THE JENSEN INEQUALITY IN AN EXTERNAL FORMULA

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Abstract. The classical Jensen inequality is expressed by internally dividing points and so is non-commutative Jensen inequalities. In this paper, considering that it is expressed by externally dividing points, we shall discuss two non-commutative Jensen inequalities and their reverse, that is, one is a vector state version and the other is a Davis-Choi-Jensen type version.

1. Introduction

The classical Jensen inequality is expressed by internally dividing points: If a real valued continuous function $f$ on an interval $J$ is concave, then

$$
\sum_{i=1}^{n} \alpha_i f(x_i) \leq f \left( \sum_{i=1}^{n} \alpha_i x_i \right)
$$

(1.1)

for all $x_i \in J$ and all $\alpha_i \geq 0$ such that $\sum_{i=1}^{n} \alpha_i = 1$. Mond-Pečarić [8] showed the following vector state version of (1.1): If $A$ is a selfadjoint operator on a Hilbert space $H$, then

$$
(f(A)x, x) \leq f((Ax, x))
$$

(1.2)

for every unit vector $x \in H$. Also, we can reform (1.2) a two variable version as follows:

$$
(f(A)x, x) + (f(B)y, y) \leq f((Ax, x) + (By, y))
$$

(1.3)

for all vectors $x$ and $y$ in $H$ such that $\|x\|^2 + \|y\|^2 = 1$, also see [4, Theorem 1.3].

On the other hand, a real valued continuous function $f$ on $J$ is said to be operator concave if

$$
(1 - t)f(A) + tf(B) \leq f((1 - t)A + tB)
$$

for all selfadjoint operators $A$ and $B$ with the spectra in $J$ and $t \in [0, 1]$. As a characterization of operator concavity, we have the following Davis-Choi-Jensen operator


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inequality [2, 1]: If \( \Phi \) is a normalized positive linear map and \( f \) is operator concave, then
\[
\Phi(f(A)) \leq f(\Phi(A))
\] (1.4)
for all selfadjoint operators \( A \) with the spectra in \( J \). Also, we can reform (1.4) a two variable version as follows:
\[
\Phi(f(A)) + \Psi(f(B)) \leq f(\Phi(A) + \Psi(B))
\] (1.5)
for all positive linear maps \( \Phi, \Psi \) such that \( \Phi(I) + \Psi(I) = I \) and for all selfadjoint operators \( A, B \) with the spectra in \( J \).

In [3], J.I.Fujii pointed out that the concavity is also expressed by externally dividing points: A real valued function \( f \) on an interval \( J \) is concave if and only if
\[
f((1 + r)x - ry) \leq (1 + r)f(x) - rf(y)
\]
for all \( x, y \in J \) and \( r > 0 \) with \( (1 + r)x - ry \in J \). Thus, an external version of the classical Jensen inequality is as follows: A real valued function \( f \) on \( J \) is concave if and only if
\[
f\left(\sum_{i=1}^{n} \alpha_i x_i - \sum_{j=1}^{k} \beta_j y_j\right) \leq \left(\sum_{i=1}^{n} \alpha_i\right)f\left(\frac{1}{\sum_{i=1}^{n} \alpha_i} \sum_{i=1}^{n} \alpha_i x_i\right) - \sum_{j=1}^{k} \beta_j f(y_j)
\] (1.6)
for \( x_1, \cdots, x_n, y_1, \cdots, y_k \in J \) and \( \alpha_i, \beta_j \geq 0 \) such that \( \sum_{i=1}^{n} \alpha_i - \sum_{j=1}^{k} \beta_j = 1 \), and \( \sum_{i=1}^{n} \alpha_i x_i - \sum_{j=1}^{k} \beta_j y_j \in J \), also see [10, p83]. Moreover, he showed the following characterization of operator concavity in terms of an external formula: A real valued continuous function \( f \) on \( J \) is operator concave if and only if
\[
f((1 + r)A - rB) \leq (1 + r)f(A) - rf(B)
\]
for all \( r > 0 \) and all selfadjoint operators \( A \) and \( B \) with \( \sigma(A), \sigma(B) \) and \( \sigma((1 + r)A - rB) \subset J \). Then we have the following external version of the Jensen operator inequality: If \( f \) is operator concave, then
\[
f\left(\sum_{k=1}^{n} r_k A - \sum_{k=1}^{n} r_k B_k\right) \leq \left(\sum_{k=1}^{n} r_k\right)f(A) - \sum_{k=1}^{n} r_k f(B_k)
\]
for all selfadjoint operators \( A \) and \( B_k \) \((k = 1, \cdots, n)\) with \( \sigma(A), \sigma(B_k) \) and \( \sigma((1 + \sum_{k=1}^{n} r_k)A - \sum_{k=1}^{n} r_k B_k) \subset J \), also see [9].

In this paper, considering that the Jensen inequality is also expressed by externally dividing points, we shall discuss two non-commutative Jensen inequalities and their reverse, that is, one is a vector state version (1.2) and the other is a Davis-Choi-Jensen type version (1.4).
2. Vector state version

First of all, we show an external version of the Jensen inequality which corresponds to a two variable vector state version (1.3):

THEOREM 2.1. Let $f$ be a real valued function on an interval $J$. Then $f$ is concave if and only if

$$f((Ax,x)-(By,y)) \leq \|x\|^2 f \left( \left( A \frac{x}{\|x\|}, \frac{x}{\|x\|} \right) \right) - (f(B)y,y)$$  \hspace{1cm} (2.1)

for all $x,y \in H$ such that $\|x\|^2 - \|y\|^2 = 1$ and for all selfadjoint operators $A$ and $B$ with the spectra in $J$ such that $(Ax,x)-(By,y) \in J$.

Proof. Suppose that $f$ is concave. Since $\|x\|^2 = \|y\|^2 + 1$, it follows that

$$\|x\|^2 f \left( \left( A \frac{x}{\|x\|}, \frac{x}{\|x\|} \right) \right)$$

$$\|x\|^2 f \left( \frac{1}{1+\|y\|^2}(Ax,x)-(By,y)) + \frac{\|y\|^2}{1+\|y\|^2} \left( \frac{y}{\|y\|}, \frac{y}{\|y\|} \right) \right)$$

$$\geq \|x\|^2 \left( \frac{1}{1+\|y\|^2}f((Ax,x)-(By,y)) + \frac{\|y\|^2}{1+\|y\|^2} f \left( \left( B \frac{y}{\|y\|}, \frac{y}{\|y\|} \right) \right) \right) \hspace{1cm} \text{by (1.1)}$$

$$= f((Ax,x)-(By,y)) + \|y\|^2 f \left( \left( B \frac{y}{\|y\|}, \frac{y}{\|y\|} \right) \right)$$

$$\geq f((Ax,x)-(By,y)) + (f(B)y,y) \hspace{1cm} \text{by (1.2)}.$$ 

Conversely, put $A = \text{diag}(x_1,\ldots,x_n)$ and $B = \text{diag}(y_1,\ldots,y_k)$ and $x = (\sqrt{\alpha_1},\ldots,\sqrt{\alpha_n})$ and $y = (\sqrt{\beta_1},\ldots,\sqrt{\beta_k})$ in (2.1). Then we have (1.6):

$$f \left( \sum_{i=1}^{n} \alpha_i x_i - \sum_{j=1}^{k} \beta_j y_j \right) \leq \sum_{i=1}^{n} \alpha_i f \left( \frac{1}{\sum_{i=1}^{n} \alpha_i} \sum_{i=1}^{n} \alpha_i x_i \right) - \sum_{j=1}^{k} \beta_j f(y_j)$$

and hence $f$ is concave.

Next, we consider the reverse Jensen inequality in an external formula. For this, we need the following result [8]:

LEMMA 2.2. Let $A$ be a positive operator on a Hilbert space $H$ such that $M \leq A \leq MI$ for some scalars $M > m > 0$. Let $f$ be a real valued continuous function on $[m,M]$ and $f(t) > 0$ for all $t \in [m,M]$. If $f$ is concave, then

$$K(m,M,f)((Ax,x)) \leq (f(A)x,x)$$ \hspace{1cm} (2.2)

for every unit vector $x \in H$, where the generalized Kantorovich constnat $K(m,M,f)$ is defined by

$$K(m,M,f) = \min \left\{ \frac{1}{f(t)} \left( \frac{f(M) - f(m)}{M-m} (t-m) + f(m) \right) : t \in [m,M] \right\} . \hspace{1cm} (2.3)$$
By Lemma 2.2, we obtain the following estimate in an external formula:

**Theorem 2.3.** Let $f$ be a real valued continuous function on $[m, M]$ and $f(t) > 0$ for all $t \in [m, M]$. If $f$ is concave, then

$$K(m, M, f)(f(A)x, x) - K(m, M, f)^{-1}(f(B)y, y) \leq f((Ax, x) - (By, y))$$

for all $x, y \in H$ such that $\|x\|^2 - \|y\|^2 = 1$ and for all selfadjoint operators $A$ and $B$ with the spectra in $J$ such that $(Ax, x) - (By, y) \in J$, where $K(m, M, f)$ is defined as (2.3).

**Proof.** For all $x, y \in H$ such that $\|x\|^2 - \|y\|^2 = 1$, it follows from Lemma 2.2 that

$$(f(A)x, x) \leq \|x\|^2 f(A \frac{x}{\|x\|}, \frac{x}{\|x\|}) \quad \text{by (1.2)}$$

$$= \|x\|^2 f\left(1 + \frac{1}{\|x\|^2}((Ax, x) - (By, y)) + \frac{y}{\|y\|^2} \left(B \frac{y}{\|y\|}, \frac{y}{\|y\|}\right)\right)$$

$$\leq \|x\|^2 K(m, M, f)^{-1}\left(1 + \frac{1}{\|x\|^2}f((Ax, x) - (By, y)) + \frac{y}{\|y\|^2} f\left(B \frac{y}{\|y\|}, \frac{y}{\|y\|}\right)\right)$$

$$= K(m, M, f)^{-1} f((Ax, x) - (By, y)) + \|y\|^2 f\left(B \frac{y}{\|y\|}, \frac{y}{\|y\|}\right)$$

and hence

$$K(m, M, f)(f(A)x, x) \leq f((Ax, x) - (By, y)) + \|y\|^2 f\left(B \frac{y}{\|y\|}, \frac{y}{\|y\|}\right)$$

$$\leq f((Ax, x) - (By, y)) + K(m, M, f)^{-1}(f(B)y, y).$$

On the other hand, by Theorem 2.1 and Lemma 2.2 again, we have

$$f((Ax, x) - (By, y)) \leq \|x\|^2 f\left(A \frac{x}{\|x\|}, \frac{x}{\|x\|}\right) - (f(B)y, y)$$

$$\leq K(m, M, f)^{-1}(f(A)x, x) - (f(B)y, y).$$

**Remark 2.4.** Put the power function $f(t) = t^p$ for $0 < p < 1$ in Theorem 2.3, then we exactly have the evaluation

$$K(m, M, f) = K(m, M, t^p) = \frac{mm^p - Mm^p}{(p - 1)(M - m)} \left(\frac{(p - 1)(M^p - m^p)}{p(mM^p - Mm^p)}\right)^p,$$
see [4], and hence

\[
\frac{mM^p - Mm^p}{(p-1)(M-m)} \left( \frac{(p-1)(M^p - m^p)}{p(mM^p - Mm^p)} \right)^p (A^p x, x) \\
- \frac{(p-1)(M-m)}{mM^p - Mm^p} \left( \frac{p(mM^p - Mm^p)}{(p-1)(M^p - m^p)} \right)^p (B^p y, y) \\
\leq ((Ax, x) - (By, y))^p \leq \frac{(p-1)(M-m)}{mM^p - Mm^p} \left( \frac{p(mM^p - Mm^p)}{(p-1)(M^p - m^p)} \right)^p (A^p x, x) - (B^p y, y)
\]

for all vectors \(x, y \in H\) such that \(\|x\|^2 - \|y\|^2 = 1\) and all selfadjoint operators \(A\) and \(B\) such that \((Ax, x) - (By, y) \in [m, M]\).

3. Davis-Choi-Jensen inequality

In this section, we denote \(P[B(H), B(K)]\) as the set of all positive linear maps \(\Phi : B(H) \to B(K)\), where \(B(H)\) is a \(C^*\)-algebra of all bounded linear operators on a Hilbert space \(H\).

The following theorem is an external version of the Davis-Choi-Jensen inequality for operator concave functions which corresponds to (1.5):

**Theorem 3.1.** Let \(f\) be a real valued continuous function on an interval \(J\). Then \(f\) is operator concave if and only if

\[
f(\Phi(A) - \Psi(B)) \leq \Phi(I)^\frac{1}{2} f \left( \Phi(I)^{-\frac{1}{2}} \Phi(A) \Phi(I)^{-\frac{1}{2}} \right) \Phi(I)^\frac{1}{2} - \Psi(f(B)) \quad (3.1)
\]

for all \(\Phi, \Psi \in P[B(H), B(K)]\) such that \(\Phi(I) - \Psi(I) = I\) and for all selfadjoint operators \(A\) and \(B\) with \(\sigma(A), \sigma(B)\) and \(\sigma(\Phi(A) - \Psi(B)) \subset J\).

**Proof.** (i) \(\implies\) (ii): By Stinespring decomposition theorem [11], \(\Phi\) restricted to a \(C^*\)-algebra \(C^*(A)\) generated by \(A\) and \(I\) admits a decomposition \(\Phi(X) = C^* \phi(X) C\) for all \(X \in C^*(A)\), where \(\phi\) is a \(*\)-representation of \(C^*(A) \subset B(H)\) and \(C\) is a bounded linear operator from \(K\) to a Hilbert space \(K'\). Similarly we have a decomposition \(\Psi(Y) = D^* \psi(Y) D\) for all \(Y \in C^*(B)\), where \(\psi\) are a \(*\)-representation and \(D\) is a bounded linear operator from \(K\) to a Hilbert space \(K''\). The assumption \(\Phi(I) - \Psi(I) = I\) implies \(C^* C - D^* D = I\) and hence \(|C|\) is invertible. Since \(|C|^{-2} + (D|C|^{-1})^* (D|C|^{-1}) = I\), it follows that

\[
\Phi(I)^\frac{1}{2} f \left( \Phi(I)^{-\frac{1}{2}} \Phi(A) \Phi(I)^{-\frac{1}{2}} \right) \Phi(I)^\frac{1}{2} \\
= |C| f(|C|^{-1} C^* \phi(A) C |C|^{-1}) |C| \\
= |C| f \left( |C|^{-1} (C^* \phi(A) C - D^* \psi(B) D) |C|^{-1} + (D|C|^{-1})^* \psi(B) D |C|^{-1} \right) |C| \\
\geq |C| \left( |C|^{-1} f(\Phi(A) - \Psi(B)) |C|^{-1} + (D|C|^{-1})^* f(\psi(B)) D |C|^{-1} \right) |C| \quad \text{by (1.5)} \\
= f(\Phi(A) - \Psi(B)) + D^* \psi(f(B)) D \\
= f(\Phi(A) - \Psi(B)) + \Psi(f(B)).
\]
(ii) $\implies$ (i): Put $\Phi(A) = C^*AC$ and $\Psi(B) = D^*BD$ for $C^*C - D^*D = I$ in (ii), then the operator concavity of $f$ follows from [3, Theorem 1].

**Remark 3.2.** If $C$ is invertible in a decomposition $\Phi(A) = C^*\phi(A)C$, then

$$f(\Phi(A) - \Psi(B)) \leq \Phi(f(A)) - \Psi(f(B)).$$

In fact, if $C$ is invertible, then $V$ is unitary in the polar decomposition $C = V|C|$ and hence

$$\Phi(I)^{\frac{1}{2}} f \left( \Phi(I)^{-\frac{1}{2}} \Phi(A) \Phi(I)^{-\frac{1}{2}} \right) \Phi(I)^{\frac{1}{2}} = |C| f(|C|^{-\frac{1}{2}} C^* \phi(A) C |C|^{-1}) |C|$$

$$= |C| f(V^* \phi(A) V) |C| = |C| V^* f(\phi(A)) V |C|$$

$$= C^* \phi(f(A)) C = \Phi(f(A)).$$

By virtue of the generalized Kantorovich constant, we consider the difference between the concavity and the operator concavity, based on an external version of the Jensen inequality:

**Theorem 3.3.** Let $f$ be a real valued continuous function on $[m, M]$ and $f(t) > 0$ for all $t \in [m, M]$. If $f$ is concave, then

$$K(m, M, f) \Phi(I)^{\frac{1}{2}} f \left( \Phi(I)^{-\frac{1}{2}} \Phi(A) \Phi(I)^{-\frac{1}{2}} \right) \Phi(I)^{\frac{1}{2}} - \Psi(f(B))$$

$$\leq f(\Phi(A) - \Psi(B))$$

$$\leq K(m, M, f)^{-1} \Phi(I)^{\frac{1}{2}} f \left( \Phi(I)^{-\frac{1}{2}} \Phi(A) \Phi(I)^{-\frac{1}{2}} \right) \Phi(I)^{\frac{1}{2}} - \Psi(f(B))$$

for all $\Phi, \Psi \in P[B(H), B(K)]$ such that $\Phi(I) - \Psi(I) = I$ and for all selfadjoint operators $A$ and $B$ with the spectra in $J$ such that $\sigma(\Phi(A) - \Psi(B)) \subset J$, where $K(m, M, f)$ is defined as (2.3).

To prove it, we need the following lemma [4, Corollary 3.21]:

**Lemma 3.4.** Let $f$ be a real valued continuous function on $[m, M]$ and $f(t) > 0$ for all $t \in [m, M]$. If $f$ is concave, then

$$K(m, M, f) f(U^*AU + W^*BW) \leq U^* f(A) U + W^* f(B) W$$

$$\leq K(m, M, f)^{-1} f(U^*AU + W^*BW)$$

for all selfadjoint operators $A, B$ with the spectra in $[m, M]$ and $U^*U + W^*W = I$, where $K(m, M, f)$ is defined as (2.3).

**Proof.** We give a proof for reader’s convenience. For any unit vector $x \in H$, we have $\|Ux\|^2 + \|Wx\|^2 = 1$ and hence

$$((U^* f(A) U + W^* f(B) W) x, x) = (f(A) Ux, Ux) + (f(B) Wx, Wx)$$

$$\leq f((AUx, Ux) + (BWx, Wx))$$

by (1.3)

$$= f(((U^*AU + W^*BW)x, x))$$

$$\leq K(m, M, f)^{-1} (f(U^*AU + W^*BW)x, x)$$

by Lemma 2.2
and this implies
\[ U^* f(A) U + W^* f(B) W \leq K(m, M, f)^{-1} f(U^* A U + W^* B W). \]

Similarly,
\[ (f(U^* A U + W^* B W)_{x,x}) \leq f(((U^* A U + W^* B W)_{x,x})) \text{ by (1.2)} \]
\[ = f((A U x, U x) + (B W x, W x)) \]
\[ \leq K(m, M, f)^{-1} ((f(A) U x, U x) + (f(B) W x, W x)) \]
\[ = K(m, M, f)^{-1} ((U^* f(A) U + W^* f(B) W)_{x,x}) \]

and this implies
\[ K(m, M, f) f(U^* A U + W^* B W) \leq U^* f(A) U + W^* f(B) W. \]

□

Proof of Theorem 3.3. As in the proof of Theorem 3.1, let \( \Phi(A) = C^* \phi(A) C \) and \( \Psi(B) = D^* \psi(B) D \) be the Stinespring decomposition. Then \( \Phi(I) - \Psi(I) = I \) implies \( C^* C - D^* D = I \) and so \( |C|^{-2} + (D |C|^{-1})^* (D |C|^{-1}) = I \). By Lemma 3.4, it follows that
\[
K(m, M, f) \Phi(I)^{1/2} f \left( \Phi(I)^{-1/2} \Phi(A) \Phi(I)^{-1/2} \right) \Phi(I)^{1/2} = K(m, M, f) \frac{1}{|C|} f \left( |C|^{-1} \Phi(A) |C|^{-1} \right) |C| \leq C \left( |C|^{-1} f(\Phi(A) - \Psi(B)) |C|^{-1} + (D |C|^{-1})^* \psi(B) D |C|^{-1} \right) |C| = f(\Phi(A) - \Psi(B)) + \Psi(f(B))
\]

and also
\[
f(\Phi(A) - \Psi(B)) + \Psi(f(B)) = |C| \left( |C|^{-1} f(\Phi(A) - \Psi(B)) |C|^{-1} + (D |C|^{-1})^* \psi(B) D |C|^{-1} \right) |C| \leq K(m, M, f)^{-1} \frac{1}{|C|} f \left( |C|^{-1} (\Phi(A) - \Psi(B)) |C|^{-1} + (D |C|^{-1})^* \psi(B) D |C|^{-1} \right) = K(m, M, f)^{-1} \Phi(I)^{1/2} f \left( \Phi(I)^{-1/2} \Phi(A) \Phi(I)^{-1/2} \right) \Phi(I)^{1/2}
\]

and this implies Theorem 3.3. □

Remark 3.5. If we put \( \Phi(A) = (Ax, x) \) and \( \Psi(B) = (By, y) \) for \( x, y \in H \) such that \( \| x \|^2 - \| y \|^2 = 1 \) in Theorem 3.3, then Theorem 3.3 does not implies Theorem 2.3, because \( f((Ax, x)) = \| x \|^2 f((A \frac{x}{\|x\|}, \frac{x}{\|x\|})) \) does not always hold.
REFERENCES


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