than or equal to $f(A)$.

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