Journal of
Mathematical
Inequalities
Volume 6, Number 3 (2012), 367–379 doi:10.7153/jmi-06-36

A NEW NONLINEAR INTEGRAL INEQUALITY OF WENDROFF TYPE WITH CONTINUOUS AND WEAKLY SINGULAR KERNEL AND ITS APPLICATION

FAHIM LAKHAL

(*Communicated by K.-L. Tseng*)

Abstract. The main objective of this paper is to establish some new explicit bounds for nonlinear integral inequalities of Wendroff type with continuous and weakly singular kernel, which generalized some known inequalities for functions in two variables and can be furnished a handy tool for the study of qualitative as well as quantitative properties of solutions of nonlinear differential equations. Some applications are also given to illustrate the usefulness of our results.

1. Introduction

It is well known that singular integral inequalities plays a very important role in the qualitative theory of partial differential, integral and integro-differential equations. During the past few years, many papers [1]-[6] and [16] have appeared in the literature which deal with integral inequalities in more than independent variable which are motivated by certain applications in the theory of hyperbolic partial differential and integral equations. Usually, the integrals concerning this type inequalities have regular or continuous kernels, but some problems of theory and practicality require us to solve integral inequalities with singular kernels. For example, D. Henry [7] used this type integral inequalities to prove a global existence and an exponential decay result for a parabolic Cauchy problem. Medved [9] presented a new method to solve Henry's type inequalities and their Bihari version. Some works can be found, for example, in [8, 12, 13, 17, 18] and some references therein. Recently,Q.-H. Ma and J. Pečarić [13] studied the inequality

$$u^p(t) \leqslant a(t) + b(t) \int_0^t (t^\alpha - s^\alpha)^{\beta-1} s^{\gamma-1} f(s) u^q(s) ds, \quad t \in \mathbb{R}_+.$$

Cheung et al. [17] investigated the inequality in two variables

$$u^p(x,y) \leqslant a(x,y) + b(x,y) \int_0^x \int_0^y (x^\alpha - s^\alpha)^{\beta-1} s^{\gamma-1} (y^\alpha - t^\alpha)^{\beta-1} t^{\gamma-1} f(s,t) u^q(s,t) dt ds,$$
$$(x,y) \in D.$$

Mathematics subject classification (2010): 42B20, 26D07, 26D15.
Keywords and phrases: Wendroff inequalities, singular integral inequalities, nonlinear differential equations.

H. Wang and K. Zheng [18] discussed the following inequality

$$u(x,y) \leqslant a(x,y) + \int_0^x \int_0^y (x^\alpha - s^\alpha)^{\beta-1} s^{\gamma-1} (y^\alpha - t^\alpha)^{\beta-1} t^{\gamma-1} f(x,y,s,t) w(u(s,t)) dt ds.$$

In this paper, motivated mainly by the work of Ma et al. [13, 15], Cheung et al. [14, 17], H. Wang and K. Zheng [18] and by applying Medved's method of disingularization of weakly singular inequalities we discuss more general form of nonlinear weakly singular integral inequalities of Wendroff type for functions in two variables

$$u^r(x,y) \leqslant a(x,y) + b(x,y) \int_0^x \int_0^y (x^{\alpha_1} - s^{\alpha_1})^{\beta-1} s^{\gamma_1-1} (y^{\alpha_2} - t^{\alpha_2})^{\beta-1} t^{\gamma_2-1} f(s,t) \omega(u(s,t)) dt ds.$$

Our paper is organized as follows. In Section 2 we prepare some tools needed to prove our theorems. Section 3 contains the statements and proofs of our main results and in Section 4 we give an application to partial integral equation with weakly singular kernel.

2. Preliminaries

Throughout the paper, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = [0,\infty)$. Let $C(M,S)$ denotes the class of all continuous functions from the set M to the set S. The partial derivatives of a function $z(x,y)$ for $x,y \in \mathbb{R}$ with respect to x, y and xy are denoted by $D_1 z(x,y)$, $D_2 z(x,y)$ and $D_1 D_2 z(x,y) = D_2 D_1 z(x,y)$ respectively. We need the following definitions and lemmas in the discussion of our main results.

DEFINITION 2.1. The function $\omega(u)$ is said to be subadditive and submultiplicative, if

$$\omega(u+v) \leqslant \omega(u) + \omega(v) \quad \text{and} \quad \omega(uv) \leqslant \omega(u)\omega(v), \text{ for } u,v \geqslant 0. \tag{1}$$

DEFINITION 2.2. Let $q > 0$ be a real number and $0 < T \leqslant \infty$. We say that a function $\omega : \mathbb{R}_+ \to \mathbb{R}$ satisfies a condition (q) if

$$e^{-qt} [\omega(u)]^q \leqslant R(t) \omega(e^{-qt} u^q), \text{ for all } u \in \mathbb{R}_+,\ t \in [0,T), \tag{q}$$

where $R(t)$ is a continuous nonnegative function.

Examples:

1. $\omega(u) = u^m$, $m > 0$ satisfies the condition (q) with $R(t) = e^{(m-1)qt}$,

2. $\omega(u) = u + au^m$, where $0 \leqslant a \leqslant 1$, $m \geqslant 1$ satisfies the condition (q) with $R(t) = 2^{q-1} e^{qmt}$.

LEMMA 2.1. (see [11]) *Assume that* $r \geqslant 1$, $a \geqslant 0$. *Then*

$$a^{\frac{1}{r}} \leqslant \frac{1}{r} K^{\frac{1-r}{r}} a + \frac{r-1}{r} K^{\frac{1}{r}}, \tag{2}$$

for any $K > 0$.

DEFINITION 2.3. (see [14]) Let $[x,y,z]$ be an ordered parameter group of nonnegative real numbers. The group is said to belong to the first class distribution and denoted by $[x,y,z] \in I$ if conditions $x \in (0,1]$, $y \in (\frac{1}{2},1)$ and $z \geqslant \frac{3}{2} - y$ are satisfied, it is said to belong to the second-class distribution and denoted by $[x,y,z] \in II$ if conditions $x \in (0,1]$, $y \in (0,\frac{1}{2}]$ and $z > \dfrac{1-2y^2}{1-y^2}$ are satisfied.

LEMMA 2.2. (see [15], page 296) *Let α,β,γ and p be positive constants. Then*

$$\int_0^t (t^\alpha - s^\alpha)^{p(\beta-1)} s^{p(\gamma-1)} ds = \frac{t^\theta}{\alpha} B\left[\frac{p(\gamma-1)+1}{\alpha}, p(\beta-1)+1\right], \quad t \in \mathbb{R}_+, \tag{3}$$

where $B[\xi,\eta] = \int_0^1 s^{\xi-1}(1-s)^{\eta-1} ds$, $(\xi,\eta \in \mathbb{C},\ \Re\xi > 0, \Re\eta > 0)$ *is the well-known beta function and* $\theta = p[\alpha(\beta-1)+\gamma-1]+1$.

LEMMA 2.3. (see [14]) *Suppose that the positive constants α,β,γ,p_1 and p_2 satisfy*

(a) if $[\alpha,\beta,\gamma] \in I$, $p_1 = \dfrac{1}{\beta}$;

(b) if $[\alpha,\beta,\gamma] \in II$, $p_2 = \dfrac{1+4\beta}{1+3\beta}$, *then*

$$B\left[\frac{p_i(\gamma-1)+1}{\alpha},\ p_i(\beta-1)+1\right] \in (0,+\infty), \tag{4}$$

$$\theta_i = p_i[\alpha(\beta-1)+\gamma-1]+1 \geqslant 0$$

are valid for $i = 1,2$.

3. The results

LEMMA 3.1. *Let $u(x,y)$, $a(x,y)$, $b(x,y)$ and $f(x,y)$ be nonnegative continuous functions defined for $x,y \in \mathbb{R}_+$, and $\omega(u)$ be a nonnegative, nondecreasing continuous function for $u \in \mathbb{R}_+$ with $\omega(u) > 0$ for $u > 0$. Assume that $a(x,y)$ and $b(x,y)$ are nondecreasing in each variable $x,y \in \mathbb{R}_+$. If*

$$u(x,y) \leqslant a(x,y) + b(x,y) \int_0^x \int_0^y f(s,t)\omega(u(s,t)) dt ds, \tag{5}$$

for all $x,y \in \mathbb{R}_+$, then

$$u(x,y) \leqslant G^{-1}\left[G(a(x,y)) + b(x,y) \int_0^x \int_0^y f(s,t) dt ds,\right] \tag{6}$$

for all $0 \leqslant x \leqslant x_1,\ 0 \leqslant y \leqslant y_1$, *where*

$$G(r) = \int_{r_0}^{r} \frac{ds}{\omega(s)}, \qquad r \geqslant r_0 > 0, \tag{7}$$

G^{-1} *is the inverse function of* G, *and* $x_1,\ y_1 \in \mathbb{R}_+$ *are chosen so that* $G(a(x,y)) + b(x,y)\int_0^x\int_0^y f(s,t)dtds \in Dom(G^{-1})$.

Proof. Fixing any numbers $\overline{x}_1$ and $\overline{y}_1$ with $0 < \overline{x}_1 \leqslant x_1$ and $0 < \overline{y}_1 \leqslant y_1$, from (5) we have

$$u(x,y) \leqslant a(\overline{x}_1,\overline{y}_1) + b(\overline{x}_1,\overline{y}_1)\int_0^x\int_0^y f(s,t)\omega(u(s,t))dtds, \tag{8}$$

for $0 \leqslant x \leqslant \overline{x}_1,\ 0 \leqslant y \leqslant \overline{y}_1$.

Defining $r_1(x,y)$ as the right hand-side of (8), then

$$\begin{gathered} r_1(0,y) = r_1(x,0) = a(\overline{x}_1,\overline{y}_1), \\ u(x,y) \leqslant r_1(x,y), \end{gathered} \tag{9}$$

$r_1(x,y)$ is nondecreasing in $y \in [0,\overline{y}_1]$ and

$$\begin{aligned} D_1r_1(x,y) &= b(\overline{x}_1,\overline{y}_1)\int_0^y f(x,t)\omega(u(x,t))dt \\ &\leqslant b(\overline{x}_1,\overline{y}_1)\int_0^y f(x,t)\omega(r_1(x,t))dt \\ &\leqslant b(\overline{x}_1,\overline{y}_1)\omega(r_1(x,y))\int_0^y f(x,t)dt. \end{aligned} \tag{10}$$

Dividing both sides of (10) by $\omega(r_1(x,y))$, we obtain

$$\frac{D_1r_1(x,y)}{\omega(r_1(x,y))} \leqslant b(\overline{x}_1,\overline{y}_1)\int_0^y f(x,t)dt. \tag{11}$$

From (7) and (11), we get

$$D_1G(r_1(x,y)) \leqslant b(\overline{x}_1,\overline{y}_1)\int_0^y f(x,t)dt. \tag{12}$$

Now setting $x = s$ in (12), and then integrating with respect to s from 0 to x, we have

$$G(r_1(x,y)) \leqslant G(r_1(0,y)) + b(\overline{x}_1,\overline{y}_1)\int_0^x\int_0^y f(s,t)dtds.$$

Noting

$$G(r_1(0,y)) = G(a(\overline{x}_1,\overline{y}_1)),$$

we have

$$G(r_1(x,y)) \leqslant G(a(\overline{x}_1,\overline{y}_1)) + b(\overline{x}_1,\overline{y}_1)\int_0^x\int_0^y f(s,t)dtds.$$

Taking $x=\overline{x}_1,\ y=\overline{y}_1$ in (9) and the last inequality, we obtain

$$\begin{aligned} &u(\overline{x}_1,\overline{y}_1) \leqslant r_1(\overline{x}_1,\overline{y}_1),\\ &G(r_1(\overline{x}_1,\overline{y}_1)) \leqslant G(a(\overline{x}_1,\overline{y}_1)) + b(\overline{x}_1,\overline{y}_1)\int_0^{\overline{x}_1}\int_0^{\overline{y}_1} f(s,t)dtds. \end{aligned} \tag{13}$$

Since $0<\ \overline{x}_1\ \leqslant x_1,\ 0<\overline{y}_1\leqslant y_1$ are arbitrary, from (13) we get

$$u(x,y) \leqslant r_1(x,y), \tag{14}$$

$$G(r_1(x,y)) \leqslant G(a(x,y)) + b(x,y)\int_0^x\int_0^y f(s,t)dtds,$$

or

$$r_1(x,y) \leqslant G^{-1}\left[G(a(x,y)) + b(x,y)\int_0^x\int_0^y f(s,t)dtds\right], \tag{15}$$

for all $0<x\leqslant x_1,\ 0<y\leqslant y_1$. Hence by (14) and (15), we have

$$u(x,y) \leqslant G^{-1}\left[G(a(x,y)) + b(x,y)\int_0^x\int_0^y f(s,t)dtds\right], \tag{16}$$

for all $0<x\leqslant x_1,\ 0<y\leqslant y_1$. By (5), (16) holds also when $x=0,\ y=0$. □

THEOREM 3.2. *Let $\psi(x,y)$, $p(x,y)$ and $q(x,y)$ be nonnegative continuous functions defined for $x,y\in\mathbb{R}_+$. Let $k(x,y,s,t)$ and its partial derivatives $D_1k(x,y,s,t)$, $D_2k(x,y,s,t)$ and $D_1D_2k(x,y,s,t)$ be nonnegative continuous functions for $0\leqslant s\leqslant x<\infty,\ 0\leqslant t\leqslant y<\infty$, and $\omega(u)$ be defined as in Lemma 3.1 and moreover we assume that it is subadditive and submultiplicative.*

If

$$\psi^r(x,y) \leqslant p(x,y) + q(x,y)\int_0^x\int_0^y k(x,y,s,t)\omega(\psi(s,t))dtds,\ \ for\ \ x,y\in\mathbb{R}_+,\ r\geqslant 1, \tag{17}$$

then

$$\psi(x,y) \leqslant \left\{p(x,y)+q(x,y)\left[G^{-1}\left(G\left(\int_0^x\int_0^y A(\sigma,\tau)d\tau d\sigma\right) + \int_0^x\int_0^y B(\sigma,\tau)d\tau d\sigma\right)\right]\right\}^{\frac{1}{r}}, \tag{18}$$

for $0 \leqslant x \leqslant x_2,\ 0 \leqslant y \leqslant y_2$ *where*

$$\begin{aligned} A(x,y) &= k(x,y,x,y)\omega(\tfrac{1}{r}K^{\frac{1-r}{r}})\omega\left(p(x,y)\right) + k(x,y,x,y)\omega(\tfrac{r-1}{r}K^{\frac{1}{r}}) \\ &+\omega(\tfrac{1}{r}K^{\frac{1-r}{r}})\int_0^y D_2k(x,y,x,t)\omega\left(p(x,t)\right)dt + \omega(\tfrac{r-1}{r}K^{\frac{1}{r}})\int_0^y D_2k(x,y,x,t)dt \\ &+\omega(\tfrac{1}{r}K^{\frac{1-r}{r}})\int_0^x D_1k(x,y,s,y)\omega\left(p(s,y)\right)ds + \omega(\tfrac{r-1}{r}K^{\frac{1}{r}})\int_0^x D_1k(x,y,s,y)ds \\ &+\omega(\tfrac{1}{r}K^{\frac{1-r}{r}})\int_0^x\int_0^y D_1D_2k(x,y,s,t)\omega\left(p(s,t)\right)dtds \\ &+\omega(\tfrac{r-1}{r}K^{\frac{1}{r}})\int_0^x\int_0^y D_1D_2k(x,y,s,t)dtds, \end{aligned} \tag{19}$$

and

$$\begin{aligned} B(x,y) &= k(x,y,x,y)\omega(\tfrac{1}{r}K^{\frac{1-r}{r}})\omega\left(q(x,y)\right) \\ &+\omega(\tfrac{1}{r}K^{\frac{1-r}{r}})\int_0^y D_2k(x,y,x,t)\omega\left(q(x,t)\right)dt \\ &+\omega(\tfrac{1}{r}K^{\frac{1-r}{r}})\int_0^x D_1k(x,y,s,y)\omega\left(q(s,y)\right)ds \\ &+\omega(\tfrac{1}{r}K^{\frac{1-r}{r}})\int_0^x\int_0^y D_1D_2k(x,y,s,t)\omega\left(q(s,t)\right)dtds, \end{aligned} \tag{20}$$

G^{-1} *is the inverse function of* G *defined in (7) and* x_2, y_2 *are chosen so that*

$$G\left(\int_0^x\int_0^y A(\sigma,\tau)d\tau d\sigma\right) + \int_0^x\int_0^y B(\sigma,\tau)d\tau d\sigma \in Dom(G^{-1}),$$

for all x,y *lying in the subintervals* $0 \leqslant x \leqslant x_2,\ 0 \leqslant y \leqslant y_2$ *of* $\mathbb{R}_+$.

Proof. Define a function $z(x,y)$ by

$$z(x,y) = \int_0^x\int_0^y k(x,y,s,t)\omega(\psi(s,t))dtds. \tag{21}$$

From (17) and (21) we observe that

$$\psi(x,y) \leqslant \left(p(x,y) + q(x,y)z(x,y)\right)^{\frac{1}{r}}. \tag{22}$$

Applying Lemma 2.1 to the inequality (22), for any $K > 0$, we obtain

$$\psi(x,y) \leqslant \frac{1}{r}K^{\frac{1-r}{r}}\left(p(x,y) + q(x,y)z(x,y)\right) + \frac{r-1}{r}K^{\frac{1}{r}}. \tag{23}$$

So, from (23) and the hypotheses on ω, we get

$$\omega(\psi(x,y)) \leqslant \omega\left(\frac{1}{r}K^{\frac{1-r}{r}}\right)\omega(p(x,y)) + \omega\left(\frac{1}{r}K^{\frac{1-r}{r}}\right)\omega(q(x,y))\omega(z(x,y)) + \omega\left(\frac{r-1}{r}K^{\frac{1}{r}}\right). \tag{24}$$

Differentiating (21) and then using (24), we have

$$\begin{aligned} D_1D_2z(x,y) \leqslant\ & k(x,y,x,y)\omega(\tfrac{1}{r}K^{\frac{1-r}{r}})\omega(p(x,y)) \\ & +k(x,y,x,y)\omega(\tfrac{1}{r}K^{\frac{1-r}{r}})\omega(q(x,y))\omega(z(x,y)) \\ & +k(x,y,x,y)\omega(\tfrac{r-1}{r}K^{\frac{1}{r}}) \\ & +\omega(\tfrac{1}{r}K^{\frac{1-r}{r}})\int_0^y D_2k(x,y,x,t)\omega(p(x,t))dt \\ & +\omega(\tfrac{1}{r}K^{\frac{1-r}{r}})\int_0^y D_2k(x,y,x,t)\omega(q(x,t))\omega(z(x,t))dt \\ & +\omega(\tfrac{r-1}{r}K^{\frac{1}{r}})\int_0^y D_2k(x,y,x,t)dt \\ & +\omega(\tfrac{1}{r}K^{\frac{1-r}{r}})\int_0^x D_1k(x,y,s,y)\omega(p(s,y))ds \\ & +\omega(\tfrac{1}{r}K^{\frac{1-r}{r}})\int_0^x D_1k(x,y,s,y)\omega(q(s,y))\omega(z(s,y))ds \\ & +\omega(\tfrac{r-1}{r}K^{\frac{1}{r}})\int_0^x D_1k(x,y,s,y)ds \\ & +\omega(\tfrac{1}{r}K^{\frac{1-r}{r}})\int_0^x\int_0^y D_1D_2k(x,y,s,t)\omega(p(s,t))dtds \\ & +\omega(\tfrac{1}{r}K^{\frac{1-r}{r}})\int_0^x\int_0^y D_1D_2k(x,y,s,t)\omega(q(s,t))\omega(z(s,t))dtds \\ & +\omega(\tfrac{r-1}{r}K^{\frac{1}{r}})\int_0^x\int_0^y D_1D_2k(x,y,s,t)dtds. \end{aligned} \tag{25}$$

From (25) and using the fact that $z(x,y)$ is monotonic nondecreasing in both the variables $x,y,$ and $\omega(u)$ is nondecreasing for $u \in \mathbb{R}_+$, we obtain

$$z_{xy}(x,y) \leqslant A(x,y) + B(x,y)\omega(z(x,y)), \tag{26}$$

where $A(x,y)$, $B(x,y)$ are defined by (19) and (20) respectively.

From (26) it is easy to observe that

$$z(x,y) \leqslant C(x,y) + \int_0^x\int_0^y B(\sigma,\tau)\omega(z(\sigma,\tau))d\tau d\sigma, \tag{27}$$

where

$$C(x,y)=\int_0^x\int_0^y A(\sigma,\tau)d\tau d\sigma.$$

Clearly $C(x,y)$ is nonegative and nondecreasing in each variable $x,y\in\mathbb{R}_+$. Then, by applying Lemma 3.1 to (27) and using (22) we get the required inequality in (18). □

REMARK 3.1. If we take $r=1$ and $q(x,y)=1$ we obtain Theorem 2 in [6].

THEOREM 3.3. *Let $u(x,y)$, $a(x,y)$, $b(x,y)$, $f(x,y)$ and ω be as in Lemma 3.1, and $r\geqslant 1$ is a constant and suppose*

$$u^r(x,y)\leqslant a(x,y)+b(x,y)\int_0^x\int_0^y(x^{\alpha_1}-s^{\alpha_1})^{\beta-1}s^{\gamma_1-1}(y^{\alpha_2}-t^{\alpha_2})^{\beta-1}t^{\gamma_2-1}f(s,t)\omega(u(s,t))\,dtds,$$
$$(x,y)\in[0,T)^2\qquad(0<T\leqslant\infty),\tag{28}$$

then for any $K>0$ the following assertions hold:

(i) If $[\alpha_1,\beta,\gamma_1]$, $[\alpha_2,\beta,\gamma_2]\in I$ and ω satisfies the condition (q) with $q=q_1=\frac{1}{1-\beta}$. Then

$$u(x,y)\leqslant e^{x+y}\left\{G^{-1}\left[G(P_1(x,y))+Q_1(x,y)\int_0^x\int_0^y k_1(s,t)dtds\right]\right\}^{1-\beta},\ \textit{for } (x,y)\in[0,T_1)^2,\tag{29}$$

where the function G and its inverse G^{-1} are as in Lemma 3.1 and

$$\begin{aligned}
M_{11}&=\tfrac{1}{\alpha_1}B\left[\tfrac{\beta+\gamma_1-1}{\alpha_1\beta},\tfrac{2\beta-1}{\beta}\right],\\
M_{21}&=\tfrac{1}{\alpha_2}B\left[\tfrac{\beta+\gamma_2-1}{\alpha_2\beta},\tfrac{2\beta-1}{\beta}\right],\\
P_1(x,y)&=3^{\frac{\beta}{1-\beta}}\left[\left(\tfrac{r-1}{r}K^{\frac{1}{r}}\right)^{\frac{1}{1-\beta}}+\left(\tfrac{1}{r}K^{\frac{1-r}{r}}a(x,y)\right)^{\frac{1}{1-\beta}}\right],\\
Q_1(x,y)&=3^{\frac{\beta}{1-\beta}}\left[\tfrac{1}{r}K^{\frac{1-r}{r}}b(x,y)\right]^{\frac{1}{1-\beta}}(M_{11}M_{21})^{\frac{\beta}{1-\beta}}(x)^{\frac{(\alpha_1+1)(\beta-1)+\gamma_1}{1-\beta}}(y)^{\frac{(\alpha_2+1)(\beta-1)+\gamma_2}{1-\beta}},\\
k_1(x,y)&=f^{\frac{1}{1-\beta}}(x,y)R_1(x+y),
\end{aligned}\tag{30}$$

and $T_1>0$ is such that the argument of G^{-1} in (29) belongs to $Dom(G^{-1})$ for all $(x,y)\in[0,T_1)^2$.

(ii) If $[\alpha_1,\beta,\gamma_1]$, $[\alpha_2,\beta,\gamma_2]\in II$, and ω satisfies the condition (q) with $q=q_2=\frac{1+4\beta}{\beta}$. Then

$$u(x,y)\leqslant e^{x+y}\left\{G^{-1}\left[G(P_2(x,y))+Q_2(x,y)\int_0^x\int_0^y k_2(s,t)dtds\right]\right\}^{\frac{\beta}{1+4\beta}},\ \textit{for } (x,y)\in[0,T_2)^2,\tag{31}$$

where

$$
\begin{aligned}
M_{12} &= \tfrac{1}{\alpha_1} B\left[\tfrac{\gamma_1(1+4\beta)-\beta}{\alpha_1(1+3\beta)}, \tfrac{4\beta^2}{1+3\beta}\right],\\
M_{22} &= \tfrac{1}{\alpha_2} B\left[\tfrac{\gamma_2(1+4\beta)-\beta}{\alpha_2(1+3\beta)}, \tfrac{4\beta^2}{1+3\beta}\right],\\
P_2(x,y) &= 3^{\frac{1+3\beta}{\beta}}\left[\left(\tfrac{r-1}{r}K^{\frac{1}{r}}\right)^{\frac{1+4\beta}{\beta}} + \left(\tfrac{1}{r}K^{\frac{1-r}{r}}a(x,y)\right)^{\frac{1+4\beta}{\beta}}\right],\\
Q_2(x,y) &= 3^{\frac{1+3\beta}{\beta}}\left[\tfrac{1}{r}K^{\frac{1-r}{r}}b(x,y)\right]^{\frac{1+4\beta}{\beta}}(M_{12}M_{22})^{\frac{(1+3\beta)}{\beta}}\\
&\quad \times (x)^{\frac{(1+4\beta)[\alpha_1(\beta-1)+\gamma_1]-\beta}{\beta}}(y)^{\frac{(1+4\beta)[\alpha_2(\beta-1)+\gamma_2]-\beta}{\beta}},\\
k_2(x,y) &= f^{\frac{1+4\beta}{\beta}}(x,y)R_2(x+y),
\end{aligned}
\tag{32}
$$

and $T_2 > 0$ is such that the argument of G^{-1} in (31) belongs to $Dom(G^{-1})$ for all $(x,y) \in [0,T_2)^2$.

Proof. Define a function $v(x,y)$ by

$$
v(x,y) = b(x,y)\int_0^x\int_0^y (x^{\alpha_1}-s^{\alpha_1})^{\beta-1}(y^{\alpha_2}-t^{\alpha_2})^{\beta-1}s^{\gamma_1-1}t^{\gamma_2-1}f(s,t)\omega(u(s,t))\,dtds,
$$
$$
(x,y) \in [0,T)^2, \tag{33}
$$

then

$$
u^r(x,y) \leqslant a(x,y) + v(x,y)
$$

or

$$
u(x,y) \leqslant (a(x,y)+v(x,y))^{\frac{1}{r}}.
$$

From the last inequality and Lemma 2.1, we have

$$
u(x,y) \leqslant \frac{r-1}{r}K^{\frac{1}{r}} + \frac{1}{r}K^{\frac{1-r}{r}}a(x,y) + \frac{1}{r}K^{\frac{1-r}{r}}v(x,y). \tag{34}
$$

If $[\alpha_1,\beta,\gamma_1]$, $[\alpha_2,\beta,\gamma_2] \in I$, let $p_1 = \frac{1}{\beta}$, $q_1 = \frac{1}{1-\beta}$, and if $[\alpha_1,\beta,\gamma_1]$, $[\alpha_2,\beta,\gamma_2] \in II$, let $p_2 = \dfrac{1+4\beta}{1+3\beta}$, $q_2 = \dfrac{1+4\beta}{\beta}$, then $\dfrac{1}{p_i} + \dfrac{1}{q_i} = 1$, for $i = 1,2$.

Applying the Hölder's inequality with indices p_i, q_i to (33) after inserting $e^{s+t}.e^{-(s+t)}$ into the integral on the right-hand side, we obtain

$$
v(x,y) \leqslant b(x,y)\left[\int_0^x\int_0^y (x^{\alpha_1}-s^{\alpha_1})^{p_i(\beta-1)}s^{p_i(\gamma_1-1)}(y^{\alpha_2}-t^{\alpha_2})^{p_i(\beta-1)}t^{p_i(\gamma_2-1)}e^{p_i(s+t)}dtds\right]^{\frac{1}{p_i}}
$$
$$
\times\left[\int_0^x\int_0^y f^{q_i}(s,t)e^{-q_i(s+t)}\omega^{q_i}(u(s,t))\,dtds\right]^{\frac{1}{q_i}}. \tag{35}
$$

By using Lemmas 2.2 and 2.3 and the condition (q) with $q = q_i$, (35) can be rewritten as

$$v(x,y) \leqslant b(x,y)e^{x+y}\left\{\frac{x^{\theta_{1i}}}{\alpha_1}B\left[\frac{p_i(\gamma_1-1)+1}{\alpha_1}, p_i(\beta-1)+1\right]\frac{y^{\theta_{2i}}}{\alpha_2}B\left[\frac{p_i(\gamma_2-1)+1}{\alpha_2}, p_i(\beta-1)+1\right]\right\}^{\frac{1}{p_i}}$$
$$\times\left[\int_0^x\int_0^y f^{q_i}(s,t)R_i(s+t)\omega\left(e^{-q_i(s+t)}u^{q_i}(s,t)\right)dtds\right]^{\frac{1}{q_i}}$$
$$= b(x,y)e^{x+y}\left(x^{\theta_{1i}}y^{\theta_{2i}}M_{1i}M_{2i}\right)^{\frac{1}{p_i}}\left[\int_0^x\int_0^y f^{q_i}(s,t)R_i(s+t)\omega\left(e^{-q_i(s+t)}u^{q_i}(s,t)\right)dtds\right]^{\frac{1}{q_i}},$$
$$(x,y)\in[0,T)^2, \tag{36}$$

where

$$\theta_{1i} = p_i\left[\alpha_1(\beta-1)+\gamma_1-1\right]+1 \text{ and } M_{1i} = \frac{1}{\alpha_1}B\left[\frac{p_i(\gamma_1-1)+1}{\alpha_1},\ p_i(\beta-1)+1\right],$$

$$\theta_{2i} = p_i\left[\alpha_2(\beta-1)+\gamma_2-1\right]+1 \text{ and } M_{2i} = \frac{1}{\alpha_2}B\left[\frac{p_i(\gamma_2-1)+1}{\alpha_2},\ p_i(\beta-1)+1\right],$$

for $i = 1,2$.

Substituting (36) in (34), we get

$$u(x,y) \leqslant \tfrac{r-1}{r}K^{\frac{1}{r}} + \tfrac{1}{r}K^{\frac{1-r}{r}}a(x,y) + \tfrac{1}{r}K^{\frac{1-r}{r}}b(x,y)e^{x+y}\left(x^{\theta_{1i}}y^{\theta_{2i}}M_{1i}M_{2i}\right)^{\frac{1}{p_i}}$$
$$\times\left[\int_0^x\int_0^y f^{q_i}(s,t)R_i(s+t)\omega\left(e^{-q_i(s+t)}u^{q_i}(s,t)\right)dtds\right]^{\frac{1}{q_i}},$$

or

$$e^{-(x+y)}u(x,y) \leqslant \tfrac{r-1}{r}K^{\frac{1}{r}} + \tfrac{1}{r}K^{\frac{1-r}{r}}a(x,y) + \tfrac{1}{r}K^{\frac{1-r}{r}}b(x,y)\left(x^{\theta_{1i}}y^{\theta_{2i}}M_{1i}M_{2i}\right)^{\frac{1}{p_i}}$$
$$\times\left[\int_0^x\int_0^y f^{q_i}(s,t)R_i(s+t)\omega\left(e^{-q_i(s+t)}u^{q_i}(s,t)\right)dtds\right]^{\frac{1}{q_i}},\ \ (x,y)\in[0,T)^2. \tag{37}$$

From (37) and the inequality

$$(A+B+C)^q \leqslant 3^{q-1}(A^q+B^q+C^q),\ A,B,C\geqslant 0,\ q\geqslant 1,$$

we obtain

$$\psi_i(x,y) \leqslant P_i(x,y) + Q_i(x,y)\int_0^x\int_0^y k_i(s,t)\omega\left(\psi_i(s,t)\right)dtds, \tag{38}$$

where

$$\psi_i(x,y) = e^{-q_i(x+y)}u^{q_i}(x,y),$$

$$P_i(x,y) = 3^{q_i-1}\left(\frac{r-1}{r}K^{\frac{1}{r}}\right)^{q_i} + 3^{q_i-1}\left(\frac{1}{r}K^{\frac{1-r}{r}}a(x,y)\right)^{q_i},$$

$$Q_i(x,y) = 3^{q_i-1}\left(\frac{1}{r}K^{\frac{1-r}{r}}b(x,y)\right)^{q_i}\left(x^{\theta_{1i}}y^{\theta_{2i}}M_{1i}M_{2i}\right)^{\frac{q_i}{p_i}},$$

$$k_i(x,y) = f^{q_i}(x,y)R_i(x+y), \text{ for } i=1,2.$$

Since $q_i \geqslant 0$ and $\theta_{1i},\ \theta_{2i} \geqslant 0\ (i=1,2)$, then $P_i(x,y)$ and $Q_i(x,y)$ are also nondecreasing in x and y.

By Lemma 3.1 and (38), considering two situations for $i=1,2,$ we can get the desired estimations (29) and (31), respectively. □

COROLLARY 3.4. *Let functions $u(x,y),\ a(x,y),\ b(x,y)$, and $f(x,y)$ be as in theorem 3.3, and $r \geqslant 1,\ m>0$ are constants and suppose*

$$u^r(x,y) \leqslant a(x,y)+b(x,y)\int_0^x\int_0^y (x^{\alpha_1}-s^{\alpha_1})^{\beta-1}s^{\gamma_1-1}(y^{\alpha_2}-t^{\alpha_2})^{\beta-1}t^{\gamma_2-1}f(s,t)u^m(s,t)dtds,$$
$$(x,y)\in[0,T)^2 \quad (0<T\leqslant\infty), \tag{39}$$

then for any $K>0$ the following assertions hold:

(i) for $[\alpha_1,\beta,\gamma_1],\ [\alpha_2,\beta,\gamma_2]\in I$,

if $m=1$,

$$u(x,y) \leqslant e^{x+y}\left[P_1(x,y)\exp\left(Q_1(x,y)\int_0^x\int_0^y f^{\frac{1}{1-\beta}}(s,t)dtds\right)\right]^{1-\beta}, \tag{40}$$

if $m>0,\ m\neq 1$,

$$u(x,y) \leqslant e^{x+y}\left[P_1^{1-m}(x,y)+(1-m)Q_1(x,y)\int_0^x\int_0^y e^{\frac{(m-1)(s+t)}{1-\beta}} f^{\frac{1}{1-\beta}}(s,t)dtds\right]^{\frac{1-\beta}{1-m}}, \tag{41}$$

for $x\geqslant 0,\ y\geqslant 0$, where $P_1(x,y),\ Q_1(x,y)$ are defined as in theorem 3.3.

(ii) for $[\alpha_1,\beta,\gamma_1],\ [\alpha_2,\beta,\gamma_2]\in II$,

if $m=1$,

$$u(x,y) \leqslant e^{x+y}\left[P_2(x,y)\exp\left(Q_2(x,y)\int_0^x\int_0^y f^{\frac{1+4\beta}{\beta}}(s,t)dtds\right)\right]^{\frac{\beta}{1+4\beta}} \tag{42}$$

if $m>0,\ m\neq 1$,

$$u(x,y) \leqslant e^{x+y}\left[P_2^{1-m}(x,y)+(1-m)Q_2(x,y)\int_0^x\int_0^y e^{\frac{(m-1)(1+4\beta)(s+t)}{\beta}} f^{\frac{1+4\beta}{\beta}}(s,t)dtds\right]^{\frac{\beta}{(1+4\beta)(1-m)}}, \tag{43}$$

for $x\geqslant 0,\ y\geqslant 0$, where $P_2(x,y),\ Q_2(x,y)$ are defined as in theorem 3.3.

4. Applications

In this section, we present applications of the inequalities (29) and (31) in theorem 3.3 for studing the boundedness of certain partial integral equations with weakly singular kernel. Consider the partial integral equation:

$$u^r(x,y) = l(x,y) + h(x,y)\int_0^x\int_0^y (x^{\alpha_1} - s^{\alpha_1})^{\beta-1} s^{\gamma_1-1} (y^{\alpha_2} - t^{\alpha_2})^{\beta-1} t^{\gamma_2-1} F(s,t,u(s,t))dtds, \tag{44}$$

for $(x,y) \in D = [0,T)^2$, where $l(x,y)$ and $h(x,y) \in C(D,\mathbb{R})$, $F \in C(D\times\mathbb{R},\mathbb{R})$, $r \geqslant 1$. Suppose that

$$\begin{aligned} |l(x,y)| &\leqslant a(x,y), \\ |h(x,y)| &\leqslant b(x,y), \\ |F(x,y,u)| &\leqslant f(x,y)\omega(|u|), \end{aligned} \tag{45}$$

where the functions, $a(x,y)$, $b(x,y)$, $f(x,y)$ and ω are as in theorem 3.3. If $u(x,y)$, $(x,y) \in D$, is any solution of (44), then by plugging (45) in (44) and applying Theorem 3.3, we obtain a bound on the solutions $u(x,y)$ of (44).

REFERENCES

[1] E. F. BECKENBACH AND R. BELLMAN, *Inequalities, Ergebnisse der Mathematik und ihrer Grenzgebiete*, Springer, Berlin, Germany, 1961.

[2] V. LAKSHMIKANTHAM AND S. LEELA, *Differential and Integral Inequalities: Theory and Applications. Vol. I: Ordinary Dfferential Equations, Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1969.

[3] D. BAĬNOV AND P. SIMEONOV, *Integral Inequalities and Applications, vol. 57 of Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1992.

[4] R. P. AGARWAL, *Difference Equations and Inequalities*, Marcel Dekker, New York, NY, USA, 1993.

[5] B. G. PACHPATTE, *Inequalities for Differential and Integral Equations, vol. 197 of Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1998.

[6] B. G. PACHPATTE, *On generalizations of Bihari's inequality*, Soochow Journal of Mathematics, Vol. 31, No. 2, pp. 261–271, April 2005.

[7] D. HENRY, *Geometric Theory of Semilinear Parabolic Equations*, vol. 840 of Lecture Notes in Mathematics, Springer, Berlin, Germany, 1981.

[8] K. M. FURATI AND N. TATAR, *Power-type estimates for a nonlinear fractional differential equation*, Nonlinear Analysis: Theory, Methods & Applications, vol. 62, no. 6, pp. 1025–1036, 2005.

[9] M. MEDVED, *A new approach to an analysis of Henry type integral inequalities and their Bihari type versions*, Journal of Mathematical Analysis and Applications, vol. 214, no. 2, pp. 349–366, 1997.

[10] M. MEDVED, *Nonlinear singular integral inequalities for functions in two and n independent variables*, Journal of Inequalities and Applications, vol. 5, no. 3, pp. 287–308, 2000.

[11] M. MEDVED, *Integral inequalities and global solutions of semilinear evolution equations*, Journal of Mathematical Analysis and Applications, vol. 267, no. 2, pp. 643–650, 2002.

[12] F. JIANG AND F. MENG, *Explicit bounds on some new nonlinear integral inequalities with delay*, Journal of Computational and Applied Mathematics, vol. 205, no. 1, pp. 479–486, 2007.

[13] Q.-H. MA AND J. PEČARIĆ, *Some new explicit bounds for weakly singular integral inequalities with applications to fractional differential and integral equations*, Journal of Mathematical Analysis and Applications, vol. 341, no. 2, pp. 894–905, 2008.

[14] W. S. CHEUNG, *On some new integrodifferential inequalities of the Gronwall and Wendroff type*, Journal of Mathematical Analysis and Applications, vol. 178, no. 2, pp. 438–449, 1993.

[15] Q. H. MA AND E. H. YANG, *Estimates on solutions of some weakly singular Volterra integral inequalities*, Acta Mathematicae Applicatae Sinica, vol. 25, no. 3, pp. 505–515, 2002.

[16] A. P. PRUDNIKOV, YU. A. BRYCHKOV, AND O. I. MARICHEV, *Integrals and Series. Elementary Functions*, vol. 1, Nauka, Moscow, Russia, 1981.

[17] W.-S. CHEUNG, Q.-H. MA, AND S. TSENG, *Some new nonlinear weakly singular integral inequalities of Wendroff type with applications*, Journal of Inequalities and Applications, vol. 2008, Article ID 909156, 13 pages, 2008.

[18] H. WANG AND K. ZHENG, *Some Nonlinear Weakly Singular Integral Inequalities with Two Variables and Applications*, Journal of Inequalities and Applications, vol. 2010, Article ID 345701, 12 pages, 2010.

(Received August 18, 2011)

Fahim Lakhal
Department of Mathematics
8 Mai 1945 University
P. O. Box 401
Guelma 24000, Algeria
e-mail: flakhal@yahoo.fr