Journal of
Mathematical
Inequalities
Volume 6, Number 3 (2012), 355–366 doi:10.7153/jmi-06-35

BOUNDS FOR SOME NEW INTEGRAL INEQUALITIES WITH DELAY ON TIME SCALES

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(*Communicated by A. Peterson*)

Abstract. The purpose of this paper is to investigate some delay integral inequalities on time scales. Our results unify and extend some delay integral inequalities and their corresponding discrete analogues. The inequalities given here can be used as handy tools in the qualities theory of certain class of delay dynamic equations on time scales.

1. Introduction

The study of dynamic equations on time scales goes back to its founder Hilger [1], which is an area of mathematics that has received a lot of attention. For example, see [2–10], and the references therein. However, there are few people studied the delay integral inequalities on time scales as far as we know. In this paper, we investigate some new delay integral inequalities on time scales, which provide explicit bounds on unknown functions, which extend some integral inequalities and their corresponding discrete analogues established by Li [7] and Xu [9]. The inequalities given here can be used as handy tools in the qualitative theory of certain classes of delay dynamic equations on time scales.

Throughout this paper, let us assume that we have already acquired the knowledge of time scales and time scales notion. For an excellent introduction to the calculus on time scales, we refer the reader to monographes [3, 4].

2. Some Preliminaries

In what follows, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = [0,\infty)$, $\mathbb{Z}$ denotes the set of integers, $\mathbb{N}_0 = \{0,1,2,\cdots,n\}$ denotes the set of nonnegative integers, $C(M,S)$ denotes the class of all continuous functions defined on set M with range in the set S, $\mathbb{T}$ is an arbitrary time scale, C_{rd} denotes the set of rd-continuous, $\mathscr{R}$ denotes the set of all regressive and rd-continuous functions, and $\mathscr{R}^+ = \{p \in \mathscr{R} : 1+\mu(t)p(t) > 0,\ t \in \mathbb{T}\}$. Throughout this paper, we always assume that $t_0 \in \mathbb{T}$, $\mathbb{T}_0 = [t_0,\infty) \cap \mathbb{T}$.

The following lemmas are very useful in our main results.

Mathematics subject classification (2010): 34C10.
Keywords and phrases: Integral inequalities, time scales, delay.
This research was supported by National Science of China (11171178), the fund of subject for doctor of ministry of education (20103705110003) and the Natural Science Foundations of Shandong Province of china (ZR2009AM011 and ZR2009AL015).

LEMMA 2.1. ([11]) *Let* $0<\alpha\leqslant 1$ *and* $x>-1$. *Then* $(1+x)^{\alpha}\leqslant 1+\alpha x$.

LEMMA 2.2. ([10]) *Let* $u(t),\ f(t),\ g(t)\in C_{rd}$, $u(t),\ f(t)$ *and* $g(t)$ *be nonnegative. If* $f(t)$ *is nondecreasing, then*

$$u(t)\leqslant f(t)+\int_{t_0}^{t} g(\tau)u(\tau)\Delta\tau,\ \ t\in\mathbb{T}_0$$

implies

$$u(t)\leqslant f(t)e_g(t,t_0),\ \ t\in\mathbb{T}_0.$$

3. Main results

In this section, we study some integral inequalities on time scales. We always assume that $t\geqslant t_0$, $t\in\mathbb{T}_0$.

THEOREM 3.1. *Assume that* $u(t),\ a(t),\ b(t),\ c(t),\ f(t),\ g(t)\in C_{rd}\ (\mathbb{T}_0,\mathbb{R}_+)$. *If* $a(t)$ *and* $b(t)$ *are nonnegative,* $a(t)$ *is nondecreasing for* $t\in\mathbb{T}_0$, *and* $u(t)$ *satisfies the following form of delay integral inequality on time scales*

$$u^p(t)\leqslant a(t)+b(t)\sum_{i=1}^{n}\int_{t_0}^{t}\left[f(s)u^{q_i}(\sigma(s))+\int_{s_0}^{s} g(\tau)u^{r_i}(\tau)\Delta\tau+c(s)\right]\Delta s,\ \ t\in\mathbb{T}_0 \quad (3.1)$$

with the initial condition

$$\begin{cases} u(t)=\varphi(t), t\in[\alpha,t_0]\cap\mathbb{T} \\ \varphi(\sigma(t))\leqslant a^{\frac{1}{p}}(t)\ \ for\ t\in\mathbb{T}_0\ with\ \sigma(t)\leqslant t_0 \end{cases} \quad (3.2)$$

implies

$$u(t)\leqslant [a(t)+b(t)A(t)e_B(t,t_0)]^{\frac{1}{p}},\ \ t\in\mathbb{T}_0, \quad (3.3)$$

where $p\neq 0,\ p\geqslant q_i\geqslant 0,\ p\geqslant r_i\geqslant 0,\ p,\ q_i,\ r_i\ (i=1,2,\cdots,n)$ *are constants.* $\sigma:\mathbb{T}_0\to\mathbb{T},\ \sigma(t)\leqslant t, -\infty<\alpha=inf\{\sigma(t),\ t\in\mathbb{T}_0\}\leqslant t_0$, *and* $\varphi(t)\in C_{rd}\{[\alpha,t_0]\cap\mathbb{T},\mathbb{R}_+\}$.

$$A(t)=\sum_{i=1}^{n}\int_{t_0}^{t}\left[f(s)a^{\frac{q_i}{p}}(s)+\int_{s_0}^{s} g(\tau)a^{\frac{r_i}{p}}(\tau)\Delta\tau+c(s)\right]\Delta s,\ \ t\in\mathbb{T}_0, \quad (3.4)$$

$$B(t)=\sum_{i=1}^{n}\left[\frac{q_i}{p}f(t)a^{\frac{q_i}{p}-1}(t)b(t)+\frac{r_i}{p}\int_{s_0}^{t} g(\tau)a^{\frac{r_i}{p}-1}(\tau)b(\tau)\Delta\tau\right],\ \ t\in\mathbb{T}_0. \quad (3.5)$$

Proof. Define a function $z(t)$ by

$$z(t)=\left\{a(t)+b(t)\sum_{i=1}^{n}\int_{t_0}^{t}\left[f(s)u^{q_i}(\sigma(s))+\int_{s_0}^{s} g(\tau)u^{r_i}(\tau)\Delta\tau+c(s)\right]\Delta s\right\}^{\frac{1}{p}}. \quad (3.6)$$

It is easy to see that $z(t)$ is a nonnegative and nondecreasing function, and

$$u(t)\leqslant z(t). \quad (3.7)$$

Therefore, for $t \in \mathbb{T}_0$ with $\sigma(t) \geqslant t_0$, we have

$$u(\sigma(t)) \leqslant z(\sigma(t)) \leqslant z(t). \tag{3.8}$$

On the other hand, using the initial condition (3.2), for $t \in \mathbb{T}_0$ with $\sigma(t) \leqslant t_0$, we have

$$u(\sigma(t)) = \varphi(\sigma(t)) \leqslant a^{\frac{1}{p}}(t) \leqslant z(t). \tag{3.9}$$

Combining (3.8) and (3.9), we obtain

$$u(\sigma(t)) \leqslant z(t). \tag{3.10}$$

It follows from $(3.6), (3.7)$ and (3.10) that

$$z^p(t) \leqslant a(t) + b(t) \sum_{i=1}^{n} \int_{t_0}^{t} \left[f(s) z^{q_i}(s) + \int_{s_0}^{s} g(\tau) z^{r_i}(\tau) \Delta\tau + c(s) \right] \Delta s. \tag{3.11}$$

Define a function $v(t)$ by

$$v(t) = \sum_{i=1}^{n} \int_{t_0}^{t} \left[f(s) z^{q_i}(s) + \int_{s_0}^{s} g(\tau) z^{r_i}(\tau) \Delta\tau + c(s) \right] \Delta s. \tag{3.12}$$

Then (3.11) can be restated as

$$z^p(t) \leqslant a(t) + b(t) v(t). \tag{3.13}$$

Using Lemma 2.1, we easily obtain

$$\begin{aligned} z^{q_i}(t) &\leqslant [a(t) + b(t)v(t)]^{\frac{q_i}{p}} \leqslant a^{\frac{q_i}{p}}(t) + \frac{q_i}{p} a^{\frac{q_i}{p}-1}(t) b(t) v(t), \\ z^{r_i}(t) &\leqslant [a(t) + b(t)v(t)]^{\frac{r_i}{p}} \leqslant a^{\frac{r_i}{p}}(t) + \frac{r_i}{p} a^{\frac{r_i}{p}-1}(t) b(t) v(t). \end{aligned} \tag{3.14}$$

It follows from (3.12) and (3.14) that

$$\begin{aligned} v(t) \leqslant \sum_{i=1}^{n} \int_{t_0}^{t} &\left[f(s) \left(a^{\frac{q_i}{p}}(s) + \frac{q_i}{p} a^{\frac{q_i}{p}-1}(s) b(s) v(s) \right) \right. \\ &\left. + \int_{s_0}^{s} g(\tau) \left(a^{\frac{r_i}{p}}(\tau) + \frac{r_i}{p} a^{\frac{r_i}{p}-1}(\tau) b(\tau) v(\tau) \right) \Delta\tau + c(s) \right] \Delta s \\ \leqslant A(t) + \int_{t_0}^{t} & B(s) v(s) \Delta s, \quad t \in \mathbb{T}_0, \end{aligned} \tag{3.15}$$

where $A(t)$ and $B(t)$ are defined by (3.4) and (3.5) respectively. Obviously, $A(t)$, $B(t) \in C_{rd}$, $A(t)$ and $B(t)$ are nonnegative, $A(t)$ is nondecreasing. Using Lemma 2.2, from (3.15) we have

$$v(t) \leqslant A(t) e_B(t, t_0). \tag{3.16}$$

Therefore, the desired inequality (3.3) follows from (3.7), (3.13) and (3.16). $\square$

REMARK 3.1. Theorem 3.1 extends some known inequalities on time scales. If $i = 1$, $q_i = 1$, $g(t) = 0$, then Theorem 3.1 reduces to [7, Theorem 1]. If $i = 1$, $\sigma(t) = t$, $g(t) = 0$, then Theorem 3.1 reduces to the form of [8, Theorem 1]. If $i = 1$, $\sigma(t) = t$, then Theorem 3.1 reduces to [9, Theorem 3.1]. Let $\mathbb{T} = \mathbb{R}$, if $i = 1$, $c(t) = 0$, then Theorem 3.1 reduces to [12, Theorem 1].

COROLLARY 3.1. *Let $\mathbb{T}=\mathbb{Z}$, assume that $u(t)$, $a(t)$, $b(t)$, $c(t)$, $f(t)$, $g(t)$ are nonnegative functions defined for $t\in\mathbb{N}_0$. If $a(t)$ and $b(t)$ are nondecreasing in $\mathbb{N}_0$, and $u(t)$ satisfies the following inequality*

$$u^p(t)\leqslant a(t)+b(t)\sum_{i=1}^{n}\sum_{s=0}^{t-1}\left[f(s)u^{q_i}(s-\rho)+\sum_{\tau=0}^{s-1}g(\tau)u^{r_i}(\tau)+c(s)\right],\ t\in\mathbb{N}_0, \tag{3.17}$$

with the initial condition

$$\begin{cases} u(t)=\varphi(t), t\in[-\rho,\cdots,-1,0] \\ \varphi(t-\rho)\leqslant a^{\frac{1}{p}}(t)\ \ for\ t\in\mathbb{N}_0\ with\ t-\rho\leqslant 0 \end{cases} \tag{3.18}$$

implies

$$u(t)\leqslant\left\{a(t)+b(t)\overline{A}(t)\prod_{s=0}^{t-1}\left[1+\overline{B}(s)\right]\right\}^{\frac{1}{p}},\ t\in\mathbb{N}_0, \tag{3.19}$$

where $p\neq 0$, $p\geqslant q_i\geqslant 0$, $p\geqslant r_i\geqslant 0$, $\rho\in\mathbb{N}_0$, p, q_i, $r_i(i=1,2,\cdots,n),\rho$ are constants, $\varphi(t)\in\mathbb{R}_+$, $t\in[-\rho,\cdots,-1,0]$,

$$\overline{A}(t)=\sum_{i=1}^{n}\sum_{s=0}^{t-1}\left[f(s)a^{\frac{q_i}{p}}(s)+\sum_{\tau=0}^{s-1}g(\tau)a^{\frac{r_i}{p}}(\tau)+c(s)\right],\ t\in\mathbb{N}_0, \tag{3.20}$$

$$\overline{B}(t)=\sum_{i=1}^{n}\left[\frac{q_i}{p}f(t)a^{\frac{q_i}{p}-1}(t)b(t)+\frac{r_i}{p}\sum_{\tau=0}^{t-1}g(\tau)a^{\frac{r_i}{p}-1}(\tau)b(\tau)\right],\ t\in\mathbb{N}_0. \tag{3.21}$$

THEOREM 3.2. *Assume that $u(t)$, $a(t)$, $b(t)$, $c(t)$, $f(t)$, $g(t)\in C_{rd}(\mathbb{T}_0,\mathbb{R}_+)$. If $a(t)$ and $b(t)$ are nonnegative, $a(t)$ is nondecreasing for $t\in\mathbb{T}_0$, and $u(t)$ satisfies the following form of delay integral inequality on time scales*

$$\begin{aligned} u^p(t)\leqslant & a(t)+\int_{t_0}^{t}b(s)u^p(s)\triangle s \\ & +\sum_{i=1}^{n}\int_{t_0}^{t}\left[f(s)u^{q_i}(\sigma(s))+\int_{s_0}^{s}g(\tau)u^{r_i}(\tau)\Delta\tau+c(s)\right]\Delta s,\quad t\in\mathbb{T}_0 \end{aligned} \tag{3.22}$$

with the initial condition (3.2), *implies*

$$u(t)\leqslant\left[e_b(t,t_0)\big(a(t)+C(t)e_D(t,t_0)\big)\right]^{\frac{1}{p}},\ t\in\mathbb{T}_0, \tag{3.23}$$

where

$$C(t)=\sum_{i=1}^{n}\int_{t_0}^{t}\left[f(s)e_b^{\frac{q_i}{p}}(s,t_0)a^{\frac{q_i}{p}}(s)+\int_{s_0}^{s}g(\tau)e_b^{\frac{r_i}{p}}(\tau,t_0)a^{\frac{r_i}{p}}(\tau)\Delta\tau+c(s)\right]\Delta s, t\in\mathbb{T}_0, \tag{3.24}$$

$$D(t)=\sum_{i=1}^{n}\left[\frac{q_i}{p}f(t)e_b^{\frac{q_i}{p}}(t,t_0)a^{\frac{q_i}{p}-1}(t)+\frac{r_i}{p}\int_{s_0}^{t}g(\tau)e_b^{\frac{r_i}{p}}(\tau,t_0)a^{\frac{r_i}{p}-1}(\tau)\Delta\tau\right],\ t\in\mathbb{T}_0. \tag{3.25}$$

Proof. Define a function $z(t)$ by

$$z(t)=\left\{a(t)+\int_{t_0}^{t}b(s)u^p(s)\triangle s+\sum_{i=1}^{n}\int_{t_0}^{t}\left[f(s)u^{q_i}(\sigma(s))+\int_{s_0}^{s}g(\tau)u^{r_i}(\tau)\Delta\tau+c(s)\right]\Delta s\right\}^{\frac{1}{p}}. \tag{3.26}$$

Using a similar way in the proof of Theorem (3.1), we easily obtain that $z(t)$ is a nonnegative and nondecreasing function, and

$$u(t)\leqslant z(t),\quad u(\sigma(t))\leqslant z(t). \tag{3.27}$$

It follows from $(3.26),(3.27)$ that

$$z^p(t)\leqslant a(t)+\int_{t_0}^{t}b(s)z^p(s)\triangle s+\sum_{i=1}^{n}\int_{t_0}^{t}\left[f(s)z^{q_i}(s)+\int_{s_0}^{s}g(\tau)z^{r_i}(\tau)\Delta\tau+c(s)\right]\Delta s. \tag{3.28}$$

Define a function $v(t)$ by

$$v(t)=a(t)+w(t), \tag{3.29}$$

where

$$w(t)=\sum_{i=1}^{n}\int_{t_0}^{t}\left[f(s)z^{q_i}(s)+\int_{s_0}^{s}g(\tau)z^{r_i}(\tau)\Delta\tau+c(s)\right]\Delta s. \tag{3.30}$$

Then (3.28) can be restated as

$$z^p(t)\leqslant v(t)+\int_{t_0}^{t}b(s)z^p(s)\triangle s. \tag{3.31}$$

Noting that $v(t)$ is nondecreasing, from (3.31) and Lemma 2.2, we obtain

$$z^p(t)\leqslant v(t)e_b(t,t_0). \tag{3.32}$$

From (3.29) and (3.32), we have

$$z(t)\leqslant e_b^{\frac{1}{p}}(t,t_0)[a(t)+w(t)]^{\frac{1}{p}}. \tag{3.33}$$

It follows from $(3.30),(3.33)$ and Lemma 2.1 that

$$\begin{aligned}w(t)\leqslant&\sum_{i=1}^{n}\int_{t_0}^{t}\left[f(s)e_b^{\frac{q_i}{p}}(s,t_0)\left(a^{\frac{q_i}{p}}(s)+\frac{q_i}{p}a^{\frac{q_i}{p}-1}(s)w(s)\right)\right.\\&\left.+\int_{s_0}^{s}g(\tau)e_b^{\frac{r_i}{p}}(\tau,t_0)\left(a^{\frac{r_i}{p}}(\tau)+\frac{r_i}{p}a^{\frac{r_i}{p}-1}(\tau)w(\tau)\right)\Delta\tau+c(s)\right]\Delta s\\&\leqslant C(t)+\int_{t_0}^{t}D(s)w(s)\Delta s,\ t\in\mathbb{T}_0,\end{aligned} \tag{3.34}$$

where $C(t)$ and $D(t)$ are defined by (3.24) and (3.25) respectively. Obviously, $C(t)$, $D(t) \in C_{rd}$, $C(t)$ and $D(t)$ are nonnegative, $C(t)$ is nondecreasing. Using Lemma 2.2, from (3.34) we have

$$w(t) \leqslant C(t)e_D(t,t_0). \tag{3.35}$$

Therefore, the desired inequality (3.23) follows from (3.27), (3.33) and (3.35). □

REMARK 3.2. If $i = 1$, $q_i = 1$, $g(t) = 0$, then Theorem 3.2 reduces to [7, Theorem 3]. Let $\mathbb{T} = \mathbb{R}$, if $i = 1$, $c(t) = 0$, then Theorem 3.2 reduces to [12, Theorem 2].

COROLLARY 3.2. *Let $\mathbb{T} = \mathbb{Z}$ and assume that $u(t)$, $a(t)$, $b(t)$, $c(t)$, $f(t)$, $g(t)$ are nonnegative functions defined for $t \in \mathbb{N}_0$. If $a(t)$ is nondecreasing in $\mathbb{N}_0$, and $u(t)$ satisfies the following inequality*

$$\begin{aligned} u^p(t) \leqslant & a(t) + \sum_{s=0}^{t-1} b(s)u^p(s) \\ & + \sum_{i=1}^{n}\sum_{s=0}^{t-1}\left[f(s)u^{q_i}(s-\rho) + \sum_{\tau=0}^{s-1} g(\tau)u^{r_i}(\tau) + c(s)\right], t \in \mathbb{N}_0 \end{aligned} \tag{3.36}$$

with the initial condition (3.18), implies

$$u(t) \leqslant R^{\frac{1}{p}}(t)\left\{a(t) + \overline{C}(t)\prod_{s=0}^{t-1}\left[1+\overline{D}(s)\right]\right\}^{\frac{1}{p}}, \quad t \in \mathbb{N}_0, \tag{3.37}$$

where

$$R(t) = \prod_{s=0}^{t-1}[1+b(s)], \quad t \in \mathbb{N}_0, \tag{3.38}$$

$$\overline{C}(t) = \sum_{i=1}^{n}\sum_{s=0}^{t-1}\left[f(s)R^{\frac{q_i}{p}}(s)a^{\frac{q_i}{p}}(s) + \sum_{\tau=0}^{s-1} g(\tau)R^{\frac{r_i}{p}}(\tau)a^{\frac{r_i}{p}}(\tau) + c(s)\right], \; t \in \mathbb{N}_0, \tag{3.39}$$

$$\overline{D}(t) = \sum_{i=1}^{n}\left[\frac{q_i}{p}f(t)R^{\frac{q_i}{p}}(t)a^{\frac{q_i}{p}-1}(t) + \frac{r_i}{p}\sum_{\tau=0}^{t-1} g(\tau)R^{\frac{r_i}{p}}(\tau)a^{\frac{r_i}{p}-1}(\tau)\right], \; t \in \mathbb{N}_0. \tag{3.40}$$

THEOREM 3.3. *Assume that $u(t)$, $a(t)$, $b(t)$, $c(t)$, $f(t)$, $g(t) \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$. If $a(t)$ and $b(t)$ are nonnegative, $a(t)$ is nondecreasing for $t \in \mathbb{T}_0$, and $u(t)$ satisfies the following form of delay integral inequality on time scales*

$$\begin{aligned} u^p(t) \leqslant & a(t) + \int_{t_0}^{t} b(s)u^p(s)\triangle s + \int_{t_0}^{t}\left[\sum_{i=1}^{n} f(s)u^{q_i}(s)\right. \\ & \left. + \int_{s_0}^{s} g(\tau)u^r(\sigma(\tau))\Delta\tau + c(s)\right]\Delta s, \quad t \in \mathbb{T}_0 \end{aligned} \tag{3.41}$$

with the initial condition (3.2), *implies*

$$u(t) \leqslant \left[e_b(t,t_0)\big(a(t)+E(t)e_F(t,t_0)\big)\right]^{\frac{1}{p}}, \ \ t \in \mathbb{T}_0, \tag{3.42}$$

where

$$E(t) = \int_{t_0}^{t} \left[\sum_{i=1}^{n} f(s)e_b^{\frac{q_i}{p}}(s,t_0)a^{\frac{q_i}{p}}(s) + \int_{s_0}^{s} g(\tau)e_b^{\frac{r}{p}}(\tau,t_0)a^{\frac{r}{p}}(\tau)\Delta\tau + c(s)\right]\Delta s, \ t \in \mathbb{T}_0, \tag{3.43}$$

$$F(t) = \sum_{i=1}^{n} \frac{q_i}{p} f(t)e_b^{\frac{q_i}{p}}(t,t_0)a^{\frac{q_i}{p}-1}(t) + \frac{r}{p}\int_{s_0}^{t} g(\tau)e_b^{\frac{r}{p}}(\tau,t_0)a^{\frac{r}{p}-1}(\tau)\Delta\tau, \ \ t \in \mathbb{T}_0. \tag{3.44}$$

THEOREM 3.4. *Assume that* $u(t),\ a(t),\ b(t),\ c(t),\ f(t),\ g(t) \in C_{rd}(\mathbb{T}_0,\mathbb{R}_+)$.*If* $a(t)$ *and* $b(t)$ *are nonnegative,* $a(t)$ *is nondecreasing for* $t \in \mathbb{T}_0$, *and* $u(t)$ *satisfies the following form of delay integral inequality on time scales*

$$\begin{aligned} u^p(t) \leqslant & a(t) + \int_{t_0}^{t} b(s)u^p(s)\triangle s \\ & + \int_{t_0}^{t}\left[f(s)u^q(s) + \sum_{i=1}^{n}\int_{s_0}^{s} g(\tau)u^{r_i}(\sigma(\tau))\Delta\tau + c(s)\right]\Delta s, t \in \mathbb{T}_0 \end{aligned} \tag{3.45}$$

with the initial condition (3.2), *implies*

$$u(t) \leqslant \left[e_b(t,t_0)\big(a(t)+\overline{E}(t)e_{\overline{F}}(t,t_0)\big)\right]^{\frac{1}{p}}, \ \ t \in \mathbb{T}_0, \tag{3.46}$$

where

$$\overline{E}(t) = \int_{t_0}^{t}\left[f(s)e_b^{\frac{q}{p}}(s,t_0)a^{\frac{q}{p}}(s) + \sum_{i=1}^{n}\int_{s_0}^{s} g(\tau)e_b^{\frac{r_i}{p}}(\tau,t_0)a^{\frac{r_i}{p}}(\tau)\Delta\tau + c(s)\right]\Delta s, \ t \in \mathbb{T}_0, \tag{3.47}$$

$$\overline{F}(t) = \frac{q}{p} f(t)e_b^{\frac{q}{p}}(t,t_0)a^{\frac{q}{p}-1}(t) + \sum_{i=1}^{n}\frac{r_i}{p}\int_{s_0}^{t} g(\tau)e_b^{\frac{r_i}{p}}(\tau,t_0)a^{\frac{r_i}{p}-1}(\tau)\Delta\tau, \ \ t \in \mathbb{T}_0. \tag{3.48}$$

THEOREM 3.5. *Let* $u(t),\ a(t),\ b(t),\ c(t),\ f(t),\ g(t) \in C_{rd}(\mathbb{T}_0,\mathbb{R}_+)$. *If* $a(t)$ *and* $b(t)$ *are nonnegative,* $a(t)$ *is nondecreasing for* $t \in \mathbb{T}_0$. *Assume that there exists of a series of positive real numbers* $q_i,\ r_i$ *such that* $p \geqslant q_i > 0,\ p \geqslant r_i > 0,\ i = 1,2,\cdots,n$. *If* $L_i : \mathbb{T}_0 \times \mathbb{R}_+ \to \mathbb{R}_+$ *is a continuous function such that*

$$0 \leqslant L_i(t,x_i) - L_i(t,y_i) \leqslant M_i(t,y_i)(x_i - y_i) \tag{3.49}$$

for $t \in \mathbb{T}_0$ *and* $x_i \geqslant y_i \geqslant 0,\ i = 1,2,\cdots,n$, *where* $M_i : \mathbb{T}_0 \times \mathbb{R}_+ \to \mathbb{R}_+$ *is a nonnegative continuous function,* $i = 1,2,\cdots,n$, *then*

$$\begin{aligned} u^p(t) \leqslant & a(t) + \int_{t_0}^{t} b(s)u^p(s)\triangle s \\ & + \sum_{i=1}^{n}\int_{t_0}^{t} L_i\left(s, f(s)u^{q_i}(\sigma(s)) + \int_{s_0}^{s} g(\tau)u^{r_i}(\tau)\Delta\tau + c(s)\right)\Delta s, \ \ t \in \mathbb{T}_0 \end{aligned} \tag{3.50}$$

with the initial condition (3.2), *implies*

$$u(t) \leqslant [e_b(t,t_0)\big(a(t)+G(t)e_H(t,t_0)\big)]^{\frac{1}{p}}, \quad t \in \mathbb{T}_0, \tag{3.51}$$

where

$$\begin{aligned} G(t) = \sum_{i=1}^{n} \int_{t_0}^{t} L_i \Bigg(& s, f(s) e_b^{\frac{q_i}{p}}(s,t_0) a^{\frac{q_i}{p}}(s) \\ & + \int_{s_0}^{s} g(\tau) e_b^{\frac{r_i}{p}}(\tau,t_0) a^{\frac{r_i}{p}}(\tau)\Delta\tau + c(s) \Bigg) \Delta s, \quad t \in \mathbb{T}_0, \end{aligned} \tag{3.52}$$

$$\begin{aligned} H(t) = \sum_{i=1}^{n} M_i & \left(t, f(t) e_b^{\frac{q_i}{p}}(t,t_0) a^{\frac{q_i}{p}}(t) + \int_{s_0}^{t} g(\tau) e_b^{\frac{r_i}{p}}(\tau,t_0) a^{\frac{r_i}{p}}(\tau)\Delta\tau + c(t) \right) \\ & \cdot \left[\frac{q_i}{p} f(t) e_b^{\frac{q_i}{p}}(t,t_0) a^{\frac{q_i}{p}-1}(t) + \frac{r_i}{p} \int_{s_0}^{t} g(\tau) e_b^{\frac{r_i}{p}}(\tau,t_0) a^{\frac{r_i}{p}-1}(\tau)\Delta\tau \right], \ t \in \mathbb{T}_0. \end{aligned} \tag{3.53}$$

Proof. Define a function $z(t)$ by

$$\begin{aligned} z(t) = \Bigg\{ & a(t) + \int_{t_0}^{t} b(s) u^p(s) \triangle s \\ & + \sum_{i=1}^{n} \int_{t_0}^{t} L_i \left(s, f(s) u^{q_i}(\sigma(s)) + \int_{s_0}^{s} g(\tau) u^{r_i}(\tau)\Delta\tau + c(s) \right) \Delta s \Bigg\}^{\frac{1}{p}}. \end{aligned} \tag{3.54}$$

Using a similar way in the proof of Theorem (3.1), we easily obtain that $z(t)$ is a nonnegative and nondecreasing function, and

$$u(t) \leqslant z(t), \quad u(\sigma(t)) \leqslant z(t). \tag{3.55}$$

It follows from (3.54) and (3.55) that

$$z^p(t) \leqslant a(t) + \int_{t_0}^{t} b(s) z^p(s) \triangle s + \sum_{i=1}^{n} \int_{t_0}^{t} L_i \left(s, f(s) z^{q_i}(s) + \int_{s_0}^{s} g(\tau) z^{r_i}(\tau)\Delta\tau + c(s) \right) \Delta s. \tag{3.56}$$

Define a function $v(t)$ by

$$v(t) = a(t) + w(t), \tag{3.57}$$

where

$$w(t) = \sum_{i=1}^{n} \int_{t_0}^{t} L_i \left(s, f(s) z^{q_i}(s) + \int_{s_0}^{s} g(\tau) z^{r_i}(\tau)\Delta\tau + c(s) \right) \Delta s. \tag{3.58}$$

Then (3.56) can be restated as

$$z^p(t) \leqslant v(t) + \int_{t_0}^{t} b(s) z^p(s) \triangle s. \tag{3.59}$$

Noting that $v(t)$ is nondecreasing, from (3.59) and Lemma 2.2, we obtain

$$z^p(t) \leqslant v(t) e_b(t,t_0). \tag{3.60}$$

From (3.57) and (3.60), we have

$$z(t) \leqslant e_b^{\frac{1}{p}}(t,t_0)[a(t)+w(t)]^{\frac{1}{p}},. \tag{3.61}$$

It follows from (3.58), (3.61) and Lemma 2.1 that

$$\begin{aligned} w(t) \leqslant \sum_{i=1}^{n} \int_{t_0}^{t} \Big\{ & L_i\left(s, f(s) e_b^{\frac{q_i}{p}}(s,t_0)\big(a^{\frac{q_i}{p}}(s) + \frac{q_i}{p} a^{\frac{q_i}{p}-1}(s) w(s)\big)\right. \\ & \left. + \int_{s_0}^{s} g(\tau) e_b^{\frac{r_i}{p}}(\tau,t_0)\big(a^{\frac{r_i}{p}}(\tau) + \frac{r_i}{p} a^{\frac{r_i}{p}-1}(\tau) w(\tau)\big)\Delta\tau + c(s)\right) \\ & -L_i\left(s, f(s) e_b^{\frac{q_i}{p}}(s,t_0) a^{\frac{q_i}{p}}(s) + \int_{s_0}^{s} g(\tau) e_b^{\frac{r_i}{p}}(\tau,t_0) a^{\frac{r_i}{p}}(\tau)\Delta\tau + c(s)\right) \\ & +L_i\left(s, f(s) e_b^{\frac{q_i}{p}}(s,t_0) a^{\frac{q_i}{p}}(s) + \int_{s_0}^{s} g(\tau) e_b^{\frac{r_i}{p}}(\tau,t_0) a^{\frac{r_i}{p}}(\tau)\Delta\tau + c(s)\right)\Big\}\Delta s \\ \leqslant G(t) + & \int_{t_0}^{t} H(s) w(s)\Delta s, \;\; t \in \mathbb{T}_0, \end{aligned} \tag{3.62}$$

where $G(t)$ and $H(t)$ are defined by (3.52) and (3.53) respectively. Obviously, $G(t)$, $H(t) \in C_{rd}$, $G(t)$ and $H(t)$ are nonnegative, $G(t)$ is nondecreasing. Using Lemma 2.2, from (3.62) we have

$$w(t) \leqslant G(t) e_H(t,t_0). \tag{3.63}$$

Therefore, the desired inequality (3.51) follows from (3.55), (3.61) and (3.62). $\square$

REMARK 3.3. If $i=1$, $q_i=1$, $f(t)=0$, $c(t)=0$, then Theorem 3.5 reduce to $[7$, Theorem $5]$. If $f(t)=1$, $g(t)=0$, $c(t)=0$, then Theorem 3.5 reduce to [8, Theorem 2.16].

COROLLARY 3.3. *Let $\mathbb{T}=\mathbb{R}$, $u(t)$, $a(t)$, $b(t)$, $c(t)$, $f(t)$, $g(t) \in C(\mathbb{R}_+,\mathbb{R}_+)$. $u(t)$ and $a(t)$ are nonnegative, $a(t)>0$ and $a(t)$ is nondecreasing for $t \in \mathbb{R}_+$. If the assume of theorem 3.5 are all hold, then*

$$\begin{aligned} u^p(t) \leqslant & a(t) + \int_0^t b(s) u^p(s) ds \\ & + \sum_{i=1}^{n} \int_0^t L_i\left(s, f(s) u^{q_i}(\sigma(s)) + \int_0^s g(\tau) u^{r_i}(\tau) d\tau + c(s)\right) ds, \quad t \in \mathbb{R}_+ \end{aligned} \tag{3.64}$$

with the initial condition (3.2) with $t \in \mathbb{R}_+$, implies

$$u(t) \leqslant \left[\overline{R}(t)\left(a(t) + \overline{G}(t) \exp \int_0^t \overline{H}(s) ds\right)\right]^{\frac{1}{p}}, \;\; t \in \mathbb{R}_+, \tag{3.65}$$

where

$$\overline{R}(t) = \exp \int_0^t b(s) ds, \;\; t \in \mathbb{R}_+, \tag{3.66}$$

$$\overline{G}(t)=\sum_{i=1}^{n}\int_{0}^{t}L_i\left(s,f(s)\overline{R}^{\frac{q_i}{p}}(s)a^{\frac{q_i}{p}}(s)+\int_{0}^{s}g(\tau)\overline{R}^{\frac{r_i}{p}}(\tau)a^{\frac{r_i}{p}}(\tau)d\tau+c(s)\right)ds,\ \ t\in\mathbb{R}_+, \tag{3.67}$$

$$\begin{aligned}\overline{H}(t)=&\sum_{i=1}^{n}M_i\left(t,f(t)\overline{R}^{\frac{q_i}{p}}(t)a^{\frac{q_i}{p}}(t)+\int_{0}^{t}g(\tau)\overline{R}^{\frac{r_i}{p}}(\tau)a^{\frac{r_i}{p}}(\tau)d\tau+c(t)\right)\\&\cdot\left[\frac{q_i}{p}f(t)\overline{R}^{\frac{q_i}{p}}(t)a^{\frac{q_i}{p}-1}(t)+\frac{r_i}{p}\int_{0}^{t}g(\tau)\overline{R}^{\frac{r_i}{p}}(\tau)a^{\frac{r_i}{p}-1}(\tau)d\tau\right],\ \ t\in\mathbb{R}_+.\end{aligned} \tag{3.68}$$

COROLLARY 3.4. *Let $\mathbb{T}=\mathbb{Z}$, Assume that $u(t)$, $a(t)$, $b(t)$, $c(t)$, $f(t)$, $g(t)$ are nonnegative functions defined for $t\in\mathbb{N}_0$. If $a(t)$ is nondecreasing in $\mathbb{N}_0$, and $u(t)$ satisfies the following inequality*

$$\begin{aligned}u^p(t)\leqslant&\, a(t)+\sum_{s=0}^{t-1}b(s)u^p(s)\\&+\sum_{i=1}^{n}\sum_{s=0}^{t-1}L_i\left(s,f(s)u^{q_i}(s-\rho)+\sum_{\tau=0}^{s-1}g(\tau)u^{r_i}(\tau)+c(s)\right),t\in\mathbb{N}_0\end{aligned} \tag{3.69}$$

with the initial condition (3.18)*, implies*

$$u(t)\leqslant R^{\frac{1}{p}}(t)\left\{a(t)+\overline{\overline{G}}(t)\prod_{s=0}^{t-1}[1+\overline{\overline{H}}(s)]\right\}^{\frac{1}{p}},\ \ t\in\mathbb{N}_0, \tag{3.70}$$

where $R(t)$ defined as (3.38)*, and*

$$\overline{\overline{G}}(t)=\sum_{i=1}^{n}\sum_{s=0}^{t-1}L_i\left(s,R^{\frac{q_i}{p}}(s)f(s)a^{\frac{q_i}{p}}(s)+\sum_{\tau=0}^{s-1}R^{\frac{r_i}{p}}(\tau)g(\tau)a^{\frac{r_i}{p}}(\tau)+c(s)\right),\ \ t\in\mathbb{N}_0, \tag{3.71}$$

$$\begin{aligned}\overline{\overline{H}}(t)=&\sum_{i=1}^{n}M_i\left(t,R^{\frac{q_i}{p}}(t)f(t)a^{\frac{q_i}{p}}(t)+\sum_{\tau=0}^{t-1}R^{\frac{r_i}{p}}(\tau)g(\tau)a^{\frac{r_i}{p}}(\tau)+c(t)\right)\\&\cdot\left[\frac{q_i}{p}R^{\frac{q_i}{p}}(t)f(t)a^{\frac{q_i}{p}-1}(t)+\frac{r_i}{p}\sum_{\tau=0}^{t-1}R^{\frac{r_i}{p}}(\tau)g(\tau)a^{\frac{r_i}{p}-1}(\tau)\right],\ \ t\in\mathbb{N}_0.\end{aligned} \tag{3.72}$$

4. Some applications

In this section, we construct some examples to demonstrate the application of our main result obtained in section 3.

Consider the delay dynamic equation

$$(u^p(t))^{\Delta}=M\left(t,K(t,u(\sigma(t))),\int_{t_0}^{t}N(\tau,u(\tau))\Delta\tau\right),\ \ t\in\mathbb{T}_0 \tag{4.1}$$

with the initial condition

$$\begin{cases} u(t) = \varphi(t), t \in [\alpha, t_0] \cap \mathbb{T} \\ \varphi(\sigma(t)) = C^{\frac{1}{p}},\ t \in \mathbb{T}_0,\ \sigma(t) \leqslant t_0 \end{cases} \tag{4.2}$$

where $M : \mathbb{T}_0 \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous function, $C = u^p(t_0)$ and $p > 0$ are constants, α and $\sigma(t)$ are defined as (3.2), and $\varphi(t) \in C_{rd}([\alpha, t_0] \cap \mathbb{T}, \mathbb{R}_+)$.

EXAMPLE 4.1. Assume that

$$\begin{aligned} &| M(t,K,V) | \leqslant | K | + | V |, \\ &| K(t,u(\sigma(t))) | \leqslant \sum_{i=1}^{n} f(t) \mid u(\sigma(t)) \mid^{q_i}, \\ &| N(t,u(t)) | \leqslant \sum_{i=1}^{n} g(t) \mid u(t) \mid^{r_i}, \end{aligned} \tag{4.3}$$

where q_i, r_i $(i = 1, 2, \cdots, n)$ are constants, $p \geqslant q_i > 0$ and $p \geqslant r_i > 0$, $f(t)$, $g(t) \in C_{rd}$, $f(t)$, $g(t)$ are nonnegative.

If every solution $u(t)$ of (4.1) satisfying the initial condition (4.2), implies

$$| u(t) | \leqslant [| C | + A(t) e_B(t,t_0)]^{\frac{1}{p}},\ \ t \in \mathbb{T}_0. \tag{4.4}$$

where $A(t)$, $B(t)$ are defined as in (3.3), (3.4) with $a(t) = C$, $b(t) = 1$, $c(t) = 0$.

Indeed, the solution $u(t)$ of (4.1) satisfies the following inequality

$$u^p(t) \leqslant C + \int_{t_0}^{t} \left(K(s, u(\sigma(s))), \int_{s_0}^{s} N(\tau, u(\tau)) \Delta\tau \right) \Delta s,\ \ t \in \mathbb{T}_0. \tag{4.5}$$

It follows from (4.3) and (4.5) that

$$| u(t) |^p \leqslant | C | + \sum_{i=1}^{n} \int_{t_0}^{t} \left(f(s) \mid u(\sigma(s)) \mid^{q_i} + \int_{s_0}^{s} g(\tau) \mid u(\tau) \mid^{r_i} \Delta\tau \right) \Delta s,\ t \in \mathbb{T}_0. \tag{4.6}$$

Using Theorem 3.1, the inequality (4.4) is obtained from (4.6).

EXAMPLE 4.2. Assume that

$$\begin{aligned} &| M(t,K,V) | \leqslant | K | + | V |, \\ &| K(t,u(\sigma(t))) | \leqslant \sum_{i=1}^{n} L_i(t, f(t) \mid u(\sigma(t)) \mid^{q_i}), \\ &| N(t,u(t)) | \leqslant b(t) \mid u(t) \mid^{p}, \end{aligned} \tag{4.7}$$

where q_i $(i = 1, 2, \cdots, n)$ are constants, $p \geqslant q_i > 0$, $b(t) \in C_{rd}$, and $b(t)$ is nonnegative.

If every solution $u(t)$ of (4.1) satisfying the initial condition (4.2), implies

$$| u(t) | \leqslant [e_b(t,t_0)(| C | + G(t) e_H(t,t_0))]^{\frac{1}{p}},\ \ t \in \mathbb{T}_0. \tag{4.8}$$

where $G(t), H(t)$ are defined as in (3.52), (3.53) with $a(t) = C,\ g(t) = 0,\ c(t) = 0$. Indeed, the solution $u(t)$ of (4.1) satisfies the following inequality

$$| u(t) |^p \leqslant | C | + \int_{t_0}^{t} b(s) | u(s) |^p \Delta s + \sum_{i=1}^{n} \int_{t_0}^{t} L\big(s, f(s) | u(\sigma(s)) |^{q_i}\big)\Delta s,\ t \in \mathbb{T}_0. \quad (4.9)$$

Using Theorem 3.1, the inequality (4.8) is obtained from (4.9).

Acknowledgements

The authors are grateful to the referees for their valuable suggestions.

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(Received August 12, 2011)

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