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WEIGHTED OSTROWSKI, TRAPEZOID AND GRÜSS TYPE INEQUALITIES ON TIME SCALES

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Abstract. In this paper we first derive a weighted Montgomery identity on time scales and then establish weighted Ostrowski, trapezoid and Grüss type inequalities on time scales, respectively. These results not only provide a generalization of the known results, but also give some other interesting inequalities on time scales as special cases.

1. Introduction

The theory of calculus on time scales was initiated by Hilger [12] in 1988 in order to unify continuous and discrete analysis, and it has a potential application in some mathematical models of real processes and phenomena studied in economics [2], population dynamics [3], space weather [15], physics [31] and so on. Nowadays many authors study the theory of certain integral inequalities on time scales (see [6, 7, 8, 11, 13, 16, 17, 19, 20, 21, 22, 23, 25, 28, 29, 30, 32, 35, 36, 38]).

In 1938, Ostrowski derived a formula to estimate the absolute deviation of a differentiable function from its integral mean [26], the so-called Ostrowski inequality, which can also be shown by using the Montgomery identity [24]. These two properties was proved by Bohner and Matthews in [7] for general time scales, which unify discrete, continuous and many other cases. By using the Montgomery identity on time scales, they established the following Ostrowski inequality on time scales.

THEOREM 1. *Let $a,b,s,t \in \mathbb{T}$, $a<b$ and $f:[a,b]\to\mathbb{R}$ be differentiable. Then*

$$\left| f(t) - \frac{1}{b-a}\int_a^b f(\sigma(s))\Delta s \right| \leqslant \frac{M}{b-a}(h_2(t,a)+h_2(t,b)), \tag{1.1}$$

where $h_2(\cdot,\cdot)$ is defined by Definition 8 below and $M = \sup_{a<t<b} \left| f^{\Delta}(t)\right| < \infty$. This inequality is sharp in the sense that the right-hand side of (1.1) cannot be replaced by a smaller one.

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Recently, Karpuz and Özkan [13] generalized Ostrowski's inequality and Montgomery's identity on arbitrary time scales by the means of generalized polynomials on time scales. By introducing a parameter, Liu, Ngô and Chen [23] also extended a generalization of the above inequality on time scales. Ngô and Liu [25] gave a sharp Grüss type inequality on time scales and then applied it to the sharp Ostrowski-Grüss inequality on time scales. Motivated by the ideas of [7, 23, 25, 37], Tuna and Daghan [35] studied generalizations of Ostrowski and Ostrowski-Grüss type inequality on time scales.

More recently, Liu [18] established the following weighted generalization of three point inequality with a parameter for mappings of bounded variation.

THEOREM 2. *Let us have* $0 \leqslant k \leqslant 1$, $f : [a,b] \to \mathbf{R}$ *be a mapping of bounded variation,* $g : [a,b] \to [0,\infty)$ *continuous and positive on* (a,b) *and let* $h : [a,b] \to \mathbb{R}$ *be differentiable such that* $h'(t) = g(t)$ *on* $[a,b]$. *Then*

$$\left| \int_a^b f(t)g(t)dt - \left\{ (1-k)f(x) + k\left[\frac{\int_a^x g(t)dt}{\int_a^b g(t)dt} f(a) + \frac{\int_x^b g(t)dt}{\int_a^b g(t)dt} f(b) \right] \right\} \int_a^b g(t)dt \right|$$
$$\leqslant \begin{cases} (1-k)\left[\dfrac{1}{2}\displaystyle\int_a^b g(t)dt + \left| h(x) - \dfrac{h(a)+h(b)}{2} \right| \right] \displaystyle\bigvee_a^b(f), \; k \in \left[0, \tfrac{1}{2}\right], \\ k\left[\dfrac{1}{2}\displaystyle\int_a^b g(t)dt + \left| h(x) - \dfrac{h(a)+h(b)}{2} \right| \right] \displaystyle\bigvee_a^b(f), \qquad k \in \left(\tfrac{1}{2}, 1\right] \end{cases}$$

for all $x \in [a,b]$, *where* $\bigvee_a^b(f)$ *denotes the total varfiation of* f *on the interval* $[a,b]$.

Motivated by the above research, the purpose of this paper is to obtain some weighted Ostrowski, trapezoid and Grüss type inequalities with a parameter on time scales based on a weighted Montgomery identity on time scales. These results not only provide a generalization of the known results, but also give some other interesting inequalities on time scales as special cases.

This paper is organized as follows. In Section 2, we briefly present the general definitions and theorems related to the time scales calculus. The weighted Montgomery identity, weighted Ostrowski type inequality, weighted trapezoid type and weighted Grüss type inequality on time scales are derived in subsections 3.1, 3.2, 3.3 and 3.4, respectively.

2. Time scales essentials

In this section we briefly introduce the time scales theory. For further details and proofs we refer the reader to Hilger's Ph.D. thesis [12], the books [4, 5, 14], and the survey [1].

DEFINITION 1. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$.

We assume throughout that $\mathbb{T}$ has the topology that is inherited from the standard topology on $\mathbb{R}$. It also assumed throughout that in $\mathbb{T}$ the interval $[a,b]$ means the set $\{t \in \mathbb{T} : \ a \leqslant t \leqslant b\}$ for the points $a < b$ in $\mathbb{T}$. Since a time scale may not be connected, we need the following concept of jump operators.

DEFINITION 2. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = \sup\{s \in \mathbb{T} : \ s < t\}$.

The jump operators σ and ρ allow the classification of points in $\mathbb{T}$ as follows.

DEFINITION 3. If $\sigma(t) > t$, *then we say that* t is right-scattered, while if $\rho(t) < t$ then we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. If $\sigma(t) = t$, then t is called right-dense, and if $\rho(t) = t$ then t is called left-dense. Points that are both right-dense and left-dense are called dense.

DEFINITION 4. The mapping $\mu : \mathbb{T} \to \mathbb{R}^+$ defined by $\mu(t) = \sigma(t) - t$ is called the *graininess function.* The set $\mathbb{T}^k$ is defined as follows: if $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$; otherwise, $\mathbb{T}^k = \mathbb{T}$.

If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$, and when $\mathbb{T} = \mathbb{Z}$, we have $\mu(t) = 1$.

DEFINITION 5. Let $f : \mathbb{T} \to \mathbb{R}$. f is called differentiable at $t \in \mathbb{T}^k$, with (delta) derivative $f^{\Delta}(t) \in \mathbb{R}$, if for any given $\varepsilon > 0$ there exists a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s] \right| \leqslant \varepsilon\, |\sigma(t) - s|, \quad \forall\, s \in U.$$

If $\mathbb{T} = \mathbb{R}$, then $f^{\Delta}(t) = \frac{df(t)}{dt}$, and if $\mathbb{T} = \mathbb{Z}$, then $f^{\Delta}(t) = f(t+1) - f(t)$.

THEOREM 3. *Assume* $f, g : \mathbb{T} \to \mathbb{R}$ *are differentiable at* $t \in \mathbb{T}^k$. *Then the product* $fg : \mathbb{T} \to \mathbb{R}$ *is differentiable at* t *with*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)).$$

DEFINITION 6. The function $f : \mathbb{T} \to \mathbb{R}$ is said to be *rd-continuous (denote* $f \in C_{rd}(\mathbb{T},\, \mathbb{R})$), if it is continuous at all right-dense points $t \in \mathbb{T}$ and its left-sided limits exist at all left-dense points $t \in \mathbb{T}$.

It follows from [4, Theorem 1.74] that every rd-continuous function has an antiderivative.

DEFINITION 7. Let $f \in C_{rd}(\mathbb{T},\mathbb{R})$. Then $F : \mathbb{T} \to \mathbb{R}$ is called the antiderivative of f on $\mathbb{T}$ if it satisfies $F^{\Delta}(t) = f(t)$ for any $t \in \mathbb{T}^k$. In this case, we define the Δ-integral of f as

$$\int_a^t f(s)\Delta s = F(t) - F(a), \ \ t \in \mathbb{T}.$$

THEOREM 4. *Let f,g be rd-continuous, $a,b,c \in \mathbb{T}$ and $\alpha,\beta \in \mathbb{R}$. Then*
(1) $\int_a^b [\alpha f(t)+\beta g(t)]\Delta t = \alpha \int_a^b f(t)\Delta t + \beta \int_a^b g(t)\Delta t$,
(2) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$,
(3) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$,
(4) $\int_a^b f(t)g^{\Delta}(t)\Delta t = (fg)(b)-(fg)(a)-\int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t$,

THEOREM 5. *If f is Δ-integrable on $[a,b]$, then so is $|f|$, and*

$$\left|\int_a^b f(t)\Delta t\right| \leqslant \int_a^b |f(t)|\Delta t.$$

DEFINITION 8. Let $h_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0$ be defined by

$$h_0(t,s) = 1 \quad \text{for all} \quad s,t \in \mathbb{T}$$

and then recursively by

$$h_{k+1}(t,s) = \int_s^t h_k(\tau,s)\Delta\tau \quad \text{for all} \quad s,t \in \mathbb{T}$$

3. Main results

In this section we first derive a weighted Montgomery identity on time scales and then establish weighted Ostrowski, trapezoid and Grüss type inequalities on time scales, respectively.

3.1. Weighted Montgomery identity on time scales

LEMMA 1. *Let $0 \leqslant k \leqslant 1$, $g : [a,b] \to [0,\infty)$ be rd continuous and positive and $h : [a,b] \to \mathbb{R}$ be differentiable such that $h^{\Delta}(t) = g(t)$ on $[a,b]$. Let $a,b,s,t \in \mathbb{T}$, $a < b$ and $f : [a,b] \to \mathbb{R}$ be differentiable. Then for all $s \in [a,b]$, we have*

$$\begin{aligned}
&\int_a^b S(t,s) f^{\Delta}(s)\Delta s \\
&= \left\{(1-k)f(t) + k\left[\frac{\int_a^t g(t)\Delta t}{\int_a^b g(t)\Delta t} f(a) + \frac{\int_t^b g(t)\Delta t}{\int_a^b g(t)\Delta t} f(b)\right]\right\}\int_a^b g(t)\Delta t \\
&\quad - \int_a^b g(s) f(\sigma(s))\Delta s
\end{aligned} \tag{3.1}$$

where

$$S(t,s) = \begin{cases} h(s) - ((1-k)h(a) + kh(t)), & a \leqslant s < t, \\ h(s) - (kh(t) + (1-k)h(b)), & t \leqslant s \leqslant b. \end{cases} \tag{3.2}$$

Proof. Using item (4) of Theorem 4, we have

$$\begin{aligned}&\int_a^t [h(s)-((1-k)h(a)+kh(t))]f^{\Delta}(s)\Delta s\\ &= [h(t)-((1-k)h(a)+kh(t))]f(t)-[h(a)-((1-k)h(a)+kh(t))]f(a)\\ &\quad -\int_a^t g(s)f(\sigma(s))\Delta s\end{aligned}$$

and

$$\begin{aligned}&\int_t^b [h(s)-(kh(t)+(1-k)h(b))]f^{\Delta}(s)\Delta s\\ &= [h(b)-(kh(t)+(1-k)h(b))]f(b)-[h(t)-(kh(t)+(1-k)h(b))]f(t)\\ &\quad -\int_t^b g(s)f(\sigma(s))\Delta s.\end{aligned}$$

Therefore, the equality (3.1) is proved by adding the above two identities. □

COROLLARY 1. *In the case of $\mathbb{T}=\mathbb{R}$ in Lemma 1, we have*

$$\begin{aligned}&\int_a^b S(t,s)f'(s)ds\\ &=\left\{(1-k)f(t)+k\left[\frac{\int_a^t g(t)dt}{\int_a^b g(t)dt}f(a)+\frac{\int_t^b g(t)dt}{\int_a^b g(t)dt}f(b)\right]\right\}\int_a^b g(t)dt-\int_a^b g(s)f(s)ds,\end{aligned}$$

where $g(t)=h'(t)$ *on* $[a,b]$ *and*

$$S(t,s)=\begin{cases} h(s)-((1-k)h(a)+kh(t)), & a\leqslant s<t,\\ h(s)-(kh(t)+(1-k)h(b)), & t\leqslant s\leqslant b.\end{cases}$$

This is the result given in [18, equation (8)].

COROLLARY 2. *In the case of $\mathbb{T}=\mathbb{Z}$ in Lemma 1, we have*

$$\begin{aligned}&\sum_{s=a}^{b-1} S(t,s)\Delta f(s)\\ &=\left\{(1-k)f(t)+k\left[\frac{\sum_{s=a}^{t-1} g(s)}{\sum_{s=a}^{b-1} g(s)}f(a)+\frac{\sum_{s=t}^{b-1} g(s)}{\sum_{s=a}^{b-1} g(s)}f(b)\right]\right\}\sum_{s=a}^{b-1} g(s)-\sum_{s=a}^{b-1} g(s)f(s+1),\end{aligned}$$

where $g(t)=h(t+1)-h(t)$ *on* $[a,b-1]$ *and*

$$S(t,s)=\begin{cases} h(s)-((1-k)h(a)+kh(t)), & a\leqslant s<t-1,\\ h(s)-(kh(t)+(1-k)h(b)), & t\leqslant s\leqslant b-1.\end{cases}$$

COROLLARY 3. *Let* $f:[a,b]\to\mathbb{R}$, $q>1$, $a=q^m$ *and* $b=q^n$ *with* $m<n$ *in Lemma* 1. *Then we have*

$$\begin{aligned}&\sum_{k=m}^{n-1} S(t,q^k)\frac{f(q^{k+1})-f(q^k)}{(q-1)q^k}\\ &=\left\{(1-k)f(t)+k\left[\frac{\int_{q^m}^{q^k} g(t)d_qt}{\int_{q^m}^{q^n} g(t)d_qt}f(q^m)+\frac{\int_{q^k}^{q^n} g(t)d_qt}{\int_{q^m}^{q^n} g(t)d_qt}f(q^n)\right]\right\}\int_{q^m}^{q^n} g(t)d_qt\\ &\quad-\int_{q^m}^{q^n} g(t)f(qt)d_qt,\end{aligned}$$

where

$$S(t,s)=\begin{cases} h(q^k)-((1-k)h(q^m)+kh(t)),\ q^m\leqslant q^k<t,\\ h(q^k)-(kh(t)+(1-k)h(q^n)),\ t\leqslant q^k\leqslant q^n.\end{cases}$$

COROLLARY 4. *In the case of* $h(t)=t$ *in Lemma* 1, *we have*

$$(1-k)f(t)=\frac{1}{b-a}\int_a^b S(t,s)f^{\Delta}(s)\Delta s-k\left[\frac{t-a}{b-a}f(a)+\frac{b-t}{b-a}f(b)\right]+\frac{1}{b-a}\int_a^b f(\sigma(s))\Delta s,$$

where

$$S(t,s)=\begin{cases} s-((1-k)a+kt),\ a\leqslant s<t,\\ s-(kt+(1-k)b),\ t\leqslant s\leqslant b.\end{cases}$$

This is the result given in [35, Lemma 1].

COROLLARY 5. *In the case of* $\mathbb{T}=\mathbb{R}$ *in Corollary* 4, *we have*

$$(1-k)f(t)=\frac{1}{b-a}\int_a^b S(t,s))f'(s)ds-k\left[\frac{t-a}{b-a}f(a)+\frac{b-t}{b-a}f(b)\right]+\frac{1}{b-a}\int_a^b f(s)ds,$$

where $g(t)=h'(t)$ *on* $[a,b]$ *and*

$$S(t,s)=\begin{cases} s-((1-k)a+kt),\ a\leqslant s<t,\\ s-(kt+(1-k)b),\ t\leqslant s\leqslant b.\end{cases}$$

COROLLARY 6. *In the case of* $\mathbb{T}=\mathbb{Z}$ *in Corollary* 4, *we have*

$$(1-k)f(t)=\frac{1}{b-a}\sum_{s=a}^{b-1} S(t,s)\Delta f(s)-k\left[\frac{t-a}{b-a}f(a)+\frac{b-t}{b-a}f(b)\right]+\frac{1}{b-a}\sum_{s=a}^{b-1} f(s+1),$$

where $g(t)=h(t+1)-h(t)$ *on* $[a,b-1]$ *and*

$$S(t,s)=\begin{cases} s-((1-k)a+kt),\ a\leqslant s<t-1,\\ s-(kt+(1-k)b),\ t\leqslant s\leqslant b-1.\end{cases}$$

COROLLARY 7. *In the case of* $k=0$ *in Corollary 4, we have*

$$f(t)=\frac{1}{b-a}\int_a^b S(t,s))f^{\Delta}(s)\Delta s+\frac{1}{b-a}\int_a^b f(\sigma(s))\Delta s, \tag{3.3}$$

where

$$S(t,s)=\begin{cases} s-a, \ a\leqslant s<t,\\ s-b, \ t\leqslant s\leqslant b.\end{cases}$$

REMARK 1. The equality (3.3) is the Montgomery identity on time scales, which can be found in [7, Lemma 3.1] and it was also studied continuous, discrete and quantum calculus cases there.

3.2. Weighted Ostrowski type inequality on time scales

THEOREM 6. *Let* $0\leqslant k\leqslant 1$, $g:[a,b]\to[0,\infty)$ *be rd continuous and positive and* $h:[a,b]\to\mathbb{R}$ *be differentiable such that* $h^{\Delta}(t)=g(t)$ *on* $[a,b]$. *Let* $a,b,s,t\in\mathbb{T}$, $a<b$ *and* $f:[a,b]\to\mathbb{R}$ *be differentiable. Then for all* $s\in[a,b]$, *we have*

$$\left|\left\{(1-k)f(t)+k\left[\frac{\int_a^t g(t)\Delta t}{\int_a^b g(t)\Delta t}f(a)+\frac{\int_t^b g(t)\Delta t}{\int_a^b g(t)\Delta t}f(b)\right]\right\}\int_a^b g(t)\Delta t - \int_a^b g(s)f(\sigma(s))\Delta s\right| \leqslant M\int_a^b |S(t,s)|\Delta s, \tag{3.4}$$

where

$$S(t,s)=\begin{cases} h(s)-((1-k)h(a)+kh(t)), \ a\leqslant s<t,\\ h(s)-(kh(t)+(1-k)h(b)), \ t\leqslant s\leqslant b\end{cases}$$

and

$$M=\sup_{a<t<b}\left|f^{\Delta}(t)\right|<\infty.$$

Proof. The proof of the Theorem 6 can be done easily from Lemma 1. □

COROLLARY 8. *In the case of* $\mathbb{T}=\mathbb{R}$ *in Theorem 6, we have*

$$\left|\left\{(1-k)f(t)+k\left[\frac{\int_a^t g(t)dt}{\int_a^b g(t)dt}f(a)+\frac{\int_t^b g(t)dt}{\int_a^b g(t)dt}f(b)\right]\right\}\int_a^b g(t)dt-\int_a^b g(s)f(s)ds\right| \leqslant M\int_a^b |S(t,s)|ds,$$

where $g(t)=h'(t)$ *on* $[a,b]$,

$$S(t,s)=\begin{cases} h(s)-((1-k)h(a)+kh(t)), \ a\leqslant s<t,\\ h(s)-(kh(t)+(1-k)h(b)), \ t\leqslant s\leqslant b\end{cases}$$

and

$$M = \sup_{a<t<b} \left|f'(t)\right| < \infty.$$

COROLLARY 9. *In the case of* $\mathbb{T} = \mathbb{Z}$ *in Theorem 6, we have*

$$\left|\left\{(1-k)f(t) + k\left[\frac{\sum_{s=a}^{t-1} g(s)}{\sum_{s=a}^{b-1} g(s)} f(a) + \frac{\sum_{s=t}^{b-1} g(s)}{\sum_{s=a}^{b-1} g(s)} f(b)\right]\right\}\sum_{s=a}^{b-1} g(s) - \sum_{s=a}^{b-1} g(s)f(s+1)\right|$$
$$\leqslant M\sum_{s=a}^{b-1}|S(t,s)|,$$

where $g(t) = h(t+1) - h(t)$ *on* $[a,b]$,

$$S(t,s) = \begin{cases} h(s) - ((1-k)h(a) + kh(t)),\ a \leqslant s < t-1, \\ h(s) - (kh(t) + (1-k)h(b)),\ t \leqslant s \leqslant b-1 \end{cases}$$

and

$$M = \sup_{a<t<b-1} |\Delta f(t)| < \infty.$$

COROLLARY 10. *Let* $f : [a,b] \to \mathbb{R}$. $q > 1$, $a = q^m$, *and* $b = q^n$ *with* $m < n$ *in Theorem 6. Then,*

$$\left|\left\{(1-k)f(t) + k\left[\frac{\int_{q^m}^{q^k} g(t)d_qt}{\int_{q^m}^{q^n} g(t)d_qt} f(a) + \frac{\int_{q^k}^{q^n} g(t)d_qt}{\int_{q^m}^{q^n} g(t)d_qt} f(b)\right]\right\}\int_{q^m}^{q^n} g(t)d_qt\right.$$
$$\left. - \int_{q^m}^{q^n} g(t)f(qt)d_qt\right|$$
$$\leqslant M\sum_{k=m}^{n-1}\left|S(t,q^k)\right|,$$

where

$$S(t,s) = \begin{cases} h(q^k) - ((1-k)h(q^m) + kh(t)),\ q^m \leqslant q^k < t, \\ h(q^k) - (kh(t) + (1-k)h(q^n)),\ t \leqslant q^k \leqslant q^n \end{cases}$$

and

$$M = \sup_{q^m<t<q^n} \left|\frac{f(q^{k+1}) - f(q^k)}{(q-1)q^k}\right| < \infty.$$

COROLLARY 11. *In the case of* $h(t) = t$ *in Theorem 6, we have*

$$\left|(1-k)f(t) + k\left[\frac{t-a}{b-a} f(a) + \frac{b-t}{b-a} f(b)\right] - \frac{1}{b-a}\int_a^b f(\sigma(s))\Delta s\right|$$
$$\leqslant \frac{M}{b-a}\left[h_2((1-k)a + kt, a) + h_2(t, (1-k)a + kt)\right.$$
$$\left. + h_2(t, kt + (1-k)b) + h_2(b, kt + (1-k)b)\right]$$

for all $k \in [0,1]$ *such that* $(1-k)a+kt$ *and* $kt+(1-k)b$ *are in* $\mathbb{T}$, *where*

$$M = \sup_{a<t<b} \left| f^{\Delta}(t) \right| < \infty.$$

COROLLARY 12. *In the case of* $\mathbb{T}=\mathbb{R}$ *in Corollary 11, we have*

$$\left|(1-k)f(t)+k\left[\frac{t-a}{b-a}f(a)+\frac{b-t}{b-a}f(b)\right]-\frac{1}{b-a}\int_a^b f(s)ds\right|$$
$$\leqslant \frac{M\left(2k^2-2k+1\right)}{2}\left[\frac{(a-t)^2+(b-t)^2}{b-a}\right],$$

where

$$M = \sup_{a<t<b} \left| f'(t) \right| < \infty.$$

COROLLARY 13. *In the case of* $\mathbb{T}=\mathbb{Z}$ *in Corollary 11, we have*

$$\left|(1-k)f(t)+k\left[\frac{t-a}{b-a}f(a)+\frac{b-t}{b-a}f(b)\right]-\frac{1}{b-a}\sum_{s=a}^{b-1} f(s+1)\right|$$
$$\leqslant \frac{M}{b-a}\left[\left(2k^2-2k+1\right)\left(t-\frac{a+b}{2}\right)^2+(2k-1)\left(t-\frac{a+b}{2}\right)+\left(\frac{b-a}{2}\right)^2\right],$$

where

$$M = \sup_{a<t<b-1} |\Delta f(t)| < \infty.$$

COROLLARY 14. *In the case of* $k=0$ *in Corollary 11, we have*

$$\left|f(t)-\frac{1}{b-a}\int_a^b f(\sigma(s))\Delta s\right| \leqslant \frac{M}{b-a}\left[h_2(t,a)+h_2(t,b)\right], \tag{3.5}$$

where

$$M = \sup_{a<t<b} \left| f^{\Delta}(t) \right| < \infty.$$

REMARK 2. The inequality (3.5) is the Ostrowski inequality on time scales, which can be found in [7, Theorem 3.5] and it was also studied continuous, discrete and quantum calculus cases there.

3.3. Weighted trapezoid type inequality on time scales

THEOREM 7. *Let $0 \leqslant k \leqslant 1$, $g:[a,b]\to[0,\infty)$ continuous and positive and $h:[a,b]\to\mathbb{R}$ be differentiable such that $h^{\Delta}(t)=g(t)$ on $[a,b]$. Let $a,b,s,t\in\mathbb{T}$, $a<b$ and $f:[a,b]\to\mathbb{R}$ be differentiable. Then for all $s\in[a,b]$, we have*

$$\begin{aligned}&\Bigg|(1-k)(f^2(b)-f^2(a))\\&+\left[(k-1)\frac{f(b)-f(a)}{\int_a^b g(t)\Delta t}+k\frac{f(\sigma(b))-f(\sigma(a))}{\int_a^b g(t)\Delta t}\right]\int_a^b g(t)f(\sigma(t))\Delta t\\&+k\left(f(\sigma(a))f(b)-f(\sigma(b))f(a)\right)-\frac{f(b)-f(a)}{\int_a^b g(t)\Delta t}\int_a^b g(t)f(\sigma^2(t))\Delta t\Bigg|\\\leqslant&\frac{M(M+N)}{\int_a^b g(t)\Delta t}\int_a^b\left(\int_a^b |S(t,s))|\,\Delta s\right)\Delta t,\end{aligned} \tag{3.6}$$

where

$$S(t,s)=\begin{cases}h(s)-((1-k)h(a)+kh(t)),\ a\leqslant s<t,\\ h(s)-(kh(t)+(1-k)h(b)),\ t\leqslant s\leqslant b,\end{cases}$$

and

$$M=\sup_{a<t<b}\left|f^{\Delta}(t)\right|<\infty \ \text{and}\ N=\sup_{a<t<b}\left|f^{\Delta}(\sigma(t))\right|<\infty.$$

Proof. We have

$$\begin{aligned}(1-k)f(t)=&-k\left[\frac{\int_a^t g(t)\Delta t}{\int_a^b g(t)\Delta t}f(a)+\frac{\int_t^b g(t)\Delta t}{\int_a^b g(t)\Delta t}f(b)\right]\\&+\frac{1}{\int_a^b g(t)\Delta t}\int_a^b g(s)f(\sigma(s))\Delta s+\frac{1}{\int_a^b g(t)\Delta t}\int_a^b S(t,s)f^{\Delta}(s)\Delta s\end{aligned} \tag{3.7}$$

and

$$(1-k)f(\sigma(t)) \tag{3.8}$$

$$\begin{aligned}=&-k\left[\frac{\int_a^t g(t)\Delta t}{\int_a^b g(t)\Delta t}f(\sigma(a))+\frac{\int_t^b g(t)\Delta t}{\int_a^b g(t)\Delta t}f(\sigma(b))\right]\\&+\frac{1}{\int_a^b g(t)\Delta t}\int_a^b g(s)f(\sigma^2(s))\Delta s+\frac{1}{\int_a^b g(t)\Delta t}\int_a^b S(t,s)f^{\Delta}(\sigma(s))\Delta s.\end{aligned} \tag{3.9}$$

Adding (3.7) and (3.9), we get

$$\begin{aligned}&(1-k)\left(f(t)+f(\sigma(t))\right)\\ =&-k\left[\frac{\int_a^t g(t)\Delta t}{\int_a^b g(t)\Delta t}\left(f(a)+f(\sigma(a))\right)+\frac{\int_t^b g(t)\Delta t}{\int_a^b g(t)\Delta t}\left(f(b)+f(\sigma(b))\right)\right]\\ &+\frac{1}{\int_a^b g(t)\Delta t}\int_a^b g(s)\left(f(\sigma(s))+f(\sigma^2(s))\right)\Delta s\\ &+\frac{1}{\int_a^b g(t)\Delta t}\int_a^b S(t,s))\left(f^\Delta(s)+f^\Delta(\sigma(s))\right)\Delta s. \qquad (3.10)\end{aligned}$$

Multiplying (3.10) by $f^\Delta(t)$, using Theorem 3 and integrating the result identity on $[a,b]$, we have

$$\begin{aligned}&(1-k)(f^2(b)-f^2(a))\\ =&-k\left[\frac{(f(a)+f(\sigma(a)))}{\int_a^b g(t)\Delta t}\int_a^b f^\Delta(t)\left(\int_a^t g(s)\Delta s\right)\Delta t\right.\\ &\left.+\frac{(f(b)+f(\sigma(b)))}{\int_a^b g(t)\Delta t}\int_a^b f^\Delta(t)\left(\int_t^b g(s)\Delta s\right)\Delta t\right]\\ &+\frac{f(b)-f(a)}{\int_a^b g(t)\Delta t}\int_a^b g(s)\left(f(\sigma(s))+f(\sigma^2(s))\right)\Delta s\\ &+\frac{1}{\int_a^b g(t)\Delta t}\int_a^b f^\Delta(t)\left(\int_a^b S(t,s)\left(f^\Delta(s)+f^\Delta(\sigma(s))\right)\Delta s\right)\Delta t. \qquad (3.11)\end{aligned}$$

From (3.11) and using the relations

$$\int_a^b f^\Delta(t)\left(\int_a^t g(s)\Delta s\right)\Delta t=f(b)\int_a^b g(s)\Delta s-\int_a^b g(t)f(\sigma(t))\Delta t$$

and

$$\int_a^b f^\Delta(t)\left(\int_t^b g(s)\Delta s\right)\Delta t=-f(a)\int_a^b g(s)\Delta s+\int_a^b g(t)f(\sigma(t))\Delta t,$$

we obtain

$$\begin{aligned}&(1-k)(f^2(b)-f^2(a))\\ &+\left[(k-1)\frac{f(b)-f(a)}{\int_a^b g(t)\Delta t}+k\frac{f(\sigma(b))-f(\sigma(a))}{\int_a^b g(t)\Delta t}\right]\int_a^b g(t)f(\sigma(t))\Delta t\\ &+k\left(f(\sigma(a))f(b)-f(\sigma(b))f(a)\right)-\frac{f(b)-f(a)}{\int_a^b g(t)\Delta t}\int_a^b g(t)f(\sigma^2(t))\Delta t\\ =&\frac{1}{\int_a^b g(t)\Delta t}\int_a^b f^\Delta(t)\left(\int_a^b S(t,s)\left(f^\Delta(s)+f^\Delta(\sigma(s))\right)\Delta s\right)\Delta t.\end{aligned}$$

Therefore, (3.6) can be established. □

COROLLARY 15. *In the case of* $\mathbb{T}=\mathbb{R}$ *in Theorem 7, we have*

$$\left|(1-k)\frac{f^2(b)-f^2(a)}{2}+(k-1)\frac{f(b)-f(a)}{2\int_a^b g(t)dt}\int_a^b g(t)f(t)dt\right|$$
$$\leqslant\frac{M^2}{\int_a^b g(t)dt}\int_a^b\left(\int_a^b|S(t,s)|\,ds\right)dt,$$

where

$$S(t,s)=\begin{cases}h(s)-((1-k)h(a)+kh(t)),\ a\leqslant s<t,\\ h(s)-(kh(t)+(1-k)h(b)),\ t\leqslant s\leqslant b.\end{cases}$$

and

$$M=\sup_{a<t<b}\left|f'(t)\right|<\infty.$$

COROLLARY 16. *In the case of* $\mathbb{T}=\mathbb{Z}$ *in Theorem 7, we have*

$$\left|(1-k)(f^2(b)-f^2(a))\right.$$
$$+\left[(k-1)\frac{f(b)-f(a)}{\sum_{s=a}^{b-1}g(s)}+k\frac{f(b+1)-f(a+1)}{\sum_{s=a}^{b-1}g(s)}\right]\sum_{s=a}^{b-1}g(s)f(s+1)$$
$$\left.+k\left(f(a+1)f(b)-f(b+1)f(a)\right)-\frac{f(b)-f(a)}{\sum_{s=a}^{b-1}g(s)}\sum_{s=a}^{b-1}g(s)f(s+2)\right|$$
$$\leqslant\frac{M(M+N)}{\sum_{s=a}^{b-1}g(s)}\sum_{t=a}^{b-1}\left(\sum_{s=a}^{b-1}|S(t,s))|\right),$$

where

$$S(t,s)=\begin{cases}h(s)-((1-k)h(a)+kh(t)),\ a\leqslant s<t-1,\\ h(s)-(kh(t)+(1-k)h(b)),\ t\leqslant s\leqslant b-1\end{cases}$$

and

$$M=\sup_{a<t<b-1}|\Delta f(t)|<\infty \text{ and } N=\sup_{a<t<b-1}|\Delta f(t+1)|<\infty.$$

COROLLARY 17. *In the case of* $h(t) = t$ *in Theorem 7, we have*

$$\begin{aligned}&\Bigg|(1-k)(f^2(b)-f^2(a))\\&+\left[(k-1)\frac{f(b)-f(a)}{b-a}+k\frac{f(\sigma(b))-f(\sigma(a))}{b-a}\right]\int_a^b f(\sigma(t))\Delta t\\&+k\left(f(\sigma(a))f(b)-f(\sigma(b))f(a)\right)-\frac{f(b)-f(a)}{b-a}\int_a^b f(\sigma^2(t))\Delta t\Bigg|\\ \leqslant&\frac{M(M+N)}{b-a}\int_a^b [h_2((1-k)a+kt,a)+h_2(t,(1-k)a+kt)\\&+h_2(t,kt+(1-k)b)+h_2(b,kt+(1-k)b)]\Delta t\end{aligned}$$

for all $k \in [0,1]$ *such that* $(1-k)a+kt$ *and* $kt+(1-k)b$ *are in* $\mathbb{T}$*, where*

$$M = \sup_{a<t<b}\left|f^{\Delta}(t)\right| < \infty \text{ and } N = \sup_{a<t<b}\left|f^{\Delta}(\sigma(t))\right| < \infty.$$

COROLLARY 18. *In the case of* $\mathbb{T} = \mathbb{R}$ *in Corollary 17, we have*

$$\begin{aligned}&\left|\frac{1}{2}(1-k)(f^2(b)-f^2(a))+(k-1)\frac{f(b)-f(a)}{b-a}\int_a^b f(t)dt\right|\\ \leqslant&\frac{M^2}{3}(b-a)^2\left(2k^2-2k+1\right)\end{aligned}$$

where

$$M = \sup_{a<t<b}\left|f'(t)\right| < \infty.$$

REMARK 3. If we set $k = 0$ in Corollary 18, we get exactly [27, Theorem 1(a_1)].

COROLLARY 19. *In the case of* $\mathbb{T} = \mathbb{Z}$ *in Corollary 17, we have*

$$\begin{aligned}&\Bigg|(1-k)(f^2(b)-f^2(a))+\left[(k-1)\frac{f(b)-f(a)}{b-a}+k\frac{f(b+1)-f(a+1)}{b-a}\right]\sum_{s=a}^{b-1}f(s+1)\\&+k\left(f(a+1)f(b)-f(b+1)f(a)\right)-\frac{f(b)-f(a)}{b-a}\sum_{s=a}^{b-1}f(s+2)\Bigg|\\ \leqslant&\frac{M(M+N)}{3}\left[(b-a)^2(2k^2-2k+1)+k^2-4k+2\right]\end{aligned}$$

where

$$M = \sup_{a<t<b-1}|\Delta f(t)| < \infty \text{ and } N = \sup_{a<t<b-1}|\Delta f(t+1)| < \infty.$$

3.4. Weighted Grüss type inequality on time scales

THEOREM 8. *Let* $0 \leqslant k \leqslant 1$, $g : [a,b] \to [0,\infty)$ *continuous and positive and* $h : [a,b] \to \mathbb{R}$ *be differentiable such that* $h^{\Delta}(t) = g(t)$ *on* $[a,b]$. *Let* $a,b,s,t \in \mathbb{T}$, $a < b$ *and* $p,q : [a,b] \to \mathbb{R}$ *be differentiable. Then for all* $s \in [a,b]$, *we have*

$$\begin{aligned}
&\Bigg|2(1-k)\left(\int_a^b g(t)\Delta t\right)\left(\int_a^b q(t)p(t)\Delta t\right) \\
&+k\int_a^b \Bigg\{q(t)\left[\left(\int_a^t g(s)\Delta s\right)p(a)+\left(\int_t^b g(s)\Delta s\right)p(b)\right] \\
&+p(t)\left[\left(\int_a^t g(s)\Delta s\right)q(a)+\left(\int_t^b g(s)\Delta s\right)q(b)\right]\Bigg\}\Delta t \\
&-\left[\left(\int_a^b q(t)\Delta t\right)\left(\int_a^b g(t)p(\sigma(t))\Delta t\right)+\left(\int_a^b p(t)\Delta t\right)\left(\int_a^b g(t)q(\sigma(t))\Delta t\right)\right]\Bigg| \\
\leqslant&\int_a^b (P|q(t)|+Q|p(t)|)\left(\int_a^b |S(t,s)|\Delta s\right)\Delta t, \qquad (3.12)
\end{aligned}$$

where

$$S(t,s) = \begin{cases} h(s)-((1-k)h(a)+kh(t)), & a \leqslant s < t, \\ h(s)-(kh(t)+(1-k)h(b)), & t \leqslant s \leqslant b, \end{cases}$$

$$P = \sup_{a<t<b}\left|p^{\Delta}(t)\right| < \infty \ \text{and} \ Q = \sup_{a<t<b}\left|q^{\Delta}(t)\right| < \infty.$$

Proof. We have

$$\begin{aligned}
(1-k)p(t) =& -k\left[\frac{\int_a^t g(s)\Delta s}{\int_a^b g(t)\Delta t}p(a)+\frac{\int_t^b g(s)\Delta s}{\int_a^b g(t)\Delta t}p(b)\right] \\
&+\frac{1}{\int_a^b g(t)\Delta t}\int_a^b g(s)p(\sigma(s))\Delta s+\frac{1}{\int_a^b g(t)\Delta t}\int_a^b S(t,s)p^{\Delta}(s)\Delta s \qquad (3.13)
\end{aligned}$$

and

$$\begin{aligned}
(1-k)q(t) =& -k\left[\frac{\int_a^t g(s)\Delta s}{\int_a^b g(t)\Delta t}q(a)+\frac{\int_t^b g(s)\Delta s}{\int_a^b g(t)\Delta t}q(b)\right] \\
&+\frac{1}{\int_a^b g(t)\Delta t}\int_a^b g(s)q(\sigma(s))\Delta s+\frac{1}{\int_a^b g(t)\Delta t}\int_a^b S(t,s)q^{\Delta}(s)\Delta s. \qquad (3.14)
\end{aligned}$$

Multiplying (3.13) by $q(t)$ and (3.14) by $p(t)$, adding and then integrating the result from a to b, we have

$$\begin{aligned}
&2(1-k)\int_a^b q(t)p(t)\Delta t\\
&=-k\int_a^b\left\{q(t)\left[\frac{\int_a^t g(s)\Delta s}{\int_a^b g(t)\Delta t}p(a)+\frac{\int_t^b g(s)\Delta s}{\int_a^b g(t)\Delta t}p(b)\right]\right.\\
&\quad\left.+p(t)\left[\frac{\int_a^t g(s)\Delta s}{\int_a^b g(t)\Delta t}q(a)+\frac{\int_t^b g(s)\Delta s}{\int_a^b g(t)\Delta t}q(b)\right]\right\}\Delta t\\
&\quad+\frac{1}{\int_a^b g(t)\Delta t}\left[\left(\int_a^b q(t)\Delta t\right)\left(\int_a^b g(s)p(\sigma(s))\Delta s\right)\right.\\
&\quad\left.+\left(\int_a^b p(t)\Delta t\right)\left(\int_a^b g(s)q(\sigma(s))\Delta s\right)\right]\\
&\quad+\frac{1}{\int_a^b g(t)\Delta t}\left[\int_a^b q(t)\left(\int_a^b S(t,s)p^{\Delta}(s)\Delta s\right)\Delta t+\int_a^b p(t)\left(\int_a^b S(t,s)q^{\Delta}(s)\Delta s\right)\Delta t\right]
\end{aligned} \tag{3.15}$$

From (3.15), we get

$$\begin{aligned}
&\left|2(1-k)\left(\int_a^b g(t)\Delta t\right)\left(\int_a^b q(t)p(t)\Delta t\right)\right.\\
&\quad+k\int_a^b\left\{q(t)\left[\left(\int_a^t g(s)\Delta s\right)p(a)+\left(\int_t^b g(s)\Delta s\right)p(b)\right]\right.\\
&\quad\left.+p(t)\left[\left(\int_a^t g(s)\Delta s\right)q(a)+\left(\int_t^b g(s)\Delta s\right)q(b)\right]\right\}\Delta t\\
&\quad\left.-\left[\left(\int_a^b q(t)\Delta t\right)\left(\int_a^b g(t)p(\sigma(t))\Delta t\right)+\left(\int_a^b p(t)\Delta t\right)\left(\int_a^b g(t)q(\sigma(t))\Delta t\right)\right]\right|\\
&\leqslant\int_a^b\left|q(t)\left(\int_a^b S(t,s)p^{\Delta}(s)\Delta s\right)+p(t)\left(\int_a^b S(t,s)q^{\Delta}(s)\Delta s\right)\right|\Delta t\\
&\leqslant\int_a^b\left(|q(t)|\int_a^b|S(t,s)|\left|p^{\Delta}(s)\right|\Delta s+|p(t)|\int_a^b|S(t,s)|\left|q^{\Delta}(s)\right|\Delta s\right)\Delta t\\
&\leqslant\int_a^b(P|q(t)|+Q|p(t)|)\left(\int_a^b|S(t,s)|\Delta s\right)\Delta t
\end{aligned}$$

This completes the proof of the inequality (3.12). □

COROLLARY 20. *In the case of* $\mathbb{T}=\mathbb{R}$ *in Theorem 8, we have*

$$\begin{aligned}
&\left|2(1-k)\left(\int_a^b g(t)dt\right)\left(\int_a^b q(t)p(t)dt\right)\right.\\
&\quad+k\int_a^b\left\{q(t)\left[\left(\int_a^t g(s)ds\right)p(a)+\left(\int_t^b g(s)ds\right)p(b)\right]\right.
\end{aligned}$$

$$+p(t)\left[\left(\int_a^t g(s)ds\right)q(a)+\left(\int_t^b g(s)ds\right)q(b)\right]\Big\}dt$$
$$-\left[\left(\int_a^b q(t)dt\right)\left(\int_a^b g(t)p(t)dt\right)+\left(\int_a^b p(t)dt\right)\left(\int_a^b g(t)q(t)dt\right)\right]\Big|$$
$$\leqslant \int_a^b (P|q(t)|+Q|p(t)|)\left(\int_a^b |S(t,s)|ds\right)dt,$$

where

$$S(t,s)=\begin{cases} h(s)-((1-k)h(a)+kh(t)),\ a\leqslant s<t,\\ h(s)-(kh(t)+(1-k)h(b)),\ t\leqslant s\leqslant b,\end{cases}$$

$$P=\sup_{a<t<b}\left|p'(t)\right|<\infty \ \text{and}\ Q=\sup_{a<t<b}\left|q'(t)\right|<\infty.$$

COROLLARY 21. *In the case of* $\mathbb{T}=\mathbb{Z}$ *in Theorem 8, we have*

$$\Big|2(1-k)\left(\sum_{t=a}^{b-1} g(t)\right)\left(\sum_{t=a}^{b-1} q(t)p(t)\right)$$
$$+k\sum_{t=a}^{b-1}\Big\{q(t)\left[\left(\sum_{s=a}^{t-1} g(s)\right)p(a)+\left(\sum_{s=t}^{b-1} g(s)\right)p(b)\right]$$
$$+p(t)\left[\left(\sum_{s=a}^{t-1} g(s)\right)q(a)+\left(\sum_{s=t}^{b-1} g(s)\right)q(b)\right]\Big\}$$
$$-\left[\left(\sum_{t=a}^{b-1} q(t)\right)\left(\sum_{t=a}^{b-1} g(t)p(t+1)\right)+\left(\sum_{t=a}^{b-1} p(t)\right)\left(\sum_{t=a}^{b-1} g(t)q(t+1)\right)\right]\Big|$$
$$\leqslant \sum_{t=a}^{b-1}(P|q(t)|+Q|p(t)|)\left(\sum_{s=a}^{b-1}|S(t,s)|\right),$$

where

$$S(t,s)=\begin{cases} h(s)-((1-k)h(a)+kh(t)),\ a\leqslant s<t-1,\\ h(s)-(kh(t)+(1-k)h(b)),\ t\leqslant s\leqslant b-1,\end{cases}$$

$$P=\sup_{a<t<b-1}|\Delta p(t)|<\infty \ \text{and}\ Q=\sup_{a<t<b-1}|\Delta q(t)|<\infty.$$

COROLLARY 22. *In the case of* $h(t)=t$ *in Theorem* 8, *we have*

$$\begin{aligned}&\Big|2(1-k)(b-a)\left(\int_a^b q(t)p(t)\Delta t\right)\\&+k\int_a^b \{q(t)[(t-a)p(a)+(b-t)p(b)]+p(t)[(t-a)q(a)+(b-t)q(b)]\}\Delta t\\&-\left[\left(\int_a^b q(t)\Delta t\right)\left(\int_a^b p(\sigma(t))\Delta t\right)+\left(\int_a^b p(t)\Delta t\right)\left(\int_a^b q(\sigma(t))\Delta t\right)\right]\Big|\\&\leqslant\int_a^b (P|q(t)|+Q|p(t)|)[h_2((1-k)a+kt,a)+h_2(t,(1-k)a+kt)\\&+h_2(t,kt+(1-k)b)+h_2(b,kt+(1-k)b)]\Delta t\end{aligned}$$

for all $k\in[0,1]$ *such that* $(1-k)a+kt$ *and* $kt+(1-k)b$ *are in* $\mathbb{T}$, *where*

$$P=\sup_{a<t<b}\left|p^{\Delta}(t)\right|<\infty \text{ and } Q=\sup_{a<t<b}\left|q^{\Delta}(t)\right|<\infty.$$

COROLLARY 23. *In the case of* $\mathbb{T}=\mathbb{R}$ *in Corollary* 22, *we have*

$$\begin{aligned}&\Big|2(1-k)(b-a)\left(\int_a^b q(t)p(t)dt\right)\\&+k\int_a^b \{q(t)[(t-a)p(a)+(b-t)p(b)]+p(t)[(t-a)q(a)+(b-t)q(b)]\}dt\\&-2\left(\int_a^b p(t)dt\right)\left(\int_a^b q(t)dt\right)\Big|\\&\leqslant\int_a^b (P|q(t)|+Q|p(t)|)\left(2k^2-2k+1\right)\left[\frac{(a-t)^2+(b-t)^2}{2}\right]dt,\end{aligned}$$

where

$$P=\sup_{a<t<b}\left|p'(t)\right|<\infty \text{ and } Q=\sup_{a<t<b}\left|q'(t)\right|<\infty.$$

REMARK 4. If we set $k=0$ in Corollary 23, we get exactly [27, Theorem 2(b_1)].

COROLLARY 24. *In the case of* $\mathbb{T}=\mathbb{Z}$ *in Corollary* 22, *we have*

$$\begin{aligned}&\Big|2(1-k)(b-a)\left(\sum_{t=a}^{b-1} q(t)p(t)\right)\\&+k\sum_{t=a}^{b-1}\{q(t)[(t-a)p(a)+(b-t)p(b)]+p(t)[(t-a)q(a)+(b-t)q(b)]\}\\&-\left[\left(\sum_{t=a}^{b-1} q(t)\right)\left(\sum_{t=a}^{b-1} p(t+1)\right)+\left(\sum_{t=a}^{b-1} p(t)\right)\left(\sum_{t=a}^{b-1} q(t+1)\right)\right]\Big|\\&\leqslant\sum_{t=a}^{b-1}(P|q(t)|+Q|p(t)|)\left[\left(2k^2-2k+1\right)\left(t-\frac{a+b}{2}\right)^2+(2k-1)\left(t-\frac{a+b}{2}\right)+\left(\frac{b-a}{2}\right)^2\right],\end{aligned}$$

where

$$P = \sup_{a<t<b-1} |\Delta p(t)| < \infty \ \textit{and}\ Q = \sup_{a<t<b-1} |\Delta q(t)| < \infty.$$

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