A PRIORI BOUNDS FOR ELLIPTIC OPERATORS IN WEIGHTED SOBOLEV SPACES

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(Communicated by D. Žubrinić)

Abstract. This paper is concerning with the study of a class of weight functions and their properties. As an application, we prove some a priori bounds for a class of uniformly elliptic second order linear differential operators in weighted Sobolev spaces.

1. Introduction

Let \( \Omega \) be an open subset of \( \mathbb{R}^n \) (not necessarily bounded), \( n \geq 3 \). Assign in \( \Omega \) the uniformly elliptic second order linear differential operator

\[
L = - \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a. \tag{1.1}
\]

The aim of this paper is to investigate about a new class of weight functions (introduced in [15]) and to obtain some a priori estimates for the operator \( L \) in weighted Sobolev spaces.

In particular, we are interested in the study of the functions \( m : \Omega \rightarrow \mathbb{R}_+ \) such that

\[
\sup_{x,y \in \Omega, |x-y| < d} \frac{m(x)}{m(y)} < +\infty, \tag{1.2}
\]

with \( d \in \mathbb{R}_+ \). Typical examples of such functions are:

\[
m(x) = e^{t|x|}, \quad m(x) = (1 + |x|^2)^t, \quad x \in \Omega, t \in \mathbb{R}.
\]

Then we study the multiplication operator

\[
u \mapsto gu \tag{1.3}
\]

defined in a weighted Sobolev space and which takes values in a weighted Lebesgue space. We give conditions on \( g \) and \( \Omega \) so that the operator defined by (1.3) is bounded and other ones in order to get its compactness.


Keywords and phrases: Weight functions, weighted Sobolev spaces, elliptic operators, a priori bounds.
As an application, we obtain some a priori estimates for the operator $L$. We recall that when $\Omega$ is bounded, the problem of determining a priori bounds has been investigated by several authors under various hypotheses on the leading coefficients. It is worth to mention the results proved in [10], [7], [8], [16], [17], where the coefficients $a_{ij}$ are required to be discontinuous. If the open set $\Omega$ is unbounded, a priori bounds are established in [12], [2] with analogous assumptions to those required in [10], while in [6], [3], [4], under similar hypotheses asked in [7], [8], the above estimates are obtained. In this paper, we extend some results of [7], [8] to a weighted case.

Actually, assuming that the coefficients $a_{ij}$ are locally $VMO$ and “close” at infinity to certain functions $e_{ij}$ of class $VMO$, and supposing that the lower – order coefficients verify suitable regularity hypotheses and have a certain behaviour at the infinity, we get the following a priori bound:

$$||u||_{W^{2,p}_s(\Omega)} \leq c \left( ||Lu||_{L^p(F)} + ||u||_{L^p(\Omega)} \right) \quad \forall u \in W^{2,p}_s(\Omega) \cap W^{1,p}_{\infty}(\Omega),$$

where $s \in \mathbb{R}$, $\Omega$ is sufficiently regular, $W^{2,p}_s(\Omega), W^{1,p}_s(\Omega)$ and $L^p_\infty(\Omega)$ are weighted Sobolev spaces in which the weight functions verify (1.2), $c \in \mathbb{R}_+$ is independent of $u$, and $\Omega_1$ is a bounded open subset of $\Omega$.

As a consequence of the above estimate we can say that the operator $L$ has closed range and finite – dimensional kernel.

We wish to stress that an analogous estimate has been obtained in [5], in a different situation. Indeed, in [5] the open set $\Omega$ has singular boundary and the coefficients of the operator $L$ are singular near a subset of $\partial \Omega$. Hence, in [5] the weight function goes to zero on such subset of $\partial \Omega$ and then also the weighted Sobolev spaces are different with respect to those considered in this paper.

2. Notation and function spaces

Let $G$ be any Lebesgue measurable subset of $\mathbb{R}^n$ and $\Sigma(G)$ the collection of all Lebesgue measurable subsets of $G$. Let $F \in \Sigma(G)$ and $|F|$ denote the Lebesgue measure of $F$. Let $\chi_F$ be the characteristic function of $F$ and $\mathcal{D}(F)$ the class of restrictions to $F$ of functions $\zeta \in C_0^\infty(\mathbb{R}^n)$ with $\overline{F} \cap \text{supp} \zeta \subseteq F$. If $X(F)$ is a space of functions defined on $F$, $X_{\text{loc}}(F)$ denotes the class of all functions $g : F \to \mathbb{R}$ such that $\zeta g \in X(F)$ for any $\zeta \in \mathcal{D}(F)$. Finally, for any $x \in \mathbb{R}^n$ and $r \in \mathbb{R}_+$, we put $B(x,r) = \{ y \in \mathbb{R}^n : |y-x| < r \}$, $B_r = B(0,r)$ and $F(x,r) = F \cap B(x,r)$. We now recall the definitions of the function spaces in which the coefficients of the operator are chosen. Indeed, if $\Omega$ has the property

$$|\Omega(x,r)| \geq A r^n \quad \forall x \in \Omega, \quad \forall r \in [0,1],$$

where $A$ is a positive constant independent of $x$ and $r$, then we can consider the space $\text{BMO}(\Omega, \tau)$ ($\tau \in \mathbb{R}_+$) of functions $g \in L^{1}_{\text{loc}}(\Omega)$ such that

$$[g]_{\text{BMO}(\Omega, \tau)} = \sup_{r \in [0,\tau]} \frac{1}{\Omega(x,r)} \int_{\Omega(x,r)} |g - \int_{\Omega(x,r)} g| < +\infty,$$
with
\[ \int_{\Omega(x,r)} g = \left| \Omega(x,r) \right|^{-1} \int_{\Omega(x,r)} g. \]

If \( g \in BMO(\Omega) = BMO(\Omega, \tau_A) \), and
\[ \tau_A = \sup \left\{ \tau \in \mathbb{R}_+ : \sup_{x \in \Omega} \frac{r^n}{\left| \Omega(x,r) \right|} \leq \frac{1}{A} \right\}, \]
we say that \( g \in VMO(\Omega) \) if \( [g]_{BMO(\Omega,\tau)} \to 0 \) for \( \tau \to 0^+ \). A function
\[ \eta[g] : [0,1] \longrightarrow \mathbb{R}_+ \]
is called a modulus of continuity of \( g \) in \( VMO(\Omega) \) if
\[ [g]_{BMO(\Omega,\tau)} \leq \eta[g](\tau) \ \forall \tau \in [0,1], \lim_{\tau \to 0^+} \eta[g](\tau) = 0. \]

For \( t \in [1, +\infty[ \) and \( \lambda \in [0,n[ \), \( M^{t,\lambda}(\Omega) \) denotes the set of all functions \( g \) in \( L^t_{loc}(\bar{\Omega}) \) endowed with the following norm:
\[ \|g\|_{M^{t,\lambda}(\Omega)} = \sup_{r \in [0,1]} \| r^{-\lambda/t} g \|_{L^t(\Omega(x,r))} < +\infty. \] (2.2)

Then we define \( \tilde{M}^{t,\lambda}(\Omega) \) as the closure of \( L^\infty(\Omega) \) in \( M^{t,\lambda}(\Omega) \) and \( M_0^{t,\lambda}(\Omega) \) as the closure of \( C_0^\infty(\Omega) \) in \( M^{t,\lambda}(\Omega) \). In particular, we put \( M^t(\Omega) = M^{t,0}(\Omega), M_0^t(\Omega) = \tilde{M}^{t,0}(\Omega) \) and \( M_0^t(\Omega) = M_0^0(\Omega) \). Recall that for a function \( g \in M^{t,\lambda}(\Omega) \) the following characterization holds:
\[ g \in M^{t,\lambda}(\Omega) \iff \lim_{\tau \to 0^+} p_g(\tau) = 0 \] (2.3)
where
\[ p_g(\tau) = \sup_{E \subset \Omega} \| \chi_E g \|_{M^{t,\lambda}(\Omega)}, \quad \tau \in \mathbb{R}_+. \]
Thus the modulus of continuity of \( g \in M^{t,\lambda}(\Omega) \) is a function
\[ \tilde{\sigma}[g] : [0,1] \longrightarrow \mathbb{R}_+ \]
such that
\[ p_g(\tau) \leq \tilde{\sigma}[g](\tau) \ \forall \tau \in [0,1], \lim_{\tau \to 0^+} \tilde{\sigma}[g](\tau) = 0. \]

Furthermore, if \( g \in M^{t,\lambda}(\Omega) \) then
\[ g \in M_0^{t,\lambda}(\Omega) \iff \lim_{\tau \to 0^+} \left( p_g(\tau) + \|(1 - \xi_{1/\tau})g\|_{M^{t,\lambda}(\Omega)} \right) = 0 \] (2.4)
where \( \xi_r, \ r \in \mathbb{R}_+ \), is a function in \( C_0^\infty(\mathbb{R}^n) \) such that
\[ 0 \leq \xi_r \leq 1, \quad \xi_r|_{B_r} = 1, \ \text{supp} \xi_r \subset B_{2r}. \]
Thus the modulus of continuity of \( g \in M^t_\lambda(\Omega) \) is a function
\[
\sigma_o[g] : [0,1] \longrightarrow \mathbb{R}_+
\]
such that
\[
p_g(\tau) + \| (1 - \zeta_{1/\tau}) g \|_{M^t_\lambda(\Omega)} \leq \sigma_o[g](\tau) \quad \forall \tau \in [0,1], \quad \lim_{\tau \to 0^+} \sigma_o[g](\tau) = 0.
\]
A more detailed account of properties of the above defined function spaces can be found in [11], [13] and [14].

3. Weight functions

Let \( \Omega \) be an open subset of \( \mathbb{R}^n \), \( d \in \mathbb{R}_+ \) and \( G_d(\Omega) \) the set of all measurable functions \( m : \Omega \rightarrow \mathbb{R}_+ \) such that
\[
\sup_{x,y \in \Omega, |x-y| < d} \frac{m(x)}{m(y)} < +\infty.
\]
(3.1)

It is easy to verify that \( m \in G_d(\Omega) \) if and only if there exists \( \gamma \in \mathbb{R}_+ \) such that
\[
\gamma^{-1} m(y) \leq m(x) \leq \gamma m(y) \quad \forall y \in \Omega, \quad \forall x \in \Omega(y,d),
\]
(3.2)
where \( \gamma \in \mathbb{R}_+ \) is independent of \( x \) and \( y \).

Hence from (3.2) we get
\[
m, m^{-1} \in L^{\infty}_{\text{loc}}(\Omega).
\]
(3.3)

Let \( G(\Omega) \) be the class of weight functions defined as follows:
\[
G(\Omega) = \bigcup_{d \in \mathbb{R}_+} G_d(\Omega).
\]

Hence, if \( m \in G(\Omega) \) then:
\[
m^s \in G(\Omega), \quad \lambda m \in G(\Omega) \quad \forall s \in \mathbb{R}, \forall \lambda \in \mathbb{R}_+.
\]

**Lemma 3.1.** Let \( m \) be a positive function defined on \( \Omega \). If \( \log m \in \text{Lip}(\Omega) \) then \( m \in G(\Omega) \).

**Proof.** By the hypothesis, there is a constant \( L \in \mathbb{R}_+ \) such that for each \( x,y \in \Omega \)
\[
|\log m(x) - \log m(y)| \leq L|x-y|.
\]
(3.4)
For \( x,y \in \Omega \) such that \( |x-y| < d \) \((d \in \mathbb{R}_+)\), from (3.4) we have
\[
\left| \log \frac{m(x)}{m(y)} \right| \leq Ld \quad \forall y \in \Omega, \quad \forall x \in \Omega(y,d),
\]
and then the claimed implication. \( \square \)
Examples of functions in $G(\Omega)$ are:

$$m(x) = e^{|x|}, \quad m(x) = (1 + |x|^2)^t, \quad x \in \Omega, \, t \in \mathbb{R}.$$ 

**Lemma 3.2.** If $m \in G(\Omega)$ and $\Omega$ has the cone property, then there exists a function $\sigma \in G(\Omega) \cap C^\infty(\Omega)$ such that

$$c_1 m(x) \leq \sigma(x) \leq c_2 m(x) \quad \forall x \in \Omega,$$  

$$\sup_{x \in \Omega} \frac{\partial^\alpha \sigma(x)}{\sigma(x)} < +\infty \quad \forall \alpha \in \mathbb{N}_0^n,$$  

where $c_1, c_2 \in \mathbb{R}_+$ are dependent only on $n, \Omega, m$.

**Proof.** Since $m \in G(\Omega)$ then there exists a positive number $d$ such that $m \in G_d(\Omega)$. Assume $g \in C^\infty(\mathbb{R}^n)$ such that $g \geq 0$, $g|_{B_1^+} = 1$, $\text{supp} \, g \subset B_1$ and

$$\sigma : x \in \Omega \rightarrow \int_\Omega m(y) g\left(\frac{x - y}{d}\right) \, dy.$$  

Since

$$\sigma(x) = \int_{\Omega(x,d)} m(y) g\left(\frac{x - y}{d}\right) \, dy \quad \forall x \in \Omega,$$  

using (3.2), it follows (3.5). Thus $\sigma \in G_d(\Omega)$.

Again by (3.2), for all $\alpha \in \mathbb{N}_0^n$ and $x \in \Omega$, we have:

$$|\partial^\alpha \sigma(x)| \leq \gamma m(x) d^{-|\alpha|} \int_{\Omega(x,d)} |g^{(|\alpha|)}\left(\frac{x - y}{d}\right)| \, dy \leq c_3 m(x),$$

where $c_3$ depends on $n, \Omega, m, \alpha$, and then (3.6) follows. $\square$

**Lemma 3.3.** If $\Omega$ has the property that there are $r_0 \in \mathbb{R}_+$ and $x_0 \in \Omega \setminus B_{r_0}$ such that for every $x \in \Omega \setminus \overline{B_{r_0}} \subset \Omega$, then for any $m \in G(\Omega)$ and for every $x \in \Omega$,

$$c_0^{-1} e^{-c|x|} \leq m(x) \leq c_0 e^{c|x|},$$

where $c$ and $c_0$ depend only on $n, \Omega$ and $m$.

**Proof.** Fix $x \in \Omega$. If $x \in \Omega \setminus B_{r_0}$ then $\overline{x_0 \Omega} \subset \Omega$ and by Lagrange’s theorem, using (3.6), we have

$$|\log \sigma(x) - \log \sigma(x_0)| \leq c|x - x_0|,$$

where $c \in \mathbb{R}_+$ depends on $n, \Omega, m$. So, by an easy computation via (3.2), we have the result. Otherwise, if $x \in \Omega \cap B_{r_0}$, the result is obtained by (3.3). $\square$
If \( m \in G(\Omega), \ k \in \mathbb{N}_0, \ 1 \leq p < +\infty \) and \( s \in \mathbb{R} \), let \( W_s^{k,p}(\Omega) \) be the space of distributions \( u \) on \( \Omega \) such that \( m^s \partial^\alpha u \in L^p(\Omega) \) for \( |\alpha| \leq k \), equipped with the norm
\[
\|u\|_{W_s^{k,p}(\Omega)} = \sum_{|\alpha| \leq k} \|m^s \partial^\alpha u\|_{L^p(\Omega)}.
\]
(3.8)
Moreover, denote by \( \overset{\diamond}{W}_s^{k,p}(\Omega) \) the closure of \( C_0^\infty(\Omega) \) in \( W_s^{k,p}(\Omega) \) and put \( \overset{\diamond}{W}_s^{0,p}(\Omega) = L^p_s(\Omega) \).

From (3.6), by induction, we can deduce the following property of the function \( \sigma \) defined in Lemma 3.2:
\[
\sup_{x \in \Omega} \frac{|\partial^\alpha \sigma^s(x)|}{\sigma^s(x)} < +\infty \quad \forall \alpha \in \mathbb{N}_0^n, \quad \forall s \in \mathbb{R}.
\]
(3.9)
Now, by (3.9), we can easily deduce the following.

**Lemma 3.4.** Let \( k \in \mathbb{N}_0, \ 1 \leq p < +\infty \) and \( s \in \mathbb{R} \). If \( \Omega \) has the cone property, \( m \in G(\Omega) \) and \( \sigma \) is the function defined in Lemma 3.2, then the map
\[
\sigma^s u
\]
defines a topological isomorphism from \( W_s^{k,p}(\Omega) \) to \( W^{k,p}(\Omega) \) and from \( \overset{\diamond}{W}_s^{k,p}(\Omega) \) to \( \overset{\diamond}{W}^{k,p}(\Omega) \).

A more detailed account of properties of the above defined spaces can be found, for instance, in [15].

**4. Some embedding results**

Let \( m \) be a function of class \( G(\Omega) \). We consider the following condition:

\( (h_0) \) \( \Omega \) has the cone property, \( p \in [1, +\infty[, s \in \mathbb{R}, k, t \) are numbers such that:
\[
k \in \mathbb{N}, \ \ t \geq p, \ t \geq \frac{n}{k}, \ t > p \text{ if } p = \frac{n}{k}, \ g \in \mathcal{M}'(\Omega).
\]

By Theorem 3.1 of [9] we easily obtain the following.

**Theorem 4.1.** If the assumption \( (h_0) \) holds, then for any \( u \in W_s^{k,p}(\Omega) \) we have \( gu \in L^p_s(\Omega) \) and
\[
\|gu\|_{L^p_s(\Omega)} \leq c \|g\|_{\mathcal{M}'(\Omega)} \|u\|_{W_s^{k,p}(\Omega)},
\]
with \( c \) dependent only on \( \Omega, n, k, p \) and \( t \).

**Corollary 4.2.** If the assumption \( (h_0) \) holds and \( g \in \mathcal{M}'(\Omega) \), then for any \( \varepsilon \in \mathbb{R}_+ \) there exists a constant \( c(\varepsilon) \in \mathbb{R}_+ \) such that
\[
\|gu\|_{L^p_s(\Omega)} \leq \varepsilon \|u\|_{W_s^{k,p}(\Omega)} + c(\varepsilon) \|u\|_{L^p_s(\Omega)} \quad \forall u \in W_s^{k,p}(\Omega),
\]
(4.2)
where \( c(\varepsilon) \) depends only on \( \varepsilon, \Omega, n, k, p, t, \sigma[g] \).
Proof. Fix $\varepsilon > 0$ and let $c$ be the constant in (4.1). Since $g \in M_0^\prime(\Omega)$, there exists $g_\varepsilon \in L^\infty(\Omega)$ such that $\|g - g_\varepsilon\|_{M_0^\prime(\Omega)} < \frac{\varepsilon}{c}$. By Theorem 4.1

$$\|gu\|_{L^p_s(\Omega)} \leq c \|g - g_\varepsilon\|_{M_0^\prime(\Omega)} \|u\|_{W_s^{k,p}(\Omega)} + \|g_\varepsilon\|_{L^\infty(\Omega)} \|u\|_{L^p_s(\Omega)}$$

for any $u$ in $W_s^{k,p}(\Omega)$, and then the result follows. \(\square\)

**Corollary 4.3.** If the assumption $(h_0)$ holds and $g \in M_0^\prime(\Omega)$, then for any $\varepsilon \in \mathbb{R}_+$ there exist a constant $c(\varepsilon) \in \mathbb{R}_+$ and a bounded open subset $\Omega_\varepsilon \subset \subset \Omega$ with the cone property such that

$$\|gu\|_{L^p_s(\Omega)} \leq \varepsilon \|u\|_{W_s^{k,p}(\Omega)} + c(\varepsilon) \|u\|_{L^p_s(\Omega_\varepsilon)} \quad \forall u \in W_s^{k,p}(\Omega), \quad (4.3)$$

where $c(\varepsilon)$ and $\Omega_\varepsilon$ depend only on $\varepsilon, \Omega, n, k, p, m, s, t, \sigma_0[g]$.

Proof. Fix $\varepsilon > 0$ and let $c$ be the constant in (4.1). Since $g \in M_0^\prime(\Omega)$, there exists $g_\varepsilon \in C_0^\infty(\Omega)$ such that $\|g - g_\varepsilon\|_{M_0^\prime(\Omega)} < \frac{\varepsilon}{c}$. Let $\Omega_\varepsilon$ be a bounded open subset of $\Omega$, with the cone property, such that $\text{supp } g_\varepsilon \subset \subset \Omega_\varepsilon$, hence by Theorem 4.1 and (3.3), it follows that

$$\|gu\|_{L^p_s(\Omega)} \leq c \|g - g_\varepsilon\|_{M_0^\prime(\Omega)} \|u\|_{W_s^{k,p}(\Omega)} + \|g_\varepsilon u\|_{L^p_s(\Omega_\varepsilon)}$$

$$\leq \varepsilon \|u\|_{W_s^{k,p}(\Omega)} + \|g_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \|u\|_{L^p_s(\Omega_\varepsilon)} \quad (4.4)$$

for any $u$ in $W_s^{k,p}(\Omega)$, and then we have the result. \(\square\)

**Theorem 4.4.** If the assumption $(h_0)$ holds and $g \in M_0^\prime(\Omega)$, then the operator

$$u \in W_s^{k,p}(\Omega) \rightarrow gu \in L^p_s(\Omega) \quad (4.5)$$

is compact.

Proof. Let $(u_n)_{n \in \mathbb{N}}$ be a sequence of functions which weakly converges to zero in $W_s^{k,p}(\Omega)$. Therefore there exists $b \in \mathbb{R}_+$ such that $\|u_n\|_{W_s^{k,p}(\Omega)} \leq b$ for every $n \in \mathbb{N}$.

For $\varepsilon > 0$, from Corollary 4.3, there exist $c(\varepsilon) \in \mathbb{R}_+$ and a bounded open subset $\Omega_\varepsilon \subset \subset \Omega$ with the cone property such that

$$\|gu_n\|_{L^p_s(\Omega)} \leq \frac{\varepsilon}{b} \|u_n\|_{W_s^{k,p}(\Omega)} + c(\varepsilon) \|u_n\|_{L^p_s(\Omega_\varepsilon)} \quad \forall n \in \mathbb{N}. \quad (4.6)$$

Since $W_s^{k,p}(\Omega) \subset W^{k,p}(\Omega_\varepsilon)$, we obtain the result from a well-known compact embedding theorem. \(\square\)
5. A priori estimates

Assume that $\Omega$ is an unbounded open subset of $\mathbb{R}^n, n \geq 3,$ with the uniform $C^{1,1}$ regularity property, $p \in ]1, +\infty[ \text{ and } s \in \mathbb{R}$.

Consider in $\Omega$ the differential operator

$$L = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a,$$  \hspace{1cm} (5.1)

with the following conditions on the coefficients:

\begin{equation}
\begin{aligned}
(a) & \quad a_{ij} = a_{ji} \in L^\infty(\Omega) \cap VMO_{loc}(\Omega), \ i, j = 1, \ldots, n, \\
(b) & \quad \exists \nu > 0 : \sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \geq \nu |\xi|^2 \text{ a.e. in } \Omega, \forall \xi \in \mathbb{R}^n, \\
(c) & \quad g \in L^\infty(\Omega), \ \lim_{r \to +\infty} \sum_{i,j=1}^{n} |e_{ij} - g a_{ij}|_{L^\infty(\Omega \setminus B_r)} = 0, \\
(d) & \quad a_i \in \tilde{M}^t(\Omega), \ i = 1, \ldots, n, \ a \in \tilde{M}^m(\Omega),
\end{aligned}
\end{equation}

where

$$t_1 \geq p, \ t_1 \geq n, \ t_1 > p \quad \text{if} \quad p = n, \\
t_2 \geq p, \ t_2 \geq n/2, \ t_2 > p \quad \text{if} \quad p = n/2.$$

Under assumptions $(a)$ - $(d)$, by Theorem 4.1, the operator $L : W^{2,p}_s(\Omega) \to L^p_s(\Omega)$ is bounded.

Let

$$L_0 = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}.$$  \hspace{1cm} (5.1)

THEOREM 5.1. Suppose that assumptions $(a), (b), (c)$ and $(d)$ hold. Then there exists $r_0, c \in \mathbb{R}_+$ such that:

$$\|u\|_{W^{2,p}_s(\Omega)} \leq c(\|Lu\|_{L^p_s(\Omega)} + \|u\|_{L^p_s(\Omega)}) \ \forall u \in W^{2,p}_s(\Omega) \cap \tilde{W}^{1,p}_s(\Omega),$$

where $c$ depends only on $n, p, t_1, t_2, \Omega, \nu, \mu, \|a_{ij}\|_{L^\infty(\Omega)}, \|e_{ij}\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)}, \eta[\zeta_{2r_0}a_{ij}], \eta[e_{ij}], \tilde{\sigma}[a], \tilde{\sigma}[a], m, s$ and $r_0$ depends only on $n, p, \Omega, \mu, \|e_{ij}\|_{L^\infty(\Omega)}, \eta[e_{ij}].$
Proof. Let \( u \in W^{2,p}_s(\Omega) \cap W^{1,p}_s(\Omega) \). By Lemma 3.4 we have that

\[ \sigma^s u \in W^{2,p}(\Omega) \cap W^{1,p}(\Omega). \]

Then, by Theorem 3.1 of [3], there exist \( r_0 \) and \( c_0 \in \mathbb{R}_+ \) such that

\[ ||\sigma^s u||_{W^{2,p}(\Omega)} \leq c_0 \left( ||L_0(\sigma^s u)||_{L^p(\Omega)} + ||\sigma^s u||_{L^p(\Omega)} \right), \]

(5.2)

where \( c_0 \) depends on \( n, p, \Omega, \nu, \mu, ||a_{ij}||_{L^\infty(\Omega)}, ||e_{ij}||_{L^\infty(\Omega)}, ||g||_{L^\infty(\Omega)}, |e_{ij}|, \) \( \eta[\zeta_{a_{ij}}], \eta[e_{ij}] \), and \( r_0 \) depends on \( n, p, \Omega, \mu, ||e_{ij}||_{L^\infty(\Omega)}, |e_{ij}|. \) Since

\[ L_0(\sigma^s u) = \sigma^s Lu - s(s - 1)\sigma^{s-2} \sum_{i,j=1}^n a_{ij}\sigma_{xi}\sigma_{xj}u - 2s\sigma^{s-1} \sum_{i,j=1}^n a_{ij}\sigma_{xi}u_{xj} \]

\[-s\sigma^{s-1} \sum_{i,j=1}^n a_{ij}\sigma_{xi}u_{xj} - \sigma^s \sum_{i=1}^n a_{ii}u_{xi} - \sigma^s au, \]

(5.3)

from (5.2) and (5.3) we have

\[ ||\sigma^s u||_{W^{2,p}(\Omega)} \leq c_1 \left( ||\sigma^s Lu||_{L^p(\Omega)} + ||\sigma^s u||_{L^p(\Omega)} \right) \]

(5.4)

\[ + \sum_{i,j=1}^n ||\sigma^{s-2}\sigma_{xi}\sigma_{xj}u||_{L^p(\Omega)} + \sum_{i,j=1}^n ||\sigma^{s-1}\sigma_{xj}u_{xj}||_{L^p(\Omega)} \]

\[ + \sum_{i,j=1}^n ||\sigma^{s-1}\sigma_{xi}u_{xj}||_{L^p(\Omega)} + \sum_{i=1}^n ||\sigma^s a_{ii}u_{xi}||_{L^p(\Omega)} + ||\sigma^s au||_{L^p(\Omega)}, \]

where \( c_1 \) depends on the same parameters as \( c_0 \) and on \( s \).

By Theorem 4.7 of [1], for all \( i = 1, \ldots, n \) we have:

\[ ||u_{xi}||_{L^p(\Omega)} \leq c_2 \left( ||u_{xx}||_{L^p(\Omega)}^{\frac{1}{2}} ||u||_{L^p(\Omega)}^{\frac{1}{2}} + ||u||_{L^p(\Omega)}^{\frac{1}{2}} \right), \]

(5.5)

where \( c_2 \) depends on \( \Omega, m, n, p. \)

Moreover, from Corollary 4.2, for any \( \varepsilon \in \mathbb{R}_+ \) and \( i = 1, \ldots, n \) there exist \( c_1(\varepsilon), c_2(\varepsilon) \in \mathbb{R}_+ \) such that:

\[ ||a_{ii}u_{xi}||_{L^p(\Omega)} \leq \varepsilon ||u||_{W^{2,p}_s(\Omega)} + c_1(\varepsilon) ||u_{xi}||_{L^p(\Omega)}; \]

(5.6)

\[ ||au||_{L^p(\Omega)} \leq \varepsilon ||u||_{W^{2,p}_s(\Omega)} + c_2(\varepsilon) ||u||_{L^p(\Omega)}; \]

(5.7)

where \( c_1(\varepsilon) \) depends on \( \varepsilon, \Omega, n, p, t_1, \tilde{\sigma}[a_{ii}] \) and \( c_2(\varepsilon) \) depends on \( \varepsilon, \Omega, n, p, t_2, \tilde{\sigma}[a] \).

From (5.4)–(5.7), Lemma 3.2 and Lemma 3.4, it follows

\[ ||u||_{W^{2,p}_s(\Omega)} \leq c_3 \left( ||Lu||_{L^p(\Omega)} + ||u||_{L^p(\Omega)} + \varepsilon ||u||_{W^{2,p}_s(\Omega)} \right) \]

\[ + c_3(\varepsilon) \left( ||u_{xx}||_{L^p(\Omega)}^{\frac{1}{2}} ||u||_{L^p(\Omega)}^{\frac{1}{2}} + ||u||_{L^p(\Omega)}^{\frac{1}{2}} \right), \]

(5.8)
where $c_3$ depends on the same parameters as $c_0$ and on $s, m,$ and $c_3(ε)$ depends on $ε$, $Ω, n, p, t_1, t_2, \bar{σ}[a_i], \bar{σ}[a]$.

For $ε = \frac{1}{c_3}$, from (5.8) we have

$$||u||_{W^{2,p}_s(Ω)} \leq c_4 \left( ||Lu||_{L^p_2(Ω)} + ||u||_{L^p_1(Ω)} + ||u_{xx}||_{L^p_2(Ω)} + ||u||_{L^p_1(Ω)} \right), \quad (5.9)$$

where $c_4$ depends on the same parameters as $c_3$ and on $t_1, t_2, \bar{σ}[a_i], \bar{σ}[a]$.

Using Young’s inequality and (5.9), we get the result. □

Add the following assumptions on the coefficients of $L$ and on the weight function:

$$(h_4) \begin{cases} 
(e_{ij})_{x_i} \in M_0^{n-1}(Ω), \text{ with } t \in [2, n], \ i, j, h = 1, \ldots, n; \\
 a_i \in M_0^{l_1}(Ω), \ i = 1, \ldots, n; \\
a = a' + b, a' \in M_0^2(Ω), b \in L^1(Ω), b_0 = \text{ess inf}_Ω b > 0, \\
g_0 = \text{ess inf}_Ω g > 0, \\
 \lim_{|x| \to +\infty} \frac{\sigma_x + \sigma_{xx}}{\sigma} = 0,
\end{cases}$$

where $t_1$ and $t_2$ are defined as in $(h_3)$.

**Theorem 5.2.** Suppose that assumptions $(h_1), (h_2)$ and $(h_4)$ hold. Then there are a real positive number $c$ and a bounded open $Ω_1 \subset Ω$ with the cone property such that:

$$||u||_{W^{2,p}_s(Ω)} \leq c \left( ||Lu||_{L^p_2(Ω)} + ||u||_{L^p(Ω_1)} \right) \quad \forall u \in W^{2,p}_s(Ω) \cap W^{1,p}_s(Ω),$$

where $c$ and $Ω_1$ are dependent only on $n, p, Ω, ν, μ, g_0, b_0, t, t_1, t_2, m, s, ||a_{ij}||_{L^p(Ω)}, ||e_{ij}||_{L^p(Ω)}, ||g||_{L^p(Ω)}, ||b||_{L^p(Ω)}, \eta[ζ_2 b_{ij}], \sigma_0[(e_{ij})_x], \sigma_0[a_1], \sigma_0[a']$.

**Proof.** Let $u \in W^{2,p}_s(Ω) \cap W^{1,p}_s(Ω)$. By Lemma 3.4 we have that

$$\sigma^su \in W^{2,p}(Ω) \cap W^{1,p}(Ω).$$

Applying Theorem 3.3 of [4] to the operator $L_0 + b$, we have that there exist a real number $c_0 \in \mathbb{R}_+$ and an open bounded subset $Ω_0 \subset Ω$ with the cone property such that

$$||\sigma^su||_{W^{2,p}(Ω)} \leq c_0 \left( ||(L_0 + b)(\sigma^su)||_{L^p(Ω)} + ||\sigma^su||_{L^p(Ω_0)} \right),$$

where $c_0$ and $Ω_0$ are dependent on $n, p, Ω, ν, μ, g_0, b_0, t, ||a_{ij}||_{L^p(Ω)}, ||e_{ij}||_{L^p(Ω)}, ||g||_{L^p(Ω)}, ||b||_{L^p(Ω)}, \eta[ζ_2 b_{ij}], \sigma_0[(e_{ij})_x], \sigma_0[a_1], \sigma_0[a']$.
Proceeding as in the proof of Theorem 5.1, we have
\[
||u||_{W^{s,p}_x(\Omega)} \leq c_1 \left( ||Lu||_{L^p(\Omega)} + ||u||_{L^p(\Omega)} + \sum_{i,j=1}^n ||\sigma^{-2}_{x_i} \sigma_{x_j} u||_{L^p(\Omega)} \right) \\
+ \sum_{i,j=1}^n ||\sigma^{-1}_{x_i} u_{x_j}||_{L^p(\Omega)} + \sum_{i,j=1}^n ||\sigma^{-1}_{x_i} u_{x_j}||_{L^p(\Omega)} \\
+ \sum_{i=1}^n ||a_i u_{x_i}||_{L^p(\Omega)} + ||a'_u||_{L^p(\Omega)},
\]
where \(c_1\) depends on the same parameters as \(c_0\) and on \(m,s\).

From Corollary 4.3 and (1.6) of [11] it follows that for any \(\varepsilon \in \mathbb{R}_+\) and \(i,j = 1, \ldots, n\) there exist \(c_1(\varepsilon), c_2(\varepsilon), c_3(\varepsilon) \in \mathbb{R}_+\) and some bounded open subsets \(\Omega_1(\varepsilon) \subset \subset \Omega, \Omega_2(\varepsilon) \subset \subset \Omega, \Omega_3(\varepsilon) \subset \subset \Omega\) with the cone property such that
\[
||\sigma^{-2}_{x_i} \sigma_{x_j} u||_{L^p(\Omega)} \leq \varepsilon ||u||_{W^{2,p}_x(\Omega)} + c_1(\varepsilon) ||u||_{L^p(\Omega_1(\varepsilon))},
\]
\[
||\sigma^{-1}_{x_i} u_{x_j}||_{L^p(\Omega)} \leq \varepsilon ||u||_{W^{2,p}_x(\Omega)} + c_2(\varepsilon) ||u_{x_j}||_{L^p(\Omega_2(\varepsilon))},
\]
\[
||\sigma^{-1}_{x_i} u_{x_j}||_{L^p(\Omega)} \leq \varepsilon ||u||_{W^{2,p}_x(\Omega)} + c_3(\varepsilon) ||u||_{L^p(\Omega_3(\varepsilon))},
\]
where \(c_1(\varepsilon), c_2(\varepsilon), c_3(\varepsilon), \Omega_1(\varepsilon), \Omega_2(\varepsilon), \Omega_3(\varepsilon)\) are dependent on \(\varepsilon, \Omega, n, p, m, s\).

Using again Corollary 4.3 and Theorem 4.7 of [1] we have that there exist \(c_4(\varepsilon), c_5(\varepsilon) \in \mathbb{R}_+\) and bounded open sets \(\Omega_4(\varepsilon) \subset \subset \Omega, \Omega_5(\varepsilon) \subset \subset \Omega\) with the cone property such that:
\[
||a_i u_{x_i}||_{L^p(\Omega)} \leq \varepsilon ||u||_{W^{2,p}_x(\Omega)} + c_4(\varepsilon) ||u||_{L^p(\Omega_4(\varepsilon))}
\]
\[
\leq \varepsilon ||u||_{W^{2,p}_x(\Omega)} + c_4(\varepsilon) \left( ||u_{x_i}||_{L^p(\Omega_4(\varepsilon))} + ||u||_{L^p(\Omega_4(\varepsilon))} \right),
\]
\[
||a'_u||_{L^p(\Omega)} \leq \varepsilon ||u||_{W^{2,p}_x(\Omega)} + c_5(\varepsilon) ||u||_{L^p(\Omega_5(\varepsilon))},
\]
where \(c_4(\varepsilon)\) and \(\Omega_4(\varepsilon)\) depend on \(\varepsilon, \Omega, n, p, m, s, t_1, \sigma_0[a_i]\), and \(c_5(\varepsilon)\) and \(\Omega_5(\varepsilon)\) depend on \(\varepsilon, \Omega, n, p, m, s, t_2, \sigma_0[a']\).

From (5.10)–(5.15) and Young’s inequality we have the result.

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(Received April 9, 2010)