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INTEGRABILITY THEOREMS FOR FOURIER–JACOBI TRANSFORMS

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Abstract. In this paper, we prove the Hardy-Littlewood-Paley inequality for the generalized Fourier transform on Chébli-Trimèche hypergroups and we study in the particular case of the Jacobi hypergroup the integrability of this transform on Besov-type spaces.

1. Introduction

We consider the Chébli-Trimèche hypergroup $(\mathbb{R}_+, *(A))$ associated with the function A which depends on a real parameter $\alpha > -\frac{1}{2}$ (see next section). We prove the Hardy-Littlewood-Paley inequality for the generalized Fourier transform $\mathscr{F}(f)$ of a function f in $L^p(\mathbb{R}_+, A(x)dx)$, $1 < p \leqslant 2$. Next, inspired by the definition of usual Besov spaces and Besov-Dunkl spaces (see [2, 5]), we define for $1 \leqslant p \leqslant 2$, $1 \leqslant q \leqslant +\infty$ and $\gamma > 0$, the Besov-type spaces for Chébli-Trimèche hypergroup denoted by $\mathscr{B}^{p,q}_{\gamma,\alpha}$ as the subspace of functions $f \in L^p(\mathbb{R}_+, A(x)dx)$ satisfying

$$\int_0^{+\infty} \left(\frac{\omega_{A,p}(f)(x)}{x^\gamma}\right)^q \frac{dx}{x} < +\infty \quad \text{if } q < +\infty$$

and

$$\sup_{x \in]0,+\infty[} \frac{\omega_{A,p}(f)(x)}{x^\gamma} < +\infty \quad \text{if } q = +\infty,$$

where $\omega_{A,p}(f)(x) = \|\tau_x(f) - f\|_{A,p}$ is the modulus of continuity of first order of f with τ_x the generalized translation operators, $x \in \mathbb{R}_+$ (see next section). We establish in the particular case of Jacobi hypergroup further results concerning integrability of the generalized Fourier transform $\mathscr{F}(f)$ of a function f when f belongs to a suitable Besov-type spaces. Analogous results have been obtained for the theory of Dunkl operators in [1, 3, 4].

The contents of this paper are as follows.

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In section 2, we collect some results about harmonic analysis on Chébli-Trimèche hypergroups.

In section 3, we prove the Hardy-Littlewood-Paley inequality for the generalized Fourier transform on Chébli-Trimèche hypergroups and we study in the particular case of the Jacobi hypergroup the integrability of this transform on Besov-type spaces.

Along this paper we use c to denote a suitable positive constant which is not necessarily the same in each occurrence. Furthermore, we denote by

- $\mathbb{C}_{*,c}(\mathbb{R})$ the space of even continuous functions on $\mathbb{R}$, with compact support.
- $\mathscr{D}_*(\mathbb{R})$ the space of even C^∞-functions on $\mathbb{R}$ with compact support.

2. Preliminaries

In this section, we recall some notations and results about harmonic analysis on Chébli-Trimèche hypergroups and we refer for more details to the articles [6, 9, 11, 12].

Let A be the Chébli-Trimèche function defined on $\mathbb{R}_+$ and satisfying the following conditions.

i) $A(x) = x^{2\alpha+1}B(x)$, with $\alpha > -\frac{1}{2}$, and B an even C^∞-function on $\mathbb{R}$ such that $B(x) \geqslant 1$ for all $x \in \mathbb{R}_+$.

ii) A is increasing.

iii) $\dfrac{A'}{A}$ is decreasing on $]0,+\infty[$ and $\lim\limits_{x\mapsto+\infty} \dfrac{A'(x)}{A(x)} = 2\rho \geqslant 0$, where ρ is a constant.

iv) There exists a constant $\eta > 0$ such that for all $x \in [x_0,+\infty[$, $x_0 > 0$, we have

$$\frac{A'(x)}{A(x)} = \begin{cases} 2\rho + e^{-\eta x}F(x) & \text{, if } \rho > 0 \\ \frac{2\alpha+1}{x} + e^{-\eta x}F(x) & \text{, if } \rho = 0, \end{cases}$$

where F is a C^∞-function bounded together with its derivatives.

We consider the Chébli-Trimèche hypergroup $(\mathbb{R}_+, *(A))$ associated with the function A. We note that it is commutative with neutral element 0 and the identity mapping is the involution. The Haar measure m on $(\mathbb{R}_+, *(A))$ is absolutely continuous with respect to the Lebesgue measure and can be choosen to have the Lebesgue density A.

Let Δ be the differential operator on $]0,+\infty[$ given by

$$\Delta = \frac{d^2}{dx^2} + \frac{A'(x)}{A(x)}\frac{d}{dx}.$$

The solution φ_λ, $\lambda \in \mathbb{C}$, of the differential equation

$$\begin{cases} \Delta u(x) = -(\lambda^2+\rho^2)u(x) \\ u(0) = 1,\ \frac{d}{dx}u(0) = 0, \end{cases}$$

is multiplicative on $(\mathbb{R}_+, *(A))$ in the sense that

$$\forall\, x,y \in \mathbb{R}_+,\ \int_{\mathbb{R}_+} \varphi_\lambda(t)\, d(\delta_x * \delta_y)(t) = \varphi_\lambda(x)\varphi_\lambda(y),$$

where δ_x is the point mass at x and $\delta_x * \delta_y$ is a probability measure which is absolutely continuous with respect to the measure m and satisfies

$$\operatorname{supp} \delta_x * \delta_y = [|x-y|, x+y].$$

We list some known properties of the characters φ_λ of the hypergroups.

i) For each $\lambda \in \mathbb{C}$, the function $x \mapsto \varphi_\lambda(x)$ is an even C^∞-function on $\mathbb{R}$ and for each $x \in \mathbb{R}_+$, the function $\lambda \mapsto \varphi_\lambda(x)$ is an entire function on $\mathbb{C}$.

ii) For every $\lambda \in \mathbb{C}$, the function φ_λ admits the integral representation

$$\varphi_\lambda(x) = \int_0^x K(x,y)\cos(\lambda y)dy, \quad \forall x > 0,$$

where $K(x,.)$ is a positive even $\mathbb{C}^\infty$-function on $]-x,x[$ with support in $[-x,x]$.

REMARK 2.1. If $A(x) = 2^{2\rho}(\sinh x)^{2\alpha+1}(\cosh x)^{2\beta+1}$, with $\alpha \geqslant \beta \geqslant -\frac{1}{2}$, $\alpha \neq -\frac{1}{2}$ and $\rho = \alpha+\beta+1$, $(\mathbb{R}_+, *(A))$ is called the Jacobi hypergroup. In this case, we have for all $x \in \mathbb{R}_+$ and $\lambda \in \mathbb{C}$,

$$\varphi_\lambda(x) = {}_2\mathrm{F}_1\left(\frac{1}{2}(\rho - i\lambda), \frac{1}{2}(\rho + i\lambda), \alpha+1; -\sinh^2 x\right),$$

where ${}_2\mathrm{F}_1$ is the Gauss hypergeometric function (see [9]). The function φ_λ is the Jacobi function and it satisfies for all $\lambda \in \mathbb{R}$ and $t > 0$

$$|1 - \varphi_\lambda(t)| \geqslant c \min\{1, (\lambda t)^2\}, \tag{2.1}$$

where c is constant which depends only on α and β (see [7, 8]).

For every $p \in [1, +\infty]$, we denote by $L^p_A(\mathbb{R}_+)$ the space $L^p(\mathbb{R}_+, A(x)dx)$ and by $L^p_{\mathbf{c}}(\mathbb{R}_+)$ the space $L^p(\mathbb{R}_+, \frac{d\lambda}{|\mathbf{c}(\lambda)|^2})$ where $|\mathbf{c}(\lambda)|^{-2}$ is an even continuous function on $\mathbb{R}$, satisfying the estimates: There exist positive constants k, k_1, k_2 such that

i) If $\rho = 0$ and $\alpha > 0$ then

$$k_1|\lambda|^{2\alpha+1} \leqslant |\mathbf{c}(\lambda)|^{-2} \leqslant k_2|\lambda|^{2\alpha+1}, \quad \lambda \in \mathbb{C}. \tag{2.2}$$

ii) If $\rho > 0$ and $\alpha > -\frac{1}{2}$ then

$$k_1|\lambda|^{2\alpha+1} \leqslant |\mathbf{c}(\lambda)|^{-2} \leqslant k_2|\lambda|^{2\alpha+1}, \quad \lambda \in \mathbb{C},\ |\lambda| > k, \tag{2.3}$$

and

$$k_1|\lambda|^{2} \leqslant |\mathbf{c}(\lambda)|^{-2} \leqslant k_2|\lambda|^{2}, \quad \lambda \in \mathbb{C},\ |\lambda| \leqslant k. \tag{2.4}$$

We use $\|.\|_{A,p}$ and $\|.\|_{\mathbf{c},p}$ as a shorthand respectively of $\|.\|_{L^p_A(\mathbb{R}_+)}$ and $\|.\|_{L^p_{\mathbf{c}}(\mathbb{R}_+)}$.

For $f \in L^1_A(\mathbb{R}_+)$ the generalized Fourier transform of f is given by

$$\mathscr{F}(f)(\lambda) = \int_{\mathbb{R}_+} f(x)\varphi_\lambda(x)A(x)dx.$$

The generalized Fourier transform satisfies the following properties.

i) For $f \in L^1_A(\mathbb{R}_+)$, we have

$$\|\mathscr{F}(f)\|_{\mathbf{c},\infty} \leqslant \|f\|_{A,1} \tag{2.5}$$

ii) For f in $L^1_A(\mathbb{R}_+)$ such that $\mathscr{F}(f)$ belongs to $L^1_{\mathbf{c}}(\mathbb{R}_+)$, we have the following inversion formula for the transform $\mathscr{F}$

$$f(x) = \int_{\mathbb{R}_+} \mathscr{F}(f)(\lambda)\varphi_\lambda(x)\frac{d\lambda}{|\mathbf{c}(\lambda)|^2},\ a.e.$$

iii) (Plancherel formula) For all $f \in \mathscr{D}_*(\mathbb{R})$, we have

$$\int_{\mathbb{R}_+} |f(x)|^2 A(x)dx = \int_{\mathbb{R}_+} |\mathscr{F}(f)(\lambda)|^2 \frac{d\lambda}{|\mathbf{c}(\lambda)|^2}. \tag{2.6}$$

The transform $\mathscr{F}$ can be uniquely extended to an isometric isomorphism from $L^2_A(\mathbb{R}_+)$ onto $L^2_{\mathbf{c}}(\mathbb{R}_+)$.

For $1 \leqslant p \leqslant 2$, we denote by p' the conjugate of p. From (2.5), (2.6) and the Marcinkiewicz interpolation theorem (see [10]), we obtain for $f \in L^p_A(\mathbb{R}_+)$

$$\|\mathscr{F}(f)\|_{\mathbf{c},p'} \leqslant c\,\|f\|_{A,p}. \tag{2.7}$$

For $x \in \mathbb{R}_+$ and $f \in \mathbb{C}_{*,c}(\mathbb{R})$, the generalized x-translate of f is defined by

$$\forall y \in \mathbb{R}_+, \quad \tau_x(f)(y) = \int_{\mathbb{R}_+} f(t)d(\delta_x * \delta_y)(t),$$

and we have $\tau_x(f)(0) = f(x)$.

The generalized translation operators τ_x, $x \in \mathbb{R}_+$, satisfy the following properties.

i) For all $x, y \in \mathbb{R}_+$ and $\lambda \in \mathbb{C}$, we have the product formula

$$\tau_x(\varphi_\lambda)(y) = \varphi_\lambda(x)\varphi_\lambda(y).$$

ii) For $f \in \mathscr{D}_*(\mathbb{R})$ and $x \in \mathbb{R}_+$, the function $y \mapsto \tau_x(f)(y)$ belongs to $\mathscr{D}_*(\mathbb{R})$ and we have

$$\forall \lambda \in \mathbb{R}_+, \quad \mathscr{F}(\tau_x f)(\lambda) = \varphi_\lambda(x)\mathscr{F}(f)(\lambda). \tag{2.8}$$

iii) Let $f \in L^p_A(\mathbb{R}_+)$, $p \in [1, +\infty]$. For all $x \in \mathbb{R}_+$, the function $\tau_x(f)$ belongs to $L^p_A(\mathbb{R}_+)$, $p \in [1, +\infty]$, and we have

$$\|\tau_x(f)\|_{A,p} \leqslant \|f\|_{A,p}.$$

3. Generalized Fourier transform

Throughout this section, k refers to the constant obtained in (2.3) and (2.4) from the estimates of $|\mathbf{c}(\lambda)|^{-2}$.

In the following lemma, we prove the Hardy-Littlewood-Paley inequality for the Fourier transform.

LEMMA 3.1. *For* $f \in L^p_A(\mathbb{R}_+)$, $1 < p \leqslant 2$, *one has*

$$\int_{\mathbb{R}_+} (g(x))^{p-2} |\mathscr{F}(f)(x)|^p \frac{dx}{|\mathbf{c}(x)|^2} \leqslant c\,\|f\|^p_{A,p} \tag{3.1}$$

where

i) $g(x) = x^{2(\alpha+1)}$ *if* $\rho = 0$ *and* $\alpha > 0$.

ii) $g(x) = \begin{cases} x^{2(\alpha+1)} & \text{for } x > k \\ x^3 & \text{for } x \leqslant k. \end{cases}$ *if* $\rho > 0$ *and* $\alpha > -\frac{1}{2}$ *where* k *refers to the constant obtained from the estimates of* $|\mathbf{c}(x)|^{-2}$.

Proof. For $f \in L^p_A(\mathbb{R}_+)$, $1 \leqslant p \leqslant 2$, we consider the operator

$$L(f)(x) = g(x)\mathscr{F}(f)(x),\ x \in \mathbb{R}_+.$$

For every $f \in L^2_A(\mathbb{R}_+)$, we have from (2.6)

$$\left(\int_{\mathbb{R}_+} |L(f)(x)|^2 \frac{dx}{(g(x))^2|\mathbf{c}(x)|^2}\right)^{\frac{1}{2}} = \|\mathscr{F}(f)\|_{\mathbf{c},2} = \|f\|_{A,2},$$

hence L is an operator of strong-type $(2,2)$ between the spaces $(\mathbb{R}_+, A(x)dx)$ and $(\mathbb{R}_+, \frac{dx}{(g(x))^2|\mathbf{c}(x)|^2})$.

i) Assume $\rho = 0$, $\alpha > 0$ and $g(x) = x^{2(\alpha+1)}$. For $\lambda \in]0,+\infty[$, $f \in L^1_A(\mathbb{R}_+)$ and using (2.2) and (2.5), we can write

$$\begin{aligned} \int_{\{x\in\mathbb{R}_+ : |L(f)(x)|>\lambda\}} \frac{dx}{(g(x))^2|\mathbf{c}(x)|^2} &= \int_{\{x\in\mathbb{R}_+ : |L(f)(x)|>\lambda\}} \frac{dx}{x^{4(\alpha+1)}|\mathbf{c}(x)|^2} \\ &\leqslant c \int_{(\frac{\lambda}{\|f\|_{A,1}})^{\frac{1}{2(\alpha+1)}}}^{+\infty} \frac{x^{2\alpha+1}}{x^{4(\alpha+1)}} dx \\ &\leqslant c\,\frac{\|f\|_{A,1}}{\lambda}. \end{aligned}$$

It yields that L is of weak-type $(1,1)$ between the spaces under consideration.

By the Marcinkiewicz interpolation theorem (see [10]), we can assert that L is an operator of strong-type (p,p), $1 < p \leqslant 2$ between the spaces $(\mathbb{R}_+, A(x)dx)$ and $(\mathbb{R}_+, \frac{dx}{(g(x))^2|\mathbf{c}(x)|^2})$.

We conclude that

$$\int_{\mathbb{R}_+} |L(f)(x)|^p \frac{dx}{(g(x))^2|\mathbf{c}(x)|^2} = \int_{\mathbb{R}_+} |g(x)|^{p-2}|\mathscr{F}(f)(x)|^p \frac{dx}{|\mathbf{c}(x)|^2}$$
$$\leqslant c\,\|f\|^p_{A,p} \ ,$$

which proves the result.

ii) Suppose now $\rho > 0,\ \alpha > -\frac{1}{2}$ and $g(x) = \begin{cases} x^{2(\alpha+1)} & \text{for } x > k \\ x^3 & \text{for } x \leqslant k, \end{cases}$ where k is the constant obtained in (2.3) and (2.4) from the estimates of $|\mathbf{c}(\lambda)|^{-2}$. Let $\lambda \in]0,+\infty[$ and $f \in L^1_A(\mathbb{R}_+)$, by (2.3), (2.4) and (2.5), we have

$$\int_{\{x\in\mathbb{R}_+:\,|L(f)(x)|>\lambda\}} \frac{dx}{(g(x))^2|\mathbf{c}(x)|^2}$$
$$\leqslant \int_{\{x\in\mathbb{R}_+:\,g(x)>\frac{\lambda}{\|f\|_{A,1}}\}} \frac{dx}{(g(x))^2|\mathbf{c}(x)|^2}$$
$$= \int_{\{x\in\mathbb{R}_+:\,g(x)>\frac{\lambda}{\|f\|_{A,1}}\}} \chi_{[0,k]}(x)\frac{dx}{(g(x))^2|\mathbf{c}(x)|^2}$$
$$+ \int_{\{x\in\mathbb{R}_+:\,g(x)>\frac{\lambda}{\|f\|_{A,1}}\}} \chi_{]k,+\infty[}(x)\frac{dx}{(g(x))^2|\mathbf{c}(x)|^2}$$
$$\leqslant c\int_{(\frac{\lambda}{\|f\|_{A,1}})^{\frac{1}{3}}}^{+\infty} \chi_{[0,k]}(x)\frac{x^2}{x^6}dx + c\int_{(\frac{\lambda}{\|f\|_{A,1}})^{\frac{1}{2(\alpha+1)}}}^{+\infty} \chi_{]k,+\infty[}(x)\frac{x^{2\alpha+1}}{x^{4(\alpha+1)}}dx$$
$$\leqslant c\int_{(\frac{\lambda}{\|f\|_{A,1}})^{\frac{1}{3}}}^{+\infty} x^{-4}dx + c\int_{(\frac{\lambda}{\|f\|_{A,1}})^{\frac{1}{2(\alpha+1)}}}^{+\infty} x^{-2\alpha-3}dx \leqslant c\,\frac{\|f\|_{A,1}}{\lambda}.$$

Hence L is of weak-type $(1,1)$ between the spaces $(\mathbb{R}_+, A(x)dx)$ and $(\mathbb{R}_+, \frac{dx}{(g(x))^2|\mathbf{c}(x)|^2})$.

We conclude by the Marcinkiewicz interpolation theorem that L is of strong-type (p,p), between the spaces under consideration.

It yields that

$$\int_{\mathbb{R}_+} |L(f)(x)|^p \frac{dx}{(g(x))^2|\mathbf{c}(x)|^2} = \int_{\mathbb{R}_+} |g(x)|^{p-2}|\mathscr{F}(f)(x)|^p \frac{dx}{|\mathbf{c}(x)|^2}$$
$$\leqslant c\,\|f\|^p_{A,p} \ ,$$

thus we obtain the result. □

In the following, we study the integrability of the generalized Fourier transform in the Jacobi hypergroup case (see Remark 2.1). For $1 \leqslant p \leqslant 2$, we denote by p' the conjugate of p.

LEMMA 3.2. *Let* $1 \leqslant p \leqslant 2$ *and* $f \in L^p_A(\mathbb{R}_+)$. *Then there exists a positive constant* c *such that for* $\delta > 0$ *one has*

$$\Big(\int_0^{+\infty} \min\{1,(\delta x)^{2p'}\}\,|\mathscr{F}(f)(x)|^{p'}\frac{dx}{|\mathbf{c}(x)|^2}\Big)^{\frac{1}{p'}} \leqslant c\,\omega_{A,p}(f)(\delta),\ \textit{if}\ 1 < p \leqslant 2$$

and

$$\operatorname*{ess\,sup}_{x>0}\Big(\min\{1,(\delta x)^2\}\,|\mathscr{F}(f)(x)|\Big)\leqslant c\,\omega_{A,1}(f)(\delta),\ \textit{if}\ p=1.$$

Proof. For $f\in L^p_A(\mathbb{R}_+)$, $1\leqslant p\leqslant 2$, we have by (2.8)

$$\mathscr{F}(\tau_\delta(f)-f)(x)=(\varphi_x(\delta)-1)\mathscr{F}(f)(x),$$

for $\delta>0$ and a.e $x\in\mathbb{R}_+$. Applying (2.7), we get

$$\begin{aligned}\|\mathscr{F}(\tau_\delta(f)-f)\|_{\mathbf{c},p'} &= \Big(\int_0^{+\infty}|1-\varphi_x(\delta)|^{p'}|\mathscr{F}(f)(x)|^{p'}\frac{dx}{|\mathbf{c}(x)|^2}\Big)^{\frac{1}{p'}}\\ &\leqslant c\,\omega_{A,p}(f)(\delta).\end{aligned}$$

From (2.1), we obtain our results. Here, when $p=1$, we make the usual modification. □

REMARK 3.1.

i) In Lemma 3.2, the gauge on the size of the generalized transform in terms of an integral modulus of continuity of f gives a quantitative form of the Riemann-Lebesgue lemma:

$$\Big(\int_{\frac{1}{\delta}}^{+\infty}|\mathscr{F}(f)(x)|^{p'}\frac{dx}{|\mathbf{c}(x)|^2}\Big)^{\frac{1}{p'}}\leqslant c\,\omega_{A,p}(f)(\delta),\ \text{if}\ 1<p\leqslant 2$$

and

$$\operatorname*{ess\,sup}_{x>\frac{1}{\delta}}|\mathscr{F}(f)(x)|\leqslant c\,\omega_{A,1}(f)(\delta),\ \text{if}\ p=1.$$

ii) We will use the following estimates deduced from Lemma 3.2 to establish the integrability of $\mathscr{F}(f)$ when f belongs to $\mathscr{B}^{p,\infty}_{\gamma,\alpha}$ for $1\leqslant p\leqslant 2$:

$$\delta^2\Big(\int_0^{\frac{1}{\delta}}x^{2p'}|\mathscr{F}(f)(x)|^{p'}\frac{dx}{|\mathbf{c}(x)|^2}\Big)^{\frac{1}{p'}}\leqslant c\,\omega_{A,p}(f)(\delta),\ \text{if}\ 1<p\leqslant 2 \tag{3.2}$$

and

$$\operatorname*{ess\,sup}_{0<x<\frac{1}{\delta}}\Big((\delta x)^2|\mathscr{F}(f)(x)|\Big)\leqslant c\,\omega_{A,1}(f)(\delta),\ \text{if}\ p=1. \tag{3.3}$$

THEOREM 3.1. *If* $f\in\mathscr{B}^{p,1}_{\frac{2(\alpha+1)}{p},\alpha}\cap\mathscr{B}^{p,1}_{\frac{3}{p},\alpha}$ *for* $1<p\leqslant 2$, *then*

$$\mathscr{F}(f)\in L^1_{\mathbf{c}}(\mathbb{R}_+).$$

Proof. For $f \in L^p_A(\mathbb{R}_+)$, $1 < p \leqslant 2$ and $\delta > 0$, we can write from (2.8) and (3.1)

$$\int_{\mathbb{R}_+} |1-\varphi_t(\delta)|^p |\mathscr{F}(\tau_\delta(f))(t)|^p (g(t))^{p-2} \frac{dt}{|\mathbf{c}(t)|^2} \leqslant c\,(\omega_{A,p}(f)(\delta))^p,$$

then by (2.1) we obtain

$$\delta^{2p} \int_0^{\frac{1}{\delta}} t^{2p} |\mathscr{F}(f)(t)|^p (g(t))^{p-2} \frac{dt}{|\mathbf{c}(t)|^2} \leqslant c\,(\omega_{A,p}(f)(\delta))^p. \tag{3.4}$$

From (2.3) and (2.4) we have

$$\begin{aligned}
&\int_0^{\frac{1}{\delta}} t|\mathscr{F}(f)(t)| \frac{dt}{|\mathbf{c}(t)|^2} \\
&= \int_0^{\frac{1}{\delta}} t|\mathscr{F}(f)(t)|\chi_{[0,k]}(t) \frac{dt}{|\mathbf{c}(t)|^2} + \int_0^{\frac{1}{\delta}} t|\mathscr{F}(f)(t)|\chi_{]k,+\infty[}(t) \frac{dt}{|\mathbf{c}(t)|^2} \\
&\leqslant c \int_0^{\frac{1}{\delta}} t|\mathscr{F}(f)(t)|\chi_{[0,k]}(t)[t^2 dt] + c \int_0^{\frac{1}{\delta}} t|\mathscr{F}(f)(t)|\chi_{]k,+\infty[}(t)[t^{2\alpha+1} dt],
\end{aligned}$$

by Hölder's inequality and (3.4), we have

$$\begin{aligned}
&\int_0^{\frac{1}{\delta}} t|\mathscr{F}(f)(t)| \frac{dt}{|\mathbf{c}(t)|^2} \\
&\leqslant c\left(\int_0^{\frac{1}{\delta}} t^{3(p-2)+2p} |\mathscr{F}(f)(t)|^p \chi_{[0,k]}(t)[t^2 dt]\right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{\delta}} t^{2(p'-2)} \chi_{[0,k]}(t) dt\right)^{\frac{1}{p'}} \\
&\quad + c\left(\int_0^{\frac{1}{\delta}} t^{2(\alpha+1)(p-2)+2p} |\mathscr{F}(f)(t)|^p \chi_{]k,+\infty[}(t)[t^{2\alpha+1} dt]\right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^{\frac{1}{\delta}} t^{(2\alpha+1)(p'-2)+2\alpha-1} \chi_{]k,+\infty[}(t) dt\right)^{\frac{1}{p'}} \\
&\leqslant c\left(\int_0^{\frac{1}{\delta}} t^{2p} |\mathscr{F}(f)(t)|^p (g(t))^{p-2} \frac{dt}{|\mathbf{c}(t)|^2}\right)^{\frac{1}{p}} \\
&\quad \times \left\{\left(\int_0^{\frac{1}{\delta}} t^{2(p'-2)} dt\right)^{\frac{1}{p'}} + \left(\int_0^{\frac{1}{\delta}} t^{(2\alpha+1)(p'-2)+2\alpha-1} dt\right)^{\frac{1}{p'}}\right\} \\
&\leqslant c\,\delta^{-2} \omega_{A,p}(f)(\delta)\left(\frac{1}{\delta^{\frac{3}{p}-1}} + \frac{1}{\delta^{\frac{2(\alpha+1)}{p}-1}}\right) \leqslant c\left(\frac{\omega_{A,p}(f)(\delta)}{\delta^{\frac{3}{p}}} \frac{1}{\delta} + \frac{\omega_{A,p}(f)(\delta)}{\delta^{\frac{2(\alpha+1)}{p}}} \frac{1}{\delta}\right).
\end{aligned}$$

Integrating with respect to δ over $\mathbb{R}_+$ for $f \in \mathscr{B}^{p,1}_{\frac{2(\alpha+1)}{p},\alpha} \cap \mathscr{B}^{p,1}_{\frac{3}{p},\alpha}$, the double integral is evaluated by interchanging the order of integration; this yields

$$\int_0^{+\infty} |\mathscr{F}(f)(t)| \frac{dt}{|\mathbf{c}(t)|^2} < +\infty.$$

This completes the proof. $\square$

THEOREM 3.2. *Let $\gamma > 0$, $1 \leqslant p \leqslant 2$ and $f \in \mathscr{B}^{p,\infty}_{\gamma,\alpha}$, then*

i) For $p \neq 1$ and $0 < \gamma \leqslant \frac{2(\alpha+1)}{p}$, one has $\mathscr{F}(f) \in L^s_{\mathbf{c}}(\mathbb{R}_+)$ provided that

$$\frac{2(\alpha+1)p}{\gamma p + 2(\alpha+1)(p-1)} < s \leqslant p'.$$

ii) For $p \neq 1$ and $\gamma > \frac{2(\alpha+1)}{p}$, one has

$$\mathscr{F}(f) \in L^1_{\mathbf{c}}(\mathbb{R}_+).$$

iii) For $p = 1$ and $\gamma > \sup\{3, 2(\alpha+1)\}$, one has

$$\mathscr{F}(f) \in L^1_{\mathbf{c}}(\mathbb{R}_+).$$

Proof. Let $f \in \mathscr{B}^{p,\infty}_{\gamma,\alpha}$, $1 \leqslant p \leqslant 2$.

i) Suppose that $p \neq 1$ and $0 < \gamma \leqslant \frac{2(\alpha+1)}{p}$. Let $\frac{2(\alpha+1)p}{\gamma p+2(\alpha+1)(p-1)} < s \leqslant p'$, we define the function

$$g(t) = \int_k^t |\mathscr{F}(f)(x)|^s x^s \frac{dx}{|\mathbf{c}(x)|^2}, \quad t > k.$$

By Hölder's inequality, (2.3) and (3.2) we have

$$\begin{aligned} g(t) &\leqslant \left(\int_k^t |\mathscr{F}(f)(x)|^{p'} x^{2p'} \frac{dx}{|\mathbf{c}(x)|^2}\right)^{\frac{s}{p'}} \left(\int_k^t \frac{dx}{|\mathbf{c}(x)|^2}\right)^{1-\frac{s}{p'}} \\ &\leqslant c t^{2s} (\omega_{A,p}(f)(\frac{1}{t}))^s \left(\int_k^t \frac{dx}{|\mathbf{c}(x)|^2}\right)^{1-\frac{s}{p'}} \\ &\leqslant c t^{(2-\gamma)s} \left(\int_k^t x^{2\alpha+1} dx\right)^{1-\frac{s}{p'}} \leqslant c t^{(2-\gamma)s+2(\alpha+1)(1-\frac{s}{p'})}. \end{aligned}$$

Then we get

$$\begin{aligned} \int_k^t |\mathscr{F}(f)(x)|^s \frac{dx}{|\mathbf{c}(x)|^2} &= \int_k^t x^{-2s} g'(x) dx \\ &= t^{-2s} g(t) + 2s \int_k^t x^{-2s-1} g(x) dx \\ &\leqslant c\left(t^{-\gamma s+2(\alpha+1)(1-\frac{s}{p'})} + \int_k^t x^{-\gamma s+2(\alpha+1)(1-\frac{s}{p'})-1} dx\right) \\ &\leqslant c\,(t^{-\gamma s+2(\alpha+1)(1-\frac{s}{p'})} + 1), \end{aligned}$$

it yields that $\mathscr{F}(f) \in L^s(]k, +\infty[, \frac{dx}{|\mathbf{c}(x)|^2})$. Since $L^{p'}([0,k], \frac{dx}{|\mathbf{c}(x)|^2}) \subset L^s([0,k], \frac{dx}{|\mathbf{c}(x)|^2})$ and $\mathscr{F}(f) \in L^{p'}([0,k], \frac{dx}{|\mathbf{c}(x)|^2})$, we deduce that $\mathscr{F}(f)$ is in $L^s_{\mathbf{c}}(\mathbb{R}_+)$.

ii) Assume now $\gamma > \frac{2(\alpha+1)}{p}$. For $p \neq 1$, by proceeding in the same manner as the proof of i) with $s = 1$, we obtain the desired result.

iii) For $p = 1$ and $\gamma > \sup\{3, 2(\alpha+1)\}$, by Hölder's inequality, (2.3), (2.4) and (3.3), we have for $t > 0$

$$\int_0^{\frac{1}{t}} |\mathscr{F}(f)(x)|x \frac{dx}{|\mathbf{c}(x)|^2} \leqslant \Big(ess \sup_{0<x\leqslant\frac{1}{t}} x^2|\mathscr{F}_k(f)(x)|\Big) \int_0^{\frac{1}{t}} \frac{1}{x} \frac{dx}{|\mathbf{c}(x)|^2}$$
$$\leqslant ct^{\gamma-2}\Big(\int_0^{\frac{1}{t}} \frac{1}{x}\chi_{[0,k]}(x)\frac{dx}{|\mathbf{c}(x)|^2} + \int_0^{\frac{1}{t}} \frac{1}{x}\chi_{]k,+\infty[}(x)\frac{dx}{|\mathbf{c}(x)|^2}\Big)$$
$$\leqslant ct^{\gamma-2}[t^{-2} + t^{-(2\alpha+1)}] \leqslant c\,[t^{(\gamma-3)-1} + t^{\gamma-2(\alpha+1)-1}].$$

Integrating with respect to t over $(0,1)$ and applying Fubini's theorem we obtain

$$\int_1^{+\infty} |\mathscr{F}(f)(x)|\frac{dx}{|\mathbf{c}(x)|^2} \leqslant c\Big(\int_0^1 t^{(\gamma-3)-1}dt + \int_0^1 t^{\gamma-2(\alpha+1)-1}dt\Big) < +\infty.$$

Since $L^\infty([0,1], \frac{dx}{|\mathbf{c}(x)|^2}) \subset L^1([0,1], \frac{dx}{|\mathbf{c}(x)|^2})$, then $\mathscr{F}(f) \in L^1_{\mathbf{c}}(\mathbb{R}_+)$. $\square$

REMARK 3.2. For $\gamma > \sup\{3, 2(\alpha+1)\}$, we can assert from Theorem 3.2, iii) that $\mathscr{B}^{1,\infty}_{\gamma,\alpha}$ is an example of space where we can apply the inversion formula.

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