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THE JENSEN INEQUALITY IN AN EXTERNAL FORMULA

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Abstract. The classical Jensen inequality is expressed by internally dividing points and so is non-commutative Jensen inequalities. In this paper, considering that it is expressed by externally dividing points, we shall discuss two non-commutative Jensen inequalities and their reverse, that is, one is a vector state version and the other is a Davis-Choi-Jensen type version.

1. Introduction

The classical Jensen inequality is expressed by internally dividing points: If a real valued continuous function f on an interval J is concave, then

$$\sum_{i=1}^{n} \alpha_i f(x_i) \leqslant f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \tag{1.1}$$

for all $x_i \in J$ and all $\alpha_i \geqslant 0$ such that $\sum_{i=1}^{n} \alpha_i = 1$. Mond-Pečarić [8] showed the following vector state version of (1.1): If A is a selfadjoint operator on a Hilbert space H, then

$$(f(A)x,x) \leqslant f((Ax,x)) \tag{1.2}$$

for every unit vector $x \in H$. Also, we can reform (1.2) a two variable version as follows:

$$(f(A)x,x) + (f(B)y,y) \leqslant f((Ax,x) + (By,y)) \tag{1.3}$$

for all vectors x and y in H such that $\| x \|^2 + \| y \|^2 = 1$, also see [4, Theorem 1.3].

On the other hand, a real valued continuous function f on J is said to be operator concave if

$$(1-t)f(A) + tf(B) \leqslant f((1-t)A + tB)$$

for all selfadjoint operators A and B with the spectra in J and $t \in [0,1]$. As a characterization of operator concavity, we have the following Davis-Choi-Jensen operator

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inequality [2, 1]: If Φ is a normalized positive linear map and f is operator concave, then

$$\Phi(f(A)) \leqslant f(\Phi(A)) \tag{1.4}$$

for all selfadjoint operators A with the spectra in J. Also, we can reform (1.4) a two variable version as follows:

$$\Phi(f(A)) + \Psi(f(B)) \leqslant f(\Phi(A) + \Psi(B)) \tag{1.5}$$

for all positive linear maps Φ, Ψ such that $\Phi(I) + \Psi(I) = I$ and for all selfadjoint operators A, B with the spectra in J.

In [3], J.I.Fujii pointed out that the concavity is also expressed by externally dividing points: A real valued function f on an interval J is concave if and only if

$$f((1+r)x - ry) \leqslant (1+r)f(x) - rf(y)$$

for all $x, y \in J$ and $r > 0$ with $(1+r)x - ry \in J$. Thus, an external version of the classical Jensen inequality is as follows: A real valued function f on J is concave if and only if

$$f\left(\sum_{i=1}^{n} \alpha_i x_i - \sum_{j=1}^{k} \beta_j y_j\right) \leqslant (\sum_{i=1}^{n} \alpha_i) f\left(\frac{1}{\sum_{i=1}^{n} \alpha_i} \sum_{i=1}^{n} \alpha_i x_i\right) - \sum_{j=1}^{k} \beta_j f(y_j) \tag{1.6}$$

for $x_1, \cdots, x_n, y_1, \cdots, y_k \in J$ and $\alpha_i, \beta_j \geqslant 0$ such that $\sum_{i=1}^{n} \alpha_i - \sum_{j=1}^{k} \beta_j = 1$, and $\sum_{i=1}^{n} \alpha_i x_i - \sum_{j=1}^{k} \beta_j y_j \in J$, also see [10, p83]. Moreover, he showed the following characterization of operator concavity in terms of an external formula: A real valued continuous function f on J is operator concave if and only if

$$f((1+r)A - rB) \leqslant (1+r)f(A) - rf(B)$$

for all $r > 0$ and all selfadjoint operators A and B with $\sigma(A), \sigma(B)$ and $\sigma((1+r)A - rB) \subset J$. Then we have the following external version of the Jensen operator inequality: If f is operator concave, then

$$f\left((1 + \sum_{k=1}^{n} r_k)A - \sum_{k=1}^{n} r_k B_k\right) \leqslant (1 + \sum_{k=1}^{n} r_k) f(A) - \sum_{k=1}^{n} r_k f(B_k)$$

for all selfadjoint operators A and B_k $(k = 1, \cdots, n)$ with $\sigma(A), \sigma(B_k)$ and $\sigma((1 + \sum_{k=1}^{n} r_k)A - \sum_{k=1}^{n} r_k B_k) \subset J$, also see [9].

In this paper, considering that the Jensen inequality is also expressed by externally dividing points, we shall discuss two non-commutative Jensen inequalities and their reverse, that is, one is a vector state version (1.2) and the other is a Davis-Choi-Jensen type version (1.4).

2. Vector state version

First of all, we show an external version of the Jensen inequality which corresponds to a two variable vector state version (1.3):

THEOREM 2.1. *Let f be a real valued function on an interval J. Then f is concave if and only if*

$$f((Ax,x)-(By,y)) \leqslant \| x \|^2 f\left(\left(A\frac{x}{\| x \|}, \frac{x}{\| x \|}\right)\right) - (f(B)y,y) \tag{2.1}$$

for all $x,y \in H$ such that $\| x \|^2 - \| y \|^2 = 1$ and for all selfadjoint operators A and B with the spectra in J such that $(Ax,x)-(By,y) \in J$.

Proof. Suppose that f is concave. Since $\| x \|^2 = \| y \|^2 + 1$, it follows that

$$\begin{aligned}
&\| x \|^2 f\left(\left(A\frac{x}{\| x \|}, \frac{x}{\| x \|}\right)\right) \\
&= \| x \|^2 f\left(\frac{1}{1+\| y \|^2}((Ax,x)-(By,y)) + \frac{\| y \|^2}{1+\| y \|^2}\left(B\frac{y}{\| y \|}, \frac{y}{\| y \|}\right)\right) \\
&\geqslant \| x \|^2 \left(\frac{1}{1+\| y \|^2} f((Ax,x)-(By,y)) + \frac{\| y \|^2}{1+\| y \|^2} f\left(\left(B\frac{y}{\| y \|}, \frac{y}{\| y \|}\right)\right)\right) \quad \text{by (1.1)} \\
&= f((Ax,x)-(By,y)) + \| y \|^2 f\left(\left(B\frac{y}{\| y \|}, \frac{y}{\| y \|}\right)\right) \\
&\geqslant f((Ax,x)-(By,y)) + (f(B)y,y) \quad \text{by (1.2)}.
\end{aligned}$$

Conversely, put $A = \operatorname{diag}(x_1, \cdots, x_n)$ and $B = \operatorname{diag}(y_1, \cdots, y_k)$ and $x = (\sqrt{\alpha_1}, \cdots, \sqrt{\alpha_n})$ and $y = (\sqrt{\beta_1}, \cdots, \sqrt{\beta_k})$ in (2.1). Then we have (1.6):

$$f\left(\sum_{i=1}^{n} \alpha_i x_i - \sum_{j=1}^{k} \beta_j y_j\right) \leqslant \left(\sum_{i=1}^{n} \alpha_i\right) f\left(\frac{1}{\sum_{i=1}^{n} \alpha_i} \sum_{i=1}^{n} \alpha_i x_i\right) - \sum_{j=1}^{k} \beta_j f(y_j)$$

and hence f is concave.

Next, we consider the reverse Jensen inequality in an external formula. For this, we need the following result [8]:

LEMMA 2.2. *Let A be a positive operator on a Hilbert space H such that $mI \leqslant A \leqslant MI$ for some scalars $M > m > 0$. Let f be a real valued continuous function on $[m,M]$ and $f(t) > 0$ for all $t \in [m,M]$. If f is concave, then*

$$K(m,M,f)f((Ax,x)) \leqslant (f(A)x,x) \tag{2.2}$$

for every unit vector $x \in H$, where the generalized Kantorovich constnat $K(m,M,f)$ is defined by

$$K(m,M,f) = \min\left\{\frac{1}{f(t)}\left(\frac{f(M)-f(m)}{M-m}(t-m)+f(m)\right) : t \in [m,M]\right\}. \tag{2.3}$$

By Lemma 2.2, we obtain the following estimate in an external formula:

THEOREM 2.3. *Let f be a real valued continuous function on $[m,M]$ and $f(t) > 0$ for all $t \in [m,M]$. If f is concave, then*

$$\begin{aligned} &K(m,M,f)(f(A)x,x) - K(m,M,f)^{-1}(f(B)y,y) \\ &\leqslant f((Ax,x)-(By,y)) \\ &\leqslant K(m,M,f)^{-1}(f(A)x,x)-(f(B)y,y) \end{aligned}$$

for all $x,y \in H$ such that $\| x \|^2 - \| y \|^2 = 1$ and for all selfadjoint operators A and B with the spectra in J such that $(Ax,x)-(By,y) \in J$, where $K(m,M,f)$ is defined as (2.3).

Proof. For all $x,y \in H$ such that $\| x \|^2 - \| y \|^2 = 1$, it follows from Lemma 2.2 that

$$\begin{aligned} &(f(A)x,x) \\ &\leqslant \| x \|^2 f((A\frac{x}{\| x \|},\frac{x}{\| x \|})) \qquad \text{by (1.2)} \\ &= \| x \|^2 f\left(\frac{1}{1+\| y \|^2}((Ax,x)-(By,y)) + \frac{\| y \|^2}{1+\| y \|^2}\left(B\frac{y}{\| y \|},\frac{y}{\| y \|}\right)\right) \\ &\leqslant \| x \|^2 K(m,M,f)^{-1}\left(\frac{1}{1+\| y \|^2}f((Ax,x)-(By,y)) + \frac{\| y \|^2}{1+\| y \|^2}f\left(\left(B\frac{y}{\| y \|},\frac{y}{\| y \|}\right)\right)\right) \\ &= K(m,M,f)^{-1}\left(f((Ax,x)-(By,y)) + \| y \|^2 f\left(\left(B\frac{y}{\| y \|},\frac{y}{\| y \|}\right)\right)\right) \end{aligned}$$

and hence

$$\begin{aligned} K(m,M,f)(f(A)x,x) &\leqslant f((Ax,x)-(By,y)) + \| y \|^2 f((B\frac{y}{\| y \|},\frac{y}{\| y \|})) \\ &\leqslant f((Ax,x)-(By,y)) + K(m,M,f)^{-1}(f(B)y,y). \end{aligned}$$

On the other hand, by Theorem 2.1 and Lemma 2.2 again, we have

$$\begin{aligned} f((Ax,x)-(By,y)) &\leqslant \| x \|^2 f((A\frac{x}{\| x \|},\frac{x}{\| x \|})) - (f(B)y,y) \\ &\leqslant K(m,M,f)^{-1}(f(A)x,x)-(f(B)y,y). \end{aligned}$$

REMARK 2.4. Put the power function $f(t) = t^p$ for $0 < p < 1$ in Theorem 2.3, then we exactly have the evaluation

$$K(m,M,f) = K(m,M,t^p) = \frac{mM^p - Mm^p}{(p-1)(M-m)}\left(\frac{(p-1)(M^p-m^p)}{p(mM^p-Mm^p)}\right)^p,$$

see [4], and hence

$$\begin{aligned}
&\frac{mM^p - Mm^p}{(p-1)(M-m)}\left(\frac{(p-1)(M^p-m^p)}{p(mM^p-Mm^p)}\right)^p (A^p x,x) \\
&\qquad - \frac{(p-1)(M-m)}{mM^p-Mm^p}\left(\frac{p(mM^p-Mm^p}{(p-1)(M^p-m^p)}\right)^p (B^p y,y) \\
&\leqslant ((Ax,x)-(By,y))^p \leqslant \frac{(p-1)(M-m)}{mM^p-Mm^p}\left(\frac{p(mM^p-Mm^p}{(p-1)(M^p-m^p)}\right)^p (A^p x,x)-(B^p y,y)
\end{aligned}$$

for all vectors $x,y \in H$ such that $\| x \|^2 - \| y \|^2 = 1$ and all selfadjoint operators A and B such that $(Ax,x)-(By,y) \in [m,M]$.

3. Davis-Choi-Jensen inequality

In this section, we denote $P[B(H),B(K)]$ as the set of all positive linear maps $\Phi : B(H) \to B(K)$, where $B(H)$ is a C^*-algebra of all bounded linear operators on a Hilbert space H.

The following theorem is an external version of the Davis-Choi-Jensen inequality for operator concave functions which corresponds to (1.5):

THEOREM 3.1. *Let f be a real valued continuous function on an interval J. Then f is operator concave if and only if*

$$f(\Phi(A)-\Psi(B)) \leqslant \Phi(I)^{\frac{1}{2}} f\left(\Phi(I)^{-\frac{1}{2}}\Phi(A)\Phi(I)^{-\frac{1}{2}}\right)\Phi(I)^{\frac{1}{2}} - \Psi(f(B)) \tag{3.1}$$

for all $\Phi,\Psi \in P[B(H),B(K)]$ such that $\Phi(I)-\Psi(I)=I$ and for all selfadjoint operators A and B with $\sigma(A),\sigma(B)$ and $\sigma(\Phi(A)-\Psi(B)) \subset J$.

Proof. (i) $\Longrightarrow$ (ii): By Stinespring decomposition theorem [11], Φ restricted to a C^*-algebra $C^*(A)$ generated by A and I admits a decomposition $\Phi(X) = C^*\phi(X)C$ for all $X \in C^*(A)$, where ϕ is a $*$-representation of $C^*(A) \subset B(H)$ and C is a bounded linear operator from K to a Hilbert space K'. Similarly we have a decomposition $\Psi(Y) = D^*\psi(Y)D$ for all $Y \in C^*(B)$, where ψ are a $*$-representation and D is a bounded linear operator from K to a Hilbert space K''. The assumption $\Phi(I)-\Psi(I)=I$ implies $C^*C - D^*D = I$ and hence $|C|$ is invertible. Since $|C|^{-2} + (D|C|^{-1})^*(D|C|^{-1}) = I$, it follows that

$$\begin{aligned}
&\Phi(I)^{\frac{1}{2}} f\left(\Phi(I)^{-\frac{1}{2}}\Phi(A)\Phi(I)^{-\frac{1}{2}}\right)\Phi(I)^{\frac{1}{2}} \\
&= |C| f(|C|^{-1}C^*\phi(A)C|C|^{-1})|C| \\
&= |C| f\left(|C|^{-1}(C^*\phi(A)C - D^*\psi(B)D)|C|^{-1} + (D|C|^{-1})^*\psi(B)D|C|^{-1}\right)|C| \\
&\geqslant |C|\left(|C|^{-1} f(\Phi(A)-\Psi(B))|C|^{-1} + (D|C|^{-1})^* f(\psi(B))D|C|^{-1}\right)|C| \qquad \text{by (1.5)} \\
&= f(\Phi(A)-\Psi(B)) + D^*\psi(f(B))D \\
&= f(\Phi(A)-\Psi(B)) + \Psi(f(B)).
\end{aligned}$$

(ii)$\Longrightarrow$(i): Put $\Phi(A) = C^*AC$ and $\Psi(B) = D^*BD$ for $C^*C - D^*D = I$ in (ii), then the operator concavity of f follows from [3, Theorem 1].

REMARK 3.2. If C is invertible in a decomposition $\Phi(A) = C^*\phi(A)C$, then

$$f(\Phi(A) - \Psi(B)) \leqslant \Phi(f(A)) - \Psi(f(B)).$$

In fact, if C is invertible, then V is unitary in the polar decomposition $C = V|C|$ and hence

$$\begin{aligned}\Phi(I)^{\frac{1}{2}} f\left(\Phi(I)^{-\frac{1}{2}}\Phi(A)\Phi(I)^{-\frac{1}{2}}\right)\Phi(I)^{\frac{1}{2}} &= |C| f(|C|^{-1}C^*\phi(A)C|C|^{-1})|C| \\ &= |C| f(V^*\phi(A)V)|C| = |C|V^* f(\phi(A))V|C| \\ &= C^*\phi(f(A))C = \Phi(f(A)).\end{aligned}$$

By virtue of the generalized Kantorovich constant, we consider the difference between the concavity and the operator concavity, based on an external version of the Jensen inequality:

THEOREM 3.3. *Let f be a real valued continuous function on $[m,M]$ and $f(t) > 0$ for all $t \in [m,M]$. If f is concave, then*

$$\begin{aligned}&K(m,M,f)\Phi(I)^{\frac{1}{2}} f\left(\Phi(I)^{-\frac{1}{2}}\Phi(A)\Phi(I)^{-\frac{1}{2}}\right)\Phi(I)^{\frac{1}{2}} - \Psi(f(B)) \\ &\qquad \leqslant f(\Phi(A) - \Psi(B)) \\ &\qquad \leqslant K(m,M,f)^{-1}\Phi(I)^{\frac{1}{2}} f\left(\Phi(I)^{-\frac{1}{2}}\Phi(A)\Phi(I)^{-\frac{1}{2}}\right)\Phi(I)^{\frac{1}{2}} - \Psi(f(B))\end{aligned}$$

for all $\Phi, \Psi \in P[B(H), B(K)]$ such that $\Phi(I) - \Psi(I) = I$ and for all selfadjoint operators A and B with the spectra in J such that $\sigma(\Phi(A) - \Psi(B)) \subset J$, where $K(m,M,f)$ is defined as (2.3).

To prove it, we need the following lemma [4, Corollary 3.21]:

LEMMA 3.4. *Let f be a real valued continuous function on $[m,M]$ and $f(t) > 0$ for all $t \in [m,M]$. If f is concave, then*

$$\begin{aligned}K(m,M,f) f(U^*AU + W^*BW) &\leqslant U^* f(A)U + W^* f(B)W \\ &\leqslant K(m,M,f)^{-1} f(U^*AU + W^*BW) \qquad (3.2)\end{aligned}$$

for all selfadjoint operators A, B with the spectra in $[m,M]$ and $U^*U + W^*W = I$, where $K(m,M,f)$ is defined as (2.3).

Proof. We give a proof for reader's convenience. For any unit vector $x \in H$, we have $\| Ux \|^2 + \| Wx \|^2 = 1$ and hence

$$\begin{aligned}((U^* f(A)U + W^* f(B)W)x, x) &= (f(A)Ux, Ux) + (f(B)Wx, Wx) \\ &\leqslant f((AUx, Ux) + (BWx, Wx)) \quad \text{by (1.3)} \\ &= f(((U^*AU + W^*BW)x, x)) \\ &\leqslant K(m,M,f)^{-1}(f(U^*AU + W^*BW)x, x) \text{ by Lemma 2.2}\end{aligned}$$

and this implies

$$U^*f(A)U + W^*f(B)W \leqslant K(m,M,f)^{-1} f(U^*AU + W^*BW).$$

Similarly,

$$\begin{aligned}(f(U^*AU + W^*BW)x,x) &\leqslant f(((U^*AU + W^*BW)x,x)) \quad \text{by (1.2)}\\ &= f((AUx,Ux) + (BWx,Wx))\\ &\leqslant K(m,M,f)^{-1}\left((f(A)Ux,Ux) + (f(B)Wx,Wx)\right)\\ &= K(m,M,f)^{-1}((U^*f(A)U + W^*f(B)W)x,x)\end{aligned}$$

and this implies

$$K(m,M,f)\ f(U^*AU + W^*BW) \leqslant U^*f(A)U + W^*f(B)W. \qquad \square$$

Proof of Theorem 3.3. As in the proof of Theorem 3.1, let $\Phi(A) = C^*\phi(A)C$ and $\Psi(B) = D^*\psi(B)D$ be the Stinespring decomposition. Then $\Phi(I) - \Psi(I) = I$ implies $C^*C - D^*D = I$ and so $|C|^{-2} + (D|C|^{-1})^*(D|C|^{-1}) = I$. By Lemma 3.4, it follows that

$$\begin{aligned}&K(m,M,f)\ \Phi(I)^{\frac{1}{2}} f\left(\Phi(I)^{-\frac{1}{2}}\Phi(A)\Phi(I)^{-\frac{1}{2}}\right)\Phi(I)^{\frac{1}{2}}\\ &= K(m,M,f)\ |C| f\left(|C|^{-1}\Phi(A)|C|^{-1}\right)|C|\\ &= K(m,M,f)\ |C| f\left(|C|^{-1}(\Phi(A) - \Psi(B))|C|^{-1} + (D|C|^{-1})^*\psi(B)D|C|^{-1}\right)|C|\\ &\leqslant |C|\left(|C|^{-1}f(\Phi(A) - \Psi(B))|C|^{-1} + (D|C|^{-1})^* f(\psi(B))D|C|^{-1}\right)|C|\\ &= f\left(\Phi(A) - \Psi(B)\right) + \Psi(f(B))\end{aligned}$$

and also

$$\begin{aligned}&f\left(\Phi(A) - \Psi(B)\right) + \Psi(f(B))\\ &= |C|\left(|C|^{-1}f(\Phi(A) - \Psi(B))|C|^{-1} + (D|C|^{-1})^* f(\psi(B))D|C|^{-1}\right)|C|\\ &\leqslant K(m,M,f)^{-1} f\left(|C|^{-1}(\Phi(A) - \Psi(B))|C|^{-1} + (D|C|^{-1})^*\psi(B)D|C|^{-1}\right)\\ &= K(m,M,f)^{-1}\Phi(I)^{\frac{1}{2}} f\left(\Phi(I)^{-\frac{1}{2}}\Phi(A)\Phi(I)^{-\frac{1}{2}}\right)\Phi(I)^{\frac{1}{2}}\end{aligned}$$

and this implies Theorem 3.3. $\square$

REMARK 3.5. If we put $\Phi(A) = (Ax,x)$ and $\Psi(B) = (By,y)$ for $x,y \in H$ such that $\| x \|^2 - \| y \|^2 = 1$ in Theorem 3.3, then Theorem 3.3 does not implies Theorem 2.3, because $f((Ax,x)) = \| x \|^2 f((A\frac{x}{\|x\|}, \frac{x}{\|x\|}))$ does not always hold.

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