Journal of
Mathematical
Inequalities
Volume 6, Number 3 (2012), 461–471 doi:10.7153/jmi-06-43

APPROXIMATE FUNCTIONAL INEQUALITIES BY ADDITIVE MAPPINGS

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(*Communicated by J. Pečarić*)

Abstract. Let n be a given positive integer, G an n-divisible abelian group, X a normed space and $f:G\to X$. We prove a generalized Hyers-Ulam stability of the following functional inequality

$$\|f(x)+f(y)+nf(z)\| \leqslant \left\|nf\left(\frac{x+y}{n}+z\right)\right\| + \varphi(x,y,z), \quad \forall x,y,z\in G,$$

which has been introduced in [3], under some conditions on $\varphi : G\times G\times G \to [0,\infty)$.

1. Introduction

The stability problem of equations originated from a question of Ulam [9] concerning the stability of group homomorphisms.

We are given a group G_1 and a metric group G_2 with metric $\rho(\cdot,\cdot)$. Given $\varepsilon>0$, does there exist a $\delta>0$ such that if $f:G_1\to G_2$ satisfies $\rho(f(xy),f(x)f(y))<\delta$ for all $x,y\in G_1$, then a homomorphism $h:G_1\to G_2$ exists with $\rho(f(x),h(x))<\varepsilon$ for all $x\in G_1$?

In 1941, D. H. Hyers [4] considered the case of approximately additive mappings between Banach spaces and proved the following result.

Suppose that E_1 and E_2 are Banach spaces and $f:E_1\to E_2$ satisfies the following condition: there if an $\varepsilon \geqslant 0$ such that

$$\|f(x+y)-f(x)-f(y)\| \leqslant \varepsilon$$

for all $x,y\in E_1$. Then the limit $h(x)=\lim_{n\to\infty}\frac{f(2^n x)}{2^n}$ exists for all $x\in E_1$ and there exists a unique additive mapping $h:E_1\to E_2$ such that

$$\|f(x)-h(x)\| \leqslant \varepsilon.$$

Moreover, if $f(tx)$ is continuous in $t\in\mathbb{R}$ for each $x\in E_1$, then the mapping h is $\mathbb{R}$-linear.

Mathematics subject classification (2010): 39B72, 39B82.
Keywords and phrases: Cauchy Jensen inequality, generalized Hyers–Ulam stability.
This study was supported by the Basic Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (No. 2011–0002614).

Paper JMI-06-43

The method which was provided by Hyers, and which produces the additive mapping h, was called a direct method. This method is the most important and most powerful tool for studying the stability of various functional equations.

In 1978, Th. M. Rassias [5] provided a generalization of Hyers Theorem which allows the Cauchy difference to be unbounded. Let E_1 and E_2 be two Banach space and $f : E_1 \to E_2$ be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed x. Assume that there exists $\varepsilon > 0$ and $0 \leqslant p < 1$ such that

$$\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon(\|x\|^p + \|y\|^p) \tag{1}$$

Then there exists a unique linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leqslant \frac{2}{2-2^p}\varepsilon\|x\|^p$$

for all $x \in E_1$.

In 1990, Th. M. Rassias [6] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geqslant 1$. This result is also true for $p < 0$.

In 1991, Z. Gajda [1] following the same approach as in [5], gave an affirmative solution to this question for $p > 1$. It was shown by Z. Gajda [1], as well as by Th. M. Rassias and P. Šemrl [6], that one cannot prove a Th. M. Rassias type theorem when $p = 1$. The counterexamples of Z. Gajda [1], as well as of Th. M. Rassias and P. Šemrl [6], have stimulated several mathematicians to invent new definitions of approximately additive or approximately linear mappings, cf. P. Găvruta [2] and S. Jung [8], who among others studied the stability of functional equations. In 1994, a generalized result of Rassias' theorem was obtained by P. Găvruta in [2].

Let G be an n-divisible abelian group $n \in \mathbf{N}$ (i.e., $a \mapsto na : G \to G$ is a surjection) and X be a normed space with norm $\|\cdot\|$. Now, for a mapping $f : G \to X$, we consider the following generalized Cauchy-Jensen equation

$$f(x) + f(y) + nf(z) = nf\left(\frac{x+y}{n} + z\right), \quad \forall x,y,z \in G, \quad n \geqq 2$$

which has been introduced in the reference [3]. First of all, we recall some result in the paper [3].

PROPOSITION 1.1. *For a mapping $f : G \to X$, the following statements are equivalent.*

(a) *f is additive.*

(b) $f(x) + f(y) + nf(z) = nf\left(\frac{x+y}{n} + z\right), \quad \forall\, x,y,z \in G.$

(c) $\|f(x) + f(y) + nf(z)\| \leqslant \left\|nf\left(\frac{x+y}{n} + z\right)\right\|, \quad \forall\, x,y,z \in G.$

The generalized Hyers-Ulam stability of functional equation (b) and functional inequality (c) has been presented in the paper [3] for a special case $n = 2$. In this paper, we are going to improve the theorems of [3] without using the oddness of approximate additive functions concerning the functional inequality (c) for the general case.

2. generalized Hyers-Ulam stability of (c)

From now on, let G be an n-divisible abelian group for some positive integer $n \geqq 2$ and $f : G \to Y$ and let Y be a Banach space.

THEOREM 2.1. *Let* $\varphi : G^3 \to \mathbb{R}^+$ *satisfy* $\lim_{k\to\infty} \frac{1}{n^k}\varphi(n^k x, n^k y, n^k z) = 0$ *for all* $x, y, z \in G$ *and*

$$\check{\varphi}(x,z) := \sum_{i=0}^{\infty} \frac{1}{2n^{i+1}} \Big(\varphi(n^{i+1}x, 0, -n^i z) + \varphi(-n^{i+1}x, 0, n^i z)\Big) < \infty,$$

for all $x, z \in G$. *Suppose that a mapping* $f : G \to Y$ *satisfies the functional inequality*

$$\|f(x) + f(y) + nf(z)\| \leqslant \left\| nf\left(\frac{x+y}{n} + z\right)\right\| + \varphi(x,y,z) \tag{2}$$

for all $x, y, z \in G$. *Then there exists a unique additive mapping* $h : G \to Y$ *such that*

$$\|f(x) - h(x)\| \leqslant \check{\varphi}(x,x) + \frac{\varphi(x,-x,0)}{2} + \frac{n^2}{n-1}\|f(0)\| \tag{3}$$

for all $x \in G$.

Proof. For all $x \in G$, letting $y = -x$, $z = 0$ in (2) and dividing both sides by 2, we have

$$\left\|\frac{f(x) + f(-x)}{2}\right\| \leqslant \frac{\varphi(x,-x,0)}{2} + n\|f(0)\| \tag{4}$$

for all $x \in G$. Replacing x by nx and letting $y = 0$ and $z = -x$ in (2), we get

$$\|f(nx) + nf(-x) + f(0)\| \leqslant \varphi(nx, 0, -x) + n\|f(0)\| \tag{5}$$

for all $x \in G$. Replacing x by $-x$ in (5), one has

$$\|f(-nx) + nf(x) + f(0)\| \leqslant \varphi(-nx, 0, x) + n\|f(0)\| \tag{6}$$

for all $x \in G$. Put $g(x) = \frac{f(x)-f(-x)}{2}$. Associating (5) with (6) yields

$$\|ng(x) - g(nx)\| \leqslant \frac{1}{2}(\varphi(nx,0,-x) + \varphi(-nx,0,x)) + n\|f(0)\|$$

that is,

$$\left\|g(x) - \frac{1}{n}g(nx)\right\| \leqslant \frac{1}{2n}(\varphi(nx,0,-x) + \varphi(-nx,0,x)) + \|f(0)\| \tag{7}$$

for all $x \in G$. It follows from (7) that

$$\begin{aligned}
&\left\|\frac{g(n^l x)}{n^l} - \frac{g(n^m x)}{n^m}\right\| \\
&\leqslant \sum_{k=l}^{m-1} \left\|\frac{1}{n^k} g(n^k x) - \frac{1}{n^{k+1}} g(n^{k+1} x)\right\| \\
&= \sum_{k=l}^{m-1} \frac{1}{n^k}\left\|g(n^k x) - \frac{1}{n} g(n^{k+1} x)\right\| \\
&\leqslant \sum_{k=l}^{m-1} \left[\frac{1}{2n^{k+1}}\Big(\varphi(n^{k+1}x, 0, -n^k x) + \varphi(-n^{k+1}x, 0, n^k x)\Big) + \frac{1}{n^k}\|f(0)\|\right]
\end{aligned} \tag{8}$$

for all nonnegative integers m and l with $m > l \geqslant 0$ and $x \in G$. Since the right hand side of (8) tends to zero as $l \to \infty$, we obtain the sequence $\{\frac{g(n^k x)}{n^k}\}$ is Cauchy for all $x \in G$. Because of the fact that Y is a Banach space it follows that the sequence $\{\frac{g(n^k x)}{n^k}\}$ converges in Y. Therefore we can define a function $h: G \to Y$ by

$$h(x) = \lim_{k\to\infty} \frac{g(n^k x)}{n^k}, \qquad x \in G.$$

Moreover, letting $l = 0$ and $m \to \infty$ in (8) yields

$$\|g(x) - h(x)\| \leqslant \check{\varphi}(x,x) + \frac{n}{n-1}\|f(0)\|$$

for all $x \in G$. Hence we have

$$\left\|\frac{f(x) - f(-x)}{2} - h(x)\right\| \leqslant \check{\varphi}(x,x) + \frac{n}{n-1}\|f(0)\| \tag{9}$$

for all $x \in G$. It follows from (4) and (9) that

$$\|f(x) - h(x)\| \leqslant \check{\varphi}(x,x) + \frac{\varphi(x,-x,0)}{2} + \frac{n^2}{n-1}\|f(0)\|$$

for all $x \in G$. It follows from (2) that

$$\begin{aligned}
&\|h(x) + h(y) - h(x+y)\| \\
&= \|h(x) + h(y) + h(-x-y)\| \\
&= \lim_{k\to\infty} \frac{1}{n^k}\|g(n^k x) + g(n^k y) + ng(-n^k(x+y))\| \\
&= \lim_{k\to\infty} \frac{1}{2n^k}\Big(\|f(n^k x) + f(n^k y) + f(-n^k(x+y)) - f(-n^k x) - f(-n^k y) - f(n^k(x+y))\|\Big) \\
&\leqslant \overline{\lim_{k\to\infty}} \frac{1}{2n^k}\Big(\|nf(0)\| + \varphi(n^k x, n^k y, -n^{k-1}(x+y)) + \|nf(0)\| + \varphi(-n^k x, -n^k y, n^{k-1}(x+y)) \\
&\quad + \|nf(n^{k-1}(x+y)) + f(-n^k(x+y))\| + \|nf(-n^{k-1}(x+y)) + f(n^k(x+y))\|\Big) \\
&\leqslant \overline{\lim_{k\to\infty}} \frac{1}{2n^k}\Big(\varphi(n^k x, n^k y, -n^{k-1}(x+y)) + \varphi(-n^k x, -n^k y, n^{k-1}(x+y)) \\
&\quad + \varphi(-n^k(x+y), 0, n^{k-1}(x+y)) + \varphi(n^k(x+y), 0, -n^{k-1}(x+y))\Big) = 0
\end{aligned}$$

for all $x, y \in G$. This implies that

$$h(x)+h(y)=h(x+y)$$

for all $x, y \in G$. Hence the mapping h is additive.

Next, let $h' : G \to Y$ be another additive mapping satisfying

$$\|f(x)-h'(x)\| \leqslant \check{\varphi}(x,x)+\frac{\varphi(x,-x,0)}{2}+\frac{n^2}{n-1}\|f(0)\|$$

for all $x \in G$.

Then we have

$$\begin{aligned}
\|h(x)-h'(x)\| &= \left\|\frac{1}{n^k}h(n^k x)-\frac{1}{n^k}h'(n^k x)\right\| \\
&\leqslant \frac{1}{n^k}(\|h(n^k x)-f(n^k x)\|+\|f(n^k x)-h'(n^k x)\|) \\
&\leqslant \frac{2}{n^k}\left(\check{\varphi}(n^k x,n^k x)+\frac{\varphi(n^k x,-n^k x,0)}{2}\right)+\frac{n^2}{n^k(n-1)}\|f(0)\| \\
&= \sum_{i=k}^{\infty}\frac{2}{n^{i+1}}\left[\varphi(n^{i+1}x,0,-n^i x)+\varphi(-n^{i+1}x,0,n^i x)\right]+\frac{\varphi(n^k x,-n^k x,0)}{n^k}+\frac{n^2}{n^k(n-1)}\|f(0)\|
\end{aligned}$$

for all $k \in \mathbb{N}$ and all $x \in G$. Taking the limit as $k \to \infty$, we conclude that

$$h(x)=h'(x)$$

for all $x \in G$. This completes the proof. $\square$

Suppose that X is a normed space in the following corollaries. If we put $\varphi(x,y,z) := \theta(\|x\|^p\|y\|^q\|z\|^t)$ and $\varphi(x,y,z) := \theta(\|x\|^p+\|y\|^q+\|z\|^t)$ in Theorem 2.1, respectively, then we get the following Corollaries 2.2 and 2.3.

COROLLARY 2.2. *Let $p+q+t<1$, $p,q,t>0$ and $\theta>0$. If a mapping $f : X \to Y$ satisfies the following functional inequality*

$$\|f(x)+f(y)+nf(z)\| \leqslant \left\|nf\left(\frac{x+y}{n}+z\right)\right\|+\theta(\|x\|^p\|y\|^q\|z\|^t)$$

for all $x,y,z \in X$, then f is additive.

COROLLARY 2.3. *Let $0<p,q,t<1$, $\theta>0$. If a mapping $f : X \to Y$ satisfies the following functional inequality*

$$\|f(x)+f(y)+nf(z)\| \leqslant \left\|nf\left(\frac{x+y}{n}+z\right)\right\|+\theta(\|x\|^p+\|y\|^q+\|z\|^t)$$

for all $x,y,z \in X$, then there exists a unique additive mapping $h : X \to Y$ such that

$$\|f(x)-h(x)\| \leqslant \theta\left[\left(\frac{n^p}{n-n^p}+\frac{1}{2}\right)\|x\|^p+\frac{1}{2}\|x\|^q+\left(\frac{1}{n-n^t}\right)\|x\|^t\right]+\frac{n^2}{n-1}\|f(0)\|$$

for all $x \in X$.

The following corollary is an immediate consequence of Theorem 2.1.

COROLLARY 2.4. *For any fixed positive integer* $n \geqslant 2$, *suppose that a mapping* $f: G \to Y$ *satisfies the inequality*

$$\left\| f(x)+f(y)+nf(z)-nf\left(\frac{x+y}{n}+z\right)\right\| \leqslant \varepsilon$$

for all $x,y,z \in G$, *where* $\varepsilon \geqslant 0$. *Then there exists a unique additive mapping* $h: G \to Y$ *satisfying the inequality*

$$\|f(x)-h(x)\| \leqslant \frac{\varepsilon}{n-1}+\frac{\varepsilon}{2}+\frac{n^2}{n-1}\|f(0)\|$$

for all $x \in G$.

We can similarly prove another stability theorem under a somewhat different condition as follows:

REMARK 2.5. Let $\varphi: G^3 \to \mathbb{R}^+$ satisfy

$$\check{\varphi}(x,y,z) := \sum_{i=0}^{\infty} \frac{1}{n^i}\left(\frac{1}{n}\varphi(n^{i+1}x,0,-n^i z)+\varphi(n^i x,-n^i y,0)\right) < \infty,$$

$\lim_{k\to\infty} \frac{1}{n^k}\varphi(n^k x,n^k y,n^k z) = 0$ for all $x,y,z \in G$. If $f: G \to Y$ is a mapping such that

$$\|f(x)+f(y)+nf(z)\| \leqslant \left\| nf\left(\frac{x+y}{n}+z\right)\right\| + \varphi(x,y,z)$$

for all $x,y,z \in G$, then there exists a unique additive mapping $h: G \to Y$ such that

$$\|f(x)-h(x)\| \leqslant \check{\varphi}(x,x,x)+\frac{2n^2+n-1}{n-1}\|f(0)\|$$

for all $x \in G$.

Proof. Replacing x by nx and letting $y=0$ and $z=-x$ in (2), we get

$$\|f(nx)+nf(-x)+f(0)\| \leqslant \varphi(nx,0,-x)+n\|f(0)\| \tag{10}$$

for all $x \in G$. For all $x \in G$, letting $y=-x$, $z=0$ in (2), we have

$$\|f(x)+f(-x)+nf(0)\| \leqslant \varphi(x,-x,0)+n\|f(0)\| \tag{11}$$

for all $x \in G$. Then we obtain the following inequality

$$\|f(nx)-nf(x)\| \leqslant \varphi(nx,0,-x)+n\varphi(x,-x,0)+(2n^2+n-1)\|f(0)\| \tag{12}$$

for all $x \in G$. The rest of the proof is similar to proof of Theorem 2.1. □

We may obtain more simple and sharp approximation than that of Theorem 2.1 for the stability result under the oddness condition.

COROLLARY 2.6. *Let* $\varphi : G^3 \to \mathbb{R}^+$ *satisfy* $\lim_{k\to\infty} \frac{1}{n^k}\varphi(n^k x, n^k y, n^k z) = 0$ *for all* $x,y,z \in G$ *and*

$$\check{\varphi}(x,z) := \sum_{i=0}^{\infty} \frac{1}{n^{i+1}} \varphi(n^{i+1}x, 0, -n^i z) < \infty,$$

for all $x,z \in G$*. Suppose that* $f : G \to Y$ *is a mapping such that* $f(-x) = -f(x)$ *for all* $x \in G$ *and*

$$\|f(x)+f(y)+nf(z)\| \leqslant \left\| nf\left(\frac{x+y}{n}+z\right)\right\| + \varphi(x,y,z)$$

for all $x,y,z \in G$*, then there exists a unique additive mapping* $h : G \to Y$ *such that*

$$\|f(x)-h(x)\| \leqslant \check{\varphi}(x,x)$$

for all $x \in G$*.*

We may obtain a stability result of functional equation (b) by the similar way as in the proof of Theorem 2.1.

COROLLARY 2.7. *Let* $\varphi : G^3 \to \mathbb{R}^+$ *satisfy*

$$\check{\varphi}(x,z) := \sum_{i=0}^{\infty} \frac{1}{2n^{i+1}} \Big(\varphi(n^{i+1}x, 0, -n^i z) + \varphi(-n^{i+1}x, 0, n^i z)\Big) < \infty,$$

for all $x,z \in G$ *and* $\lim_{k\to\infty} \frac{1}{n^k}\varphi(n^k x, n^k y, n^k z) = 0$ *for all* $x,y,z \in G$*. If* $f : G \to Y$ *is a mapping such that* $f(0) = 0$ *and*

$$\left\| f(x)+f(y)+nf(z) - nf\left(\frac{x+y}{n}+z\right)\right\| \leqslant \varphi(x,y,z)$$

for all $x,y,z \in G$*, then there exists a unique additive mapping* $h : G \to Y$ *such that*

$$\|f(x)-h(x)\| \leqslant \check{\varphi}(x,x) + \frac{\varphi(x,-x,0)}{2}$$

for all $x \in G$*.*

Now, we consider another stability result of functional inequality (c) in the followings.

THEOREM 2.8. *Let* $\varphi : G^3 \to \mathbb{R}^+$ *satisfy*

$$\tilde{\varphi}(x,z) := \frac{1}{2}\sum_{i=0}^{\infty} n^i \left(\varphi\left(\frac{x}{n^i}, 0, -\frac{z}{n^{i+1}}\right) + \varphi\left(-\frac{x}{n^i}, 0, \frac{z}{n^{i+1}}\right)\right) < \infty,$$

for all $x,z \in G$ *and* $\lim_{k\to\infty} n^k \varphi(\frac{x}{n^k}, \frac{y}{n^k}, \frac{z}{n^k}) = 0$ *for all* $x,y,z \in G$*. If* $f : G \to Y$ *is a mapping such that*

$$\|f(x)+f(y)+nf(z)\| \leqslant \left\| nf\left(\frac{x+y}{n}+z\right)\right\| + \varphi(x,y,z) \tag{13}$$

for all $x,y,z \in G$. Then there exists a unique additive mapping $h: G \to Y$ such that

$$\|f(x)-h(x)\| \leqslant \tilde{\varphi}(x,x)+\frac{\varphi(x,-x,0)}{2} \tag{14}$$

for all $x \in G$.

Proof. Note that $f(0)=0$ since $\varphi(0,0,0)=0$. Let $y=-x$, $z=0$ in (13) and dividing both sides by 2, we have

$$\left\|\frac{f(x)+f(-x)}{2}\right\| \leqslant \frac{\varphi(x,-x,0)}{2} \tag{15}$$

for all $x \in G$. Replacing x by nx and letting $y=0$ and $z=-x$ in (13), we get

$$\|f(nx)+nf(-x)\| \leqslant \varphi(nx,0,-x) \tag{16}$$

for all $x \in G$. Replacing x by $-x$ in (16), we get

$$\|f(-nx)+nf(x)\| \leqslant \varphi(-nx,0,x) \tag{17}$$

for all $x \in G$. Put $g(x)=\frac{f(x)-f(-x)}{2}$. Using (16) and (17) yields the functional inequality

$$\|ng(x)-g(nx)\| \leqslant \frac{1}{2}(\varphi(nx,0,-x)+\varphi(-nx,0,x))$$

for all $x \in G$. Replacing x by $\frac{x}{n}$, we get

$$\left\|g(x)-ng\left(\frac{x}{n}\right)\right\| \leqslant \frac{1}{2}\left(\varphi\left(x,0,-\frac{x}{n}\right)+\varphi\left(-x,0,\frac{x}{n}\right)\right) \tag{18}$$

for all $x \in G$. The remaining roof is similar to the corresponding proof of Theorem 2.1. This completes the proof. □

Suppose that X is a normed space in the following corollaries. If we put $\varphi(x,y,z) := \theta(\|x\|^p\|y\|^q\|z\|^t)$ and $\varphi(x,y,z) := \theta(\|x\|^p+\|y\|^q+\|z\|^t)$ in Theorem 2.8, respectively, then we get the following Corollaries 2.9 and 2.10.

COROLLARY 2.9. *Let $p+q+t>1$, $p,q,t>0$, $\theta>0$. If a mapping $f: X \to Y$ satisfies the following functional inequality*

$$\|f(x)+f(y)+nf(z)\| \leqslant \left\|nf\left(\frac{x+y}{n}+z\right)\right\|+\theta(\|x\|^p\|y\|^q\|z\|^t)$$

for all $x,y,z \in X$, then f is additive.

COROLLARY 2.10. *Let $p,q,t > 1$, $\theta > 0$. If a mapping $f : X \to Y$ satisfies the following functional inequality*

$$\|f(x)+f(y)+nf(z)\| \leqslant \left\| nf\left(\frac{x+y}{n}+z\right)\right\| + \theta(\|x\|^p + \|y\|^q + \|z\|^t)$$

for all $x,y,z \in X$, then there exists a unique additive mapping $h : X \to Y$ such that

$$\|f(x)-h(x)\| \leqslant \theta\Big[\Big(\frac{n^p}{n^p-n}+\frac{1}{2}\Big)\|x\|^p + \frac{1}{2}\|x\|^q + \Big(\frac{1}{n^t-n}\Big)\|x\|^t\Big]$$

for all $x \in X$.

We can similarly prove another stability theorem under somewhat different conditions as follows:

REMARK 2.11. Let $\varphi : G^3 \to \mathbb{R}^+$ satisfy

$$\tilde{\varphi}(x,y,z) := \sum_{i=0}^{\infty} n^i \Big(\varphi\left(\frac{x}{n^i},0,-\frac{z}{n^{i+1}}\right) + n\varphi\left(\frac{x}{n^{i+1}},-\frac{y}{n^{i+1}},0\right)\Big) < \infty,$$

and $\lim_{k\to\infty} n^k \varphi\left(\frac{x}{n^k},\frac{y}{n^k},\frac{z}{n^k}\right) = 0$ for all $x,y,z \in G$. If $f : G \to Y$ is a mapping such that

$$\|f(x)+f(y)+nf(z)\| \leqslant \left\| nf\left(\frac{x+y}{n}+z\right)\right\| + \varphi(x,y,z)$$

for all $x,y,z \in G$, then there exists a unique additive mapping $h : G \to Y$ such that

$$\|f(x)-h(x)\| \leqslant \tilde{\varphi}(x,x,x)$$

for all $x \in G$.

We may obtain more simple and sharp approximation than that of Theorem 2.8 for the stability result under the oddness condition.

REMARK 2.12. Let $\varphi : G^3 \to \mathbb{R}^+$ satisfy

$$\tilde{\varphi}(x,z) := \sum_{i=0}^{\infty} n^i \varphi\left(\frac{x}{n^i},0,-\frac{z}{n^{i+1}}\right) < \infty,$$

for all $x,z \in G$ and $\lim_{k\to\infty} n^k \varphi\left(\frac{x}{n^k},\frac{y}{n^k},\frac{z}{n^k}\right) = 0$ for all $x,y,z \in G$. If a mapping $f : G \to Y$ is odd and

$$\|f(x)+f(y)+nf(z)\| \leqslant \left\| nf\left(\frac{x+y}{n}+z\right)\right\| + \varphi(x,y,z)$$

for all $x,y,z \in G$, then there exists a unique additive mapping $h : G \to Y$ such that

$$\|f(x)-h(x)\| \leqslant \widetilde{\varphi}(x,x)$$

for all $x \in G$.

We may alternatively obtain a stability result of functional equation (b) by the similar way as in the proof of Theorem 2.8.

COROLLARY 2.13. *Let* $\varphi : G^3 \to \mathbb{R}^+$ *satisfy*

$$\tilde{\varphi}(x,z) := \frac{1}{2}\sum_{i=0}^{\infty} n^i\left(\varphi\left(\frac{x}{n^i},0,-\frac{z}{n^{i+1}}\right)+\varphi\left(-\frac{x}{n^i},0,\frac{z}{n^{i+1}}\right)\right) < \infty,$$

for all $x,z \in G$ *and* $\lim_{k\to\infty} n^k\varphi(\frac{x}{n^k},\frac{y}{n^k},\frac{z}{n^k}) = 0$ *for all* $x,y,z \in G$. *If* $f : G \to Y$ *is a mapping such that*

$$\left\| f(x)+f(y)+nf(z)-nf\left(\frac{x+y}{n}+z\right)\right\| \leqslant \varphi(x,y,z)$$

for all $x,y,z \in G$, *then there exists a unique additive mapping* $h : G \to Y$ *such that*

$$\|f(x)-h(x)\| \leqslant \tilde{\varphi}(x,x)+\frac{\varphi(x,-x,0)}{2}$$

for all $x \in G$.

Acknowledgement. The authors are very grateful for the referees valuable comment. Correspondence should be addressed to Eunyoung Son.

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(Received October 26, 2011)

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