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ON IMPROVED ARITHMETIC–GEOMETRIC MEAN AND HEINZ INEQUALITIES FOR MATRICES

CHUANJIANG HE, LIMIN ZOU AND SHAHID QAISAR

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Abstract. In this paper, we first generalize an inequality and improve another one for unitarily invariant norms, which are established by Kittaneh and Manasrah in [Improved Young and Heinz inequalities for matrices. J. Math. Anal. Appl.361(2010)262-269]. Then we present a new inequality for unitarily invariant norms, which is equivalent to an inequality presented by Kittaneh and Manasrah in the case of the Hilbert-Schmidt norm.

1. Introduction

Let $M_{m,n}$ be the space of $m \times n$ complex matrices and $M_n = M_{n,n}$. Let $\|\cdot\|$ denote any unitarily invariant norm on M_n. So, $\|UAV\| = \|A\|$ for all $A \in M_n$ and for all unitary matrices U, $V \in M_n$. For $A = (a_{ij}) \in M_n$, the Hilbert-Schmidt norm of A is defined by

$$\|A\|_2 = \sqrt{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left|a_{ij}\right|^2\right)} = \sqrt{tr|A|^2} = \sqrt{\sum_{j=1}^{n} s_j^2(A)},$$

where $s_1(A) \geqslant s_2(A) \geqslant \cdots \geqslant s_{n-1}(A) \geqslant s_n(A)$ are the singular values of A, that is, the eigenvalues of the positive semidefinite matrix $|A| = (AA^*)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity. For $A = (a_{ij}) \in M_n$, the trace norm is defined by

$$\|A\|_1 = \sum_{j=1}^{n} s_j(A) = tr|A|.$$

It is known that the Hilbert-Schmidt and trace norms are unitarily invariant.

Let A, B, $X \in M_n$ such that A and B are positive semidefinite, Kittaneh and Manasrah [1] have obtained the following inequalities:

$$2\left\|A^{1/2}XB^{1/2}\right\|_2 + \left(\sqrt{\|AX\|_2} - \sqrt{\|XB\|_2}\right)^2 \leqslant \|AX + XB\|_2, \tag{1.1}$$

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$$\left\|A^{v}B^{1-v}+A^{1-v}B^{v}\right\|_1 \leqslant \|A+B\|_1 - 2r_0\left(\sqrt{\|A\|_1}-\sqrt{\|B\|_1}\right)^2, \tag{1.2}$$

$$\left\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\right\|_2^2 \leqslant \|AX+XB\|_2^2 - 2r_0\|AX-XB\|_2^2, \tag{1.3}$$

where $0 \leqslant v \leqslant 1$ and $r_0 = \min\{v,\ 1-v\}$.

The inequality (1.1) [1, Theorem 3.3] is a refinement of the arithmetic-geometric mean inequality [2, Theorem 2]:

$$2\left\|A^{1/2}XB^{1/2}\right\| \leqslant \|AX+XB\|$$

for the Hilbert-Schmidt norm. The inequality (1.2) [1, Corollary 3.9] is a refinement of the Heinz inequality [2, Theorem 2]:

$$\left\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\right\| \leqslant \|AX+XB\|$$

for $X = I$ and the trace norm. The inequality (1.3) [1, Theorem 3.5] is an improvement of the Heinz inequality for the Hilbert-Schmidt norm.

In this paper, we first generalize the inequality (1.1) and improve the inequality (1.2). Then we present a new inequality for unitarily invariant norms, which is equivalent to the inequality (1.3) in the case of the Hilbert-Schmidt norm.

2. Main results

First, we generalize the inequality (1.1). To do this, we need the following lemma.

Lemma 2.1. [3] *Let A, B, $X \in M_n$ such that A and B are positive semidefinite. If $0 \leqslant v \leqslant 1$, then*

$$\left\|A^{v}XB^{1-v}\right\| \leqslant \|AX\|^{v}\,\|XB\|^{1-v}.$$

Theorem 2.1. *Let A, B, $X \in M_n$ such that A and B are positive semidefinite. If $0 \leqslant v \leqslant 1$, then*

$$\begin{aligned} 2\sqrt{v(1-v)}\left\|A^{1/2}XB^{1/2}\right\|_2 + \left(\sqrt{v\|AX\|_2}-\sqrt{(1-v)\|XB\|_2}\right)^2 \\ \leqslant \|vAX+(1-v)XB\|_2. \end{aligned} \tag{2.1}$$

Proof. By Lemma 2.1, we have

$$\begin{aligned}
&\left(2\sqrt{v(1-v)}\left\|A^{1/2}XB^{1/2}\right\|_2+\left(\sqrt{v\|AX\|_2}-\sqrt{(1-v)\|XB\|_2}\right)^2\right)^2\\
&\quad-\|vAX+(1-v)XB\|_2^2\\
&=4\sqrt{v(1-v)}\left\|A^{1/2}XB^{1/2}\right\|_2\left(\sqrt{v\|AX\|_2}-\sqrt{(1-v)\|XB\|_2}\right)^2\\
&\quad+2v(1-v)\left\|A^{1/2}XB^{1/2}\right\|_2^2+\left(\sqrt{v\|AX\|_2}-\sqrt{(1-v)\|XB\|_2}\right)^4\\
&\quad-v^2\|AX\|_2^2+(1-v)^2\|XB\|_2^2\\
&\leqslant 4\sqrt{v(1-v)}\left\|A^{1/2}XB^{1/2}\right\|_2\left(\sqrt{v\|AX\|_2}-\sqrt{(1-v)\|XB\|_2}\right)^2\\
&\quad+\left(\sqrt{v\|AX\|_2}-\sqrt{(1-v)\|XB\|_2}\right)-(v\|AX\|-(1-v)\|XB\|_2)^2\\
&=-4\sqrt{v(1-v)}\left(\sqrt{v\|AX\|_2}-\sqrt{(1-v)\|XB\|_2}\right)^2\\
&\quad\times\left(\sqrt{\|AX\|_2\|XB\|_2}-\left\|A^{1/2}XB^{1/2}\right\|_2\right)\\
&\leqslant 0.
\end{aligned} \tag{2.2}$$

For the proof of the last inequality in (2.2), we have used Lemma 2.1 again. This completes the proof. □

REMARKS.

1. Taking $v=\frac{1}{2}$ in the inequality (2.1), we obtain the inequality (1.1).

2. The inequality (2.1) is related to the following inequality presented by Kosaki [4] and Bhatia and Parthasarathy [5]: if $A,\ B,\ X\in M_n$ such that A and B are positive semidefinite and if $0\leqslant v\leqslant 1$, then

$$\left\|A^vXB^{1-v}\right\|_2\leqslant\|vAX+(1-v)XB\|_2. \tag{2.3}$$

It should be noticed that neither (2.1) nor (2.3) is uniformly better than the other.

Second, we improve the inequality (1.2). To do this, we need the two lemmas as follows.

LEMMA 2.2. [6] *Suppose that* $a,\ b\geqslant 0$. *If* $0\leqslant v\leqslant 1$, *then*

$$\frac{a^vb^{1-v}+a^{1-v}b^v}{2}\leqslant ta^{1/2}b^{1/2}+(1-t)\frac{a+b}{2},$$

where $t=4\left(v-v^2\right)$.

LEMMA 2.3. [7, p. 94] *Let $A,B \in M_n$, then*

$$\sum_{j=1}^{n} s_j(AB) \leqslant \sum_{j=1}^{n} s_j(A) s_j(B).$$

THEOREM 2.2. *If A, $B \in M_n$ are positive semidefinite, then*

$$\left\|A^v B^{1-v}\right\|_1 + \left\|A^{1-v} B^v\right\|_1 \leqslant (1-t)\|A+B\|_1 + 2t\sqrt{\|A\|_1 \cdot \|B\|_1}, \tag{2.4}$$

where $0 \leqslant v \leqslant 1,\ t = 4\left(v - v^2\right)$.

Proof. By Lemma 2.2, we have

$$\frac{s_j^v(A) s_j^{1-v}(B) + s_j^{1-v}(A) s_j^v(B)}{2} \leqslant t s_j^{1/2}(A) s_j^{1/2}(B) + (1-t)\frac{s_j(A)+s_j(B)}{2},$$

for all $j = 1,\cdots,n$. By Lemma 2.3 and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&\frac{\|A^v B^{1-v}\|_1 + \|A^{1-v}B^v\|_1}{2} \\
&= \frac{\sum_{j=1}^{n} s_j(A^v B^{1-v}) + \sum_{j=1}^{n} s_j(A^{1-v}B^v)}{2} \\
&\leqslant \frac{\sum_{j=1}^{n} s_j^v(A) s_j^{1-v}(B) + \sum_{j=1}^{n} s_j^{1-v}(A) s_j^v(B)}{2} \\
&\leqslant t\sum_{j=1}^{n} s_j^{1/2}(A) s_j^{1/2}(B) + \left(\tfrac{1-t}{2}\right)\sum_{j=1}^{n}\left(s_j(A)+s_j(B)\right) \\
&\leqslant t\sqrt{\sum_{j=1}^{n} s_j(A)\sum_{j=1}^{n} s_j(B)} + \left(\tfrac{1-t}{2}\right)\sum_{j=1}^{n}\left(s_j(A)+s_j(B)\right) \\
&= (1-t)\,tr\left(\tfrac{A+B}{2}\right) + t\sqrt{trA \cdot trB}.
\end{aligned} \tag{2.5}$$

Note that

$$(1-t)\|A+B\|_1 + 2t\sqrt{\|A\|_1 \cdot \|B\|_1} = (1-t)\,tr(A+B) + 2t\sqrt{trA \cdot trB}. \tag{2.6}$$

If follows from (2.5) and (2.6) that

$$\left\|A^v B^{1-v}\right\|_1 + \left\|A^{1-v} B^v\right\|_1 \leqslant (1-t)\|A+B\|_1 + 2t\sqrt{\|A\|_1 \cdot \|B\|_1}.$$

This completes the proof. □

REMARK. By the triangle inequality, we have

$$\left\|A^v B^{1-v} + A^{1-v} B^v\right\|_1 \leqslant \left\|A^v B^{1-v}\right\|_1 + \left\|A^{1-v} B^v\right\|_1. \tag{2.7}$$

So, it follows from (2.4) and (2.7) that

$$\left\|A^{v}B^{1-v}+A^{1-v}B^{v}\right\|_{1}\leqslant(1-t)\left\|A+B\right\|_{1}+2t\sqrt{\left\|A\right\|_{1}\cdot\left\|B\right\|_{1}}. \tag{2.8}$$

Now, we compare the inequality (1.2) with the inequality (2.8). Let

$$K=\left\|A+B\right\|_{1}-2r_{0}\left(\sqrt{\left\|A\right\|_{1}}-\sqrt{\left\|B\right\|_{1}}\right)^{2}-(1-t)\left\|A+B\right\|_{1}-2t\sqrt{\left\|A\right\|_{1}\cdot\left\|B\right\|_{1}},$$

where $r_0=\min\{v,\ 1-v\}$. It easily follows that

$$K=2v(1-2v)\left(\left\|A+B\right\|_{1}-2\sqrt{\left\|A\right\|_{1}\cdot\left\|B\right\|_{1}}\right)\geqslant 0,\quad 0\leqslant v\leqslant\frac{1}{2}$$

and

$$K=2\left(-2v^{2}+3v-1\right)\left(\left\|A+B\right\|_{1}-2\sqrt{\left\|A\right\|_{1}\cdot\left\|B\right\|_{1}}\right)\geqslant 0,\quad \frac{1}{2}\leqslant v\leqslant 1$$

Therefore, the inequality (2.8) is a refinement of the inequality (1.2).

Next, we present a new inequality for unitarily invariant norms. To do this, we need the following lemmas on convex functions.

LEMMA 2.4. *If $A,\ B,\ X\in M_n$ such that A and B are positive semidefinite, then the function $f(v)=\left\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\right\|^{2}$ is continuous and convex on* $[0,1]$.

Proof. For each unitarily invariant norm, the function

$$\varphi(v)=\left\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\right\|$$

is continuous and convex on $[0,1]$ [7, p. 265].

Since $h(x)=x^2\ (x\geqslant 0)$ is continuous and nondecreasing, $h(\phi(v))=\phi^2(v)$ is also continuous and convex, which implies that the function $f(v)$ is continuous and convex on $[0,1]$. This completes the proof. □

LEMMA 2.5. [8,9] *Let $\varphi(x)$ be a real valued convex function on an interval $[a,b]$. For any $x_1,x_2\in[a,b]$, we have*

$$\varphi(x)\leqslant\frac{\varphi(x_{2})-\varphi(x_{1})}{x_{2}-x_{1}}x-\frac{x_{1}\varphi(x_{2})-x_{2}\varphi(x_{1})}{x_{2}-x_{1}},\quad x\in(x_{1},x_{2}).$$

THEOREM 2.3. *Let $A,\ B,\ X\in M_n$ such that A and B are positive semidefinite. For unitarily invariant norms, we have*

$$\left\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\right\|^{2}\leqslant(1-2r_{0})\left\|AX+XB\right\|^{2}+8r_{0}\left\|A^{1/2}XB^{1/2}\right\|^{2}, \tag{2.9}$$

where $0\leqslant v\leqslant 1,\ r_0=\min\{v,\ 1-v\}$.

Proof. By Lemma 2.4, from Lemma 2.5 we obtain:

If $0 \leqslant v \leqslant \frac{1}{2}$, then

$$\begin{aligned} f(v) &\leqslant \frac{f\left(\frac{1}{2}\right)-f(0)}{\frac{1}{2}-0}v-\frac{0\cdot f\left(\frac{1}{2}\right)-\frac{1}{2}f(0)}{\frac{1}{2}-0} \\ &= (1-2v)\|AX+XB\|^2+8v\left\|A^{1/2}XB^{1/2}\right\|^2 \\ &= (1-2r_0)\|AX+XB\|^2+8r_0\left\|A^{1/2}XB^{1/2}\right\|^2. \end{aligned}$$

If $\frac{1}{2} \leqslant v \leqslant 1$, then

$$\begin{aligned} f(v) &\leqslant \frac{f(1)-f\left(\frac{1}{2}\right)}{1-\frac{1}{2}}v-\frac{\frac{1}{2}\cdot f(1)-1\cdot f\left(\frac{1}{2}\right)}{1-\frac{1}{2}} \\ &= (2v-1)\|AX+XB\|^2+8(1-v)\left\|A^{1/2}XB^{1/2}\right\|^2 \\ &= (1-2r_0)\|AX+XB\|^2+8r_0\left\|A^{1/2}XB^{1/2}\right\|^2. \end{aligned}$$

This completes the proof. □

REMARK. For the Hilbert-Schmidt norm, the inequality (2.9) is equivalent to the inequality (1.3). In fact, it easily follows that

$$\|AX+XB\|_2^2 = \|AX\|_2^2+\|XB\|_2^2+2\left\|A^{1/2}XB^{1/2}\right\|_2^2$$

and

$$\|AX-XB\|_2^2 = \|AX\|_2^2+\|XB\|_2^2-2\left\|A^{1/2}XB^{1/2}\right\|_2^2.$$

As a result, we have

$$(1-2r_0)\|AX+XB\|_2^2+8r_0\left\|A^{1/2}XB^{1/2}\right\|_2^2 = \|AX+XB\|_2^2-2r_0\|AX-XB\|_2^2.$$

So, from (2.9) we obtain

$$\left\|A^vXB^{1-v}+A^{1-v}XB^v\right\|_2^2 \leqslant \|AX+XB\|_2^2-2r_0\|AX-XB\|_2^2,$$

which is the inequality (1.3).

REFERENCES

[1] F. KITTANEH, Y. MANASRAH, *Improved Young and Heinz inequalities for matrices*, J. Math. Anal. Appl. **361** (2010), 262–269.

[2] R. BHATIA, C. DAVIS, *More matrix forms of the arithmetic-geometric mean inequality*, SIAM J. Matrix Anal. Appl. **14** (1993), 132–136.

[3] F. KITTANEH, *Norm inequalities for fractional powers of positive operators*, Lett. Math. Phys. **27** (1993), 279–285.

[4] H. KOSAKI, *Arithmetic-geometric mean and related inequalities for operators*, J. Funct. Anal. **156** (1998), 429–451.

[5] R. BHATIA, K. R. PARTHASARATHY, *Positive definite functions and operator inequalities*, Bull. London Math. Soc. **32** (2000), 214–228.
[6] R. BHATIA, *Interpolating the arithmetic-geometric mean inequality and its operator version*, Linear Algebra Appl. **413** (2006), 355–363.
[7] R. BHATIA, *Matrix Analysis*, Springer-Verlag, New York, 1997.
[8] S. WANG, L. ZOU, Y. JIANG, *Some inequalities for unitarily invariant norms of matrices*, J. Inequal. Appl. (2011), **2011**:10.
[9] R. BHATIA, R. SHARMA, *Some inequalities for positive linear maps*, Linear Algebra Appl. **436** (2012), 1562–1571.

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Chuanjiang He
College of Mathematics and Statistics
Chongqing University
Chongqing, 401331
P. R. China

Limin Zou
College of Mathematics and Statistics
Chongqing University
Chongqing, 401331
P. R. China
e-mail: limin-zou@163.com

Shahid Qaisar
College of Mathematics and Statistics
Chongqing University
Chongqing, 401331
P. R. China