ON IMPROVED ARITHMETIC–GEOMETRIC MEAN AND HEINZ INEQUALITIES FOR MATRICES

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Abstract. In this paper, we first generalize an inequality and improve another one for unitarily invariant norms, which are established by Kittaneh and Manasrah in [Improved Young and Heinz inequalities for matrices. J. Math. Anal. Appl. 361(2010)262-269]. Then we present a new inequality for unitarily invariant norms, which is equivalent to an inequality presented by Kittaneh and Manasrah in the case of the Hilbert-Schmidt norm.

1. Introduction

Let $M_{m,n}$ be the space of $m \times n$ complex matrices and $M_n = M_{n,n}$. Let $\| \cdot \|$ denote any unitarily invariant norm on $M_n$. So, $\| UAV \| = \| A \|$ for all $A \in M_n$ and for all unitary matrices $U, V \in M_n$. For $A = (a_{ij}) \in M_n$, the Hilbert-Schmidt norm of $A$ is defined by

$$\| A \|_2 = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2} = \sqrt{\text{tr}|A|^2} = \sqrt{\sum_{j=1}^{n} s_j^2(A)},$$

where $s_1(A) \geq s_2(A) \geq \cdots \geq s_{n-1}(A) \geq s_n(A)$ are the singular values of $A$, that is, the eigenvalues of the positive semidefinite matrix $|A| = (A^*A)^{1/2}$, arranged in decreasing order and repeated according to multiplicity. For $A = (a_{ij}) \in M_n$, the trace norm is defined by

$$\| A \|_1 = \sum_{j=1}^{n} s_j(A) = \text{tr}|A|.$$

It is known that the Hilbert-Schmidt and trace norms are unitarily invariant.

Let $A, B, X \in M_n$ such that $A$ and $B$ are positive semidefinite, Kittaneh and Manasrah [1] have obtained the following inequalities:

$$2 \left\| A^{1/2}X B^{1/2} \right\|_2 + \left( \sqrt{\|AX\|_2} - \sqrt{\|XB\|_2} \right)^2 \leq \|AX + XB\|_2, \quad (1.1)$$


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\[ ||A^vB^{1-v} + A^{1-v}B^v||_1 \leq ||A + B||_1 - 2r_0 \left( \sqrt{||A||_1} - \sqrt{||B||_1} \right)^2, \quad (1.2) \]

\[ ||A^vXB^{1-v} + A^{1-v}XB^v||_2^2 \leq ||AX + XB||_2^2 - 2r_0 ||AX - XB||_2^2, \quad (1.3) \]

where \( 0 \leq v \leq 1 \) and \( r_0 = \min \{ v, 1 - v \} \).

The inequality (1.1) [1, Theorem 3.3] is a refinement of the arithmetic-geometric mean inequality [2, Theorem 2]:

\[ 2 \left\| A^{1/2}XB^{1/2} \right\| \leq \| AX + XB \| \]

for the Hilbert-Schmidt norm. The inequality (1.2) [1, Corollary 3.9] is a refinement of the Heinz inequality [2, Theorem 2]:

\[ \| A^vXB^{1-v} + A^{1-v}XB^v \| \leq \| AX + XB \| \]

for \( X = I \) and the trace norm. The inequality (1.3) [1, Theorem 3.5] is an improvement of the Heinz inequality for the Hilbert-Schmidt norm.

In this paper, we first generalize the inequality (1.1) and improve the inequality (1.2). Then we present a new inequality for unitarily invariant norms, which is equivalent to the inequality (1.3) in the case of the Hilbert-Schmidt norm.

2. Main results

First, we generalize the inequality (1.1). To do this, we need the following lemma.

**Lemma 2.1.** [3] Let \( A, B, X \in M_n \) such that \( A \) and \( B \) are positive semidefinite. If \( 0 \leq v \leq 1 \), then

\[ \| A^vXB^{1-v} \| \leq \| AX \|^v \| XB \|^{1-v}. \]

**Theorem 2.1.** Let \( A, B, X \in M_n \) such that \( A \) and \( B \) are positive semidefinite. If \( 0 \leq v \leq 1 \), then

\[
2\sqrt{v(1-v)} \left\| A^{1/2}XB^{1/2} \right\|_2 + \left( \sqrt{v\| AX \|_2} - \sqrt{(1-v)\| XB \|_2} \right)^2 \\
\leq \| vAX + (1-v)XB \|_2. \quad (2.1)
\]
**Proof.** By Lemma 2.1, we have

\[
\frac{2}{\sqrt{v}} (1 - v) \left( A^{1/2} X B^{1/2} \right) + \left( \sqrt{v} \|A\|_2 - \sqrt{(1 - v) \|B\|_2} \right)^2
- \|vAX + (1 - v) XB\|_2^2
= \frac{4}{\sqrt{v}} (1 - v) \left( A^{1/2} X B^{1/2} \right) + \left( \sqrt{v} \|A\|_2 - \sqrt{(1 - v) \|B\|_2} \right)^2
+ 2v (1 - v) \left( A^{1/2} X B^{1/2} \right) + \left( \sqrt{v} \|A\|_2 - \sqrt{(1 - v) \|B\|_2} \right)^4
- v^2 \|A\|_2^2 + (1 - v) \|B\|_2^2
\leq 4 \frac{v}{\sqrt{v}} (1 - v) \left( A^{1/2} X B^{1/2} \right) + \left( \sqrt{v} \|A\|_2 - \sqrt{(1 - v) \|B\|_2} \right)^2
+ \left( \sqrt{v} \|A\|_2 - \sqrt{(1 - v) \|B\|_2} \right) - \|AX\| - (1 - v) \|B\|_2^2
= -4 \frac{v}{\sqrt{v}} (1 - v) \left( \sqrt{v} \|A\|_2 - \sqrt{(1 - v) \|B\|_2} \right)^2
\times \left( \sqrt{\|A\|_2^2 \|B\|_2^2 - \|A^{1/2} X B^{1/2}\|_2^2} \right)
\leq 0.
\]

(2.2)

For the proof of the last inequality in (2.2), we have used Lemma 2.1 again. This completes the proof. \(\square\)

**Remarks.**

1. Taking \(v = \frac{1}{2}\) in the inequality (2.1), we obtain the inequality (1.1).

2. The inequality (2.1) is related to the following inequality presented by Kosaki [4] and Bhatia and Parthasarathy [5]: if \(A, B, X \in M_n\) such that \(A\) and \(B\) are positive semidefinite and if \(0 \leq v \leq 1\), then

\[
\|A^{1/2} X B^{1/2}\|_2 \leq \|vAX + (1 - v) XB\|_2.
\]

(2.3)

It should be noticed that neither (2.1) nor (2.3) is uniformly better than the other.

Second, we improve the inequality (1.2). To do this, we need the two lemmas as follows.

**Lemma 2.2.** [6] Suppose that \(a, b \geq 0\). If \(0 \leq v \leq 1\), then

\[
\frac{a^v b^{1-v} + a^{1-v} b^v}{2} \leq t a^{1/2} b^{1/2} + (1 - t) \frac{a + b}{2},
\]

where \(t = 4 (v - v^2)\).
Lemma 2.3. [7, p. 94] Let \( A, B \in M_n \), then
\[
\sum_{j=1}^{n} s_j(AB) \leq \sum_{j=1}^{n} s_j(A)s_j(B).
\]

Theorem 2.2. If \( A, B \in M_n \) are positive semidefinite, then
\[
\|A^vB^{1-v}\|_1 + \|A^{1-v}B^v\|_1 \leq (1-t)\|A+B\|_1 + 2t\sqrt{\|A\|_1 \cdot \|B\|_1}, \tag{2.4}
\]
where \( 0 \leq v \leq 1 \), \( t = 4(v - v^2) \).

Proof. By Lemma 2.2, we have
\[
\frac{s_j^v(A)s_j^{1-v}(B) + s_j^{1-v}(A)s_j^v(B)}{2} \leq ts_j^{1/2}(A)s_j^{1/2}(B) + \frac{(1-t)}{2}(s_j(A) + s_j(B)),
\]
for all \( j = 1, \cdots, n \). By Lemma 2.3 and the Cauchy-Schwarz inequality, we obtain
\[
\|A^vB^{1-v}\|_1 + \|A^{1-v}B^v\|_1 = 2\sum_{j=1}^{n} s_j(A^vB^{1-v}) + \sum_{j=1}^{n} s_j(A^{1-v}B^v) \leq \frac{2}{2} \sum_{j=1}^{n} s_j^v(A)s_j^{1-v}(B) + \frac{2}{2} \sum_{j=1}^{n} s_j^{1-v}(A)s_j^v(B) \leq t \sum_{j=1}^{n} s_j^{1/2}(A)s_j^{1/2}(B) + \frac{(1-t)}{2} \sum_{j=1}^{n} (s_j(A) + s_j(B)) \leq t \sqrt{\sum_{j=1}^{n} s_j(A)\sum_{j=1}^{n} s_j(B) + \frac{(1-t)}{2} \sum_{j=1}^{n} (s_j(A) + s_j(B))} = (1-t)\text{tr}(\frac{A+B}{2}) + t\sqrt{\text{tr}A \cdot \text{tr}B}. \tag{2.5}
\]

Note that
\[
(1-t)\|A+B\|_1 + 2t\sqrt{\|A\|_1 \cdot \|B\|_1} = (1-t)\text{tr}(A+B) + 2t\sqrt{\text{tr}A \cdot \text{tr}B}. \tag{2.6}
\]

If follows from (2.5) and (2.6) that
\[
\|A^vB^{1-v}\|_1 + \|A^{1-v}B^v\|_1 \leq (1-t)\|A+B\|_1 + 2t\sqrt{\|A\|_1 \cdot \|B\|_1}.
\]

This completes the proof. \( \Box \)

Remark. By the triangle inequality, we have
\[
\|A^vB^{1-v} + A^{1-v}B^v\|_1 \leq \|A^vB^{1-v}\|_1 + \|A^{1-v}B^v\|_1. \tag{2.7}
\]
So, it follows from (2.4) and (2.7) that

$$\|A^vB^{1-v} + A^{1-v}B^v\|_1 \leq (1-t)\|A + B\|_1 + 2t\sqrt{\|A\|_1 \cdot \|B\|_1}. \quad (2.8)$$

Now, we compare the inequality (1.2) with the inequality (2.8). Let

$$K = \|A + B\|_1 - 2r_0\left(\sqrt{\|A\|_1} - \sqrt{\|B\|_1}\right)^2 - (1-t)\|A + B\|_1 - 2t\sqrt{\|A\|_1 \cdot \|B\|_1},$$

where $r_0 = \min\{v, 1-v\}$. It easily follows that

$$K = 2v(1-2v)\left(\|A + B\|_1 - 2\sqrt{\|A\|_1 \cdot \|B\|_1}\right) \geq 0, \quad 0 \leq v \leq \frac{1}{2}$$

and

$$K = 2\left(-2v^2 + 3v - 1\right)\left(\|A + B\|_1 - 2\sqrt{\|A\|_1 \cdot \|B\|_1}\right) \geq 0, \quad \frac{1}{2} \leq v \leq 1$$

Therefore, the inequality (2.8) is a refinement of the inequality (1.2).

Next, we present a new inequality for unitarily invariant norms. To do this, we need the following lemmas on convex functions.

**Lemma 2.4.** If $A$, $B$, $X \in M_n$ such that $A$ and $B$ are positive semidefinite, then the function $f(v) = \|A^vXB^{1-v} + A^{1-v}XB^v\|^2$ is continuous and convex on $[0, 1]$.

**Proof.** For each unitarily invariant norm, the function

$$\phi(v) = \|A^vXB^{1-v} + A^{1-v}XB^v\|$$

is continuous and convex on $[0, 1]$ [7, p. 265].

Since $h(x) = x^2$ ($x \geq 0$) is continuous and nondecreasing, $h(\phi(v)) = \phi^2(v)$ is also continuous and convex, which implies that the function $f(v)$ is continuous and convex on $[0, 1]$. This completes the proof. □

**Lemma 2.5.** [8, 9] Let $\phi(x)$ be a real valued convex function on an interval $[a, b]$. For any $x_1, x_2 \in [a, b]$, we have

$$\phi(x) \leq \frac{\phi(x_2) - \phi(x_1)}{x_2 - x_1}x - \frac{x_1\phi(x_2) - x_2\phi(x_1)}{x_2 - x_1}, \quad x \in (x_1, x_2).$$

**Theorem 2.3.** Let $A$, $B$, $X \in M_n$ such that $A$ and $B$ are positive semidefinite. For unitarily invariant norms, we have

$$\|A^vXB^{1-v} + A^{1-v}XB^v\|^2 \leq (1-2r_0)\|AX + XB\|^2 + 8r_0\|A^{1/2}XB^{1/2}\|^2, \quad (2.9)$$

where $0 \leq v \leq 1$, $r_0 = \min\{v, 1-v\}$.

**Proof.** By Lemma 2.4, from Lemma 2.5 we obtain:
If $0 \leq v \leq \frac{1}{2}$, then
\[
f(v) \leq \frac{f\left(\frac{1}{2}\right) - f(0)}{\frac{1}{2} - 0}v - \frac{0 \cdot f\left(\frac{1}{2}\right) - f(0)}{\frac{1}{2} - 0} \\
= (1 - 2v)\|AX + XB\|^2 + 8v\|A^{1/2}XB^{1/2}\|^2 \\
= (1 - 2r_0)\|AX + XB\|^2 + 8r_0\|A^{1/2}XB^{1/2}\|^2.
\]
If $\frac{1}{2} \leq v \leq 1$, then
\[
f(v) \leq \frac{f(1) - f\left(\frac{1}{2}\right)}{1 - \frac{1}{2}}v - \frac{1 - f\left(\frac{1}{2}\right)}{1 - \frac{1}{2}} \\
= (2v - 1)\|AX + XB\|^2 + 8(1 - v)\|A^{1/2}XB^{1/2}\|^2 \\
= (1 - 2r_0)\|AX + XB\|^2 + 8r_0\|A^{1/2}XB^{1/2}\|^2.
\]
This completes the proof. □

REMARK. For the Hilbert-Schmidt norm, the inequality (2.9) is equivalent to the inequality (1.3). In fact, it easily follows that
\[
\|AX + XB\|_2^2 = \|AX\|_2^2 + \|XB\|_2^2 + 2\|A^{1/2}XB^{1/2}\|_2^2
\]
and
\[
\|AX - XB\|_2^2 = \|AX\|_2^2 + \|XB\|_2^2 - 2\|A^{1/2}XB^{1/2}\|_2^2.
\]
As a result, we have
\[
(1 - 2r_0)\|AX + XB\|_2^2 + 8r_0\|A^{1/2}XB^{1/2}\|_2^2 = \|AX + XB\|_2^2 - 2r_0\|AX - XB\|_2^2.
\]
So, from (2.9) we obtain
\[
\|A^vX^{1-v}B^{1-v}X^vB^v\|_2^2 \leq \|AX + XB\|_2^2 - 2r_0\|AX - XB\|_2^2,
\]
which is the inequality (1.3).

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