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MATRIX INEQUALITIES INCLUDING FURUTA INEQUALITY VIA RIEMANNIAN MEAN OF n–MATRICES

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Abstract. Very recently, Yamazaki has obtained an excellent generalization of Ando-Hiai inequality and a characterization of chaotic order (so called Furuta inequality for chaotic order) via weighted Riemannian mean, a kind of geometric mean, of n positive definite matrices.

In this paper, by discussing extensions of Yamazaki's results, we shall obtain a generalization of Furuta inequality via weighted Riemannian mean of n-matrices.

1. Introduction

We frequently use the weighted geometric mean of two positive definite matrices A and B defined by $A\sharp_\alpha B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^\alpha A^{\frac{1}{2}}$ for $\alpha \in [0,1]$. In particular, we call $A\sharp_{\frac{1}{2}} B$ (denoted by $A\sharp B$ simply) the geometric mean of A and B.

It has been a longstanding problem to extend the (weighted) geometric mean for three or more positive definite matrices. Many authors attempt to find a natural extension, for example, Ando-Li-Mathias' mean and its refinement [2, 5, 15, 16] and Riemannian mean (or the least squares mean) [4, 18, 19]. We remark that Ando-Li-Mathias [2] originally proposed ten properties ((P1)–(P10) stated below) which should be required for a reasonable geometric mean of positive definite matrices.

Let $P_m(\mathbb{C})$ be the set of $m\times m$ positive definite matrices on $\mathbb{C}$, and also we recall that $\omega = (w_1,\ldots,w_n)$ is a probability vector if the components satisfy $\sum_i w_i = 1$ and $w_i > 0$ for $i = 1,\ldots,n$. For $A,B \in P_m(\mathbb{C})$, Riemannian metric between A and B is defined as $\delta_2(A,B) = \|\log A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\|_2$, where $\|X\|_2 = (\mathrm{tr}X^*X)^{\frac{1}{2}}$ (details are in [3]). By using Riemannian metric, Riemannian mean is defined as follows:

DEFINITION 1. ([3, 4, 18, 19]) Let $A_1,\ldots,A_n \in P_m(\mathbb{C})$ and $\omega = (w_1,\ldots,w_n)$ be a probability vector. Then weighted Riemannian mean $\mathfrak{G}_\delta(\omega;A_1,\ldots,A_n) \in P_m(\mathbb{C})$ is defined by

$$\mathfrak{G}_\delta(\omega;A_1,\ldots,A_n) = \mathop{\arg\min}_{X\in P_m(\mathbb{C})} \sum_{i=1}^n w_i\delta_2^2(A_i,X),$$

where $\arg\min f(X)$ means the point X_0 which attains minimum value of the function $f(X)$. In particular, we call $\mathfrak{G}_\delta(\omega;A_1,\ldots,A_n)$ (denoted by $\mathfrak{G}_\delta(A_1,\ldots,A_n)$ simply) Riemannian mean if $\omega = (\frac{1}{n},\ldots,\frac{1}{n})$.

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We remark that $\mathfrak{G}_\delta(\omega;A,B) = A \sharp_\alpha B$ for $\alpha \in [0,1]$ and $\omega = (1-\alpha,\alpha)$ since the property $\delta_2(A, A\sharp_\alpha B) = \alpha\delta_2(A,B)$ holds.

On the other hand, the weighted geometric mean sometimes appears in famous matrix inequalities, for example, Furuta inequality [10] and Ando-Hiai inequality [1]. We remark that these inequalities hold even in the case of bounded linear operators on a complex Hilbert space. In what follows, we denote $A \geqslant 0$ if A is a positive semidefinite matrix (or operator), and we denote $A > 0$ if A is a positive definite matrix (or operator).

THEOREM 1.A. (Satellite form of Furuta inequality [10, 17])

$$A \geqslant B \geqslant 0 \textit{ with } A > 0 \quad \textit{implies} \quad A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \leqslant B \leqslant A \textit{ for } p \geqslant 1 \textit{ and } r \geqslant 0.$$

THEOREM 1.B. (Ando-Hiai inequality [1]) *For* $A,B > 0$,

$$A \sharp_\alpha B \leqslant I \textit{ for } \alpha \in (0,1) \quad \textit{implies} \quad A^r \sharp_\alpha B^r \leqslant I \textit{ for } r \geqslant 1.$$

For $A,B > 0$, it is well known that chaotic order $\log A \geqslant \log B$ is weaker than usual order $A \geqslant B$ since $\log t$ is a matrix (or operator) monotone function. The following result is known as the Furuta inequality for chaotic order.

THEOREM 1.C. (Furuta inequality for chaotic order [7, 12]) *Let* $A,B > 0$. *Then the following assertions are mutually equivalent;*

(i) $\log A \geqslant \log B$,

(ii) $A^{-p} \sharp B^p \leqslant I$ *for all* $p \geqslant 0$,

(iii) $A^{-r} \sharp_{\frac{r}{p+r}} B^p \leqslant I$ *for all* $p \geqslant 0$ *and* $r \geqslant 0$.

Very recently, Yamazaki [21] has obtained an excellent generalization of Theorems 1.B and 1.C via weighted Riemannian mean $\mathfrak{G}_\delta$ of n-matrices.

THEOREM 1.D. ([21]) *Let* $A_1,\ldots,A_n \in P_m(\mathbb{C})$ *and* $\omega = (w_1,\ldots,w_n)$ *be a probability vector. Then*

$$\mathfrak{G}_\delta(\omega;A_1,\ldots,A_n) \leqslant I \quad \textit{implies} \quad \mathfrak{G}_\delta(\omega;A_1^p,\ldots,A_n^p) \leqslant I \textit{ for } p \geqslant 1.$$

THEOREM 1.E. ([21]) *Let* $A_1,\ldots,A_n \in P_m(\mathbb{C})$. *Then the following assertions are mutually equivalent;*

(i) $\log A_1 + \cdots + \log A_n \leqslant 0$,

(ii) $\mathfrak{G}_\delta(A_1^p,\ldots,A_n^p) \leqslant I$ *for all* $p > 0$,

(iii) $\mathfrak{G}_\delta(\omega;A_1^{p_1},\ldots,A_n^{p_n}) \leqslant I$ *for all* $p_1,\ldots,p_n > 0$, *where* $p_{\neq i} = \prod_{j\neq i} p_j$ *and* $\omega = \left(\frac{p_{\neq 1}}{\sum_i p_{\neq i}},\ldots,\frac{p_{\neq n}}{\sum_i p_{\neq i}}\right)$.

Theorems 1.D and 1.E imply Theorems 1.B and 1.C, respectively, since $\mathfrak{G}_\delta(\omega;A,B) = A\sharp_\alpha B$ for $\omega = (1-\alpha,\alpha)$. Moreover, it has been shown in [21] that Theorem 1.D does not hold for other geometric means satisfying (P1)–(P10).

In this paper, corresponding to Theorem 1.E, we shall obtain a generalization of Furuta inequality (Theorem 1.A) via weighted Riemannian mean of n-matrices. Moreover we shall show an extension of Theorem 1.D.

2. Preliminaries

Ando-Li-Mathias [2] originally proposed the following ten properties (P1)–(P10) which should be required for a reasonable geometric mean of positive definite matrices. It is shown in [3, 4, 18, 19] that weighted Riemannian mean satisfies (P1)–(P10) (see also [21]).

Let $A_i, A_i', B_i \in P_m(\mathbb{C})$ for $i = 1,\dots,n$ and let $\omega = (w_1,\dots,w_n)$ be a probability vector. Then

(P1) Consistency with scalars. If $A_1,\dots,A_n$ commute with each other, then

$$\mathfrak{G}_\delta(\omega;A_1,\dots,A_n) = A_1^{w_1}\dots A_n^{w_n}.$$

(P2) Joint homogeneity. For positive numbers $a_i > 0$ $(i = 1,\dots n)$,

$$\mathfrak{G}_\delta(\omega;a_1A_1,\dots,a_nA_n) = a_1^{w_1}\dots a_n^{w_n}\mathfrak{G}_\delta(\omega;A_1,\dots,A_n).$$

(P3) Permutation invariance. For any permutation π on $\{1,\dots n\}$,

$$\mathfrak{G}_\delta(\omega;A_1,\dots,A_n) = \mathfrak{G}_\delta(\pi(\omega);A_{\pi(1)},\dots,A_{\pi(n)}),$$

where $\pi(\omega) = (w_{\pi(1)},\cdots,w_{\pi(n)})$.

(P4) Monotonicity. If $B_i \leqslant A_i$ for each $i = 1,\dots n$, then

$$\mathfrak{G}_\delta(\omega;B_1,\dots,B_n) \leqslant \mathfrak{G}_\delta(\omega;A_1,\dots,A_n).$$

(P5) Continuity. For each $i = 1,\dots n$, let $\{A_i^{(k)}\}_{k=1}^\infty$ be positive definite matrix sequences such that $A_i^{(k)} \to A_i$ as $k \to \infty$. Then

$$\mathfrak{G}_\delta(\omega;A_1^{(k)},\dots,A_n^{(k)}) \to \mathfrak{G}_\delta(\omega;A_1,\dots,A_n) \quad \text{as } k \to \infty.$$

(P6) Congruence invariance. For any invertible matrix S,

$$\mathfrak{G}_\delta(\omega;S^*A_1S,\dots,S^*A_nS) = S^*\mathfrak{G}_\delta(\omega;A_1,\dots,A_n)S.$$

(P7) Joint concavity.

$$\begin{aligned}&\mathfrak{G}_\delta(\omega;\lambda A_1+(1-\lambda)A_1',\dots,\lambda A_n+(1-\lambda)A_n')\\ &\geqslant \lambda\mathfrak{G}_\delta(\omega;A_1,\dots,A_n)+(1-\lambda)\mathfrak{G}_\delta(\omega;A_1',\dots,A_n') \quad \text{for } 0\leqslant\lambda\leqslant 1.\end{aligned}$$

(P8) Self-duality. $\mathfrak{G}_\delta(\omega;A_1^{-1},\ldots,A_n^{-1})^{-1}=\mathfrak{G}_\delta(\omega;A_1,\ldots,A_n)$.

(P9) Determinant identity. $\det\mathfrak{G}_\delta(\omega;A_1,\ldots,A_n)=\prod_{i=1}^{n}(\det A_i)^{w_i}$.

(P10) The arithmetic-geometric-harmonic mean inequality.

$$\left(\sum_{i=1}^{n} w_iA_i^{-1}\right)^{-1}\leqslant\mathfrak{G}_\delta(\omega;A_1,\ldots,A_n)\leqslant\sum_{i=1}^{n}w_iA_i.$$

We remark that, in [2], they require continuity from above as (P5). Riemannian mean has a stronger property (P5') than (P5).

(P5') Non-expansive.

$$\delta_2(\mathfrak{G}_\delta(\omega;A_1,\ldots,A_n),\mathfrak{G}_\delta(\omega;B_1,\ldots,B_n))\leqslant\sum_{i=1}^{n}w_i\delta_2(A_i,B_i).$$

It was obtained in [18, 19] that Riemannian mean has a useful characterization via a matrix equation.

THEOREM 2.A. ([18, 19]) *Let $A_1,\ldots,A_n\in P_m(\mathbb{C})$ and $\omega=(w_1,\ldots,w_n)$ be a probability vector. Then $X=\mathfrak{G}_\delta(\omega;A_1,\ldots,A_n)$ is the unique positive solution of the following matrix equation:*

$$w_1\log X^{\frac{-1}{2}}A_1X^{\frac{-1}{2}}+\cdots+w_n\log X^{\frac{-1}{2}}A_nX^{\frac{-1}{2}}=0.$$

3. Main results

Firstly, we show an extension of Theorem 1.D. Theorem 1.D follows from Theorem 3.1 by putting $p_1=\cdots=p_n=p$.

THEOREM 3.1. *Let $A_1,\ldots,A_n\in P_m(\mathbb{C})$ and $\omega=(w_1,\ldots,w_n)$ be a probability vector. If $\mathfrak{G}_\delta(\omega;A_1,\ldots,A_n)\leqslant I$, then*

$$\mathfrak{G}_\delta(\omega';A_1^{p_1},\ldots,A_n^{p_n})\leqslant\mathfrak{G}_\delta(\omega;A_1,\ldots,A_n)\leqslant I\quad\text{for } p_1,\ldots,p_n\geqslant 1,$$

where $\widehat{\omega'}=(\frac{w_1}{p_1},\ldots,\frac{w_n}{p_n})$ and $\omega'=\frac{\widehat{\omega'}}{\|\widehat{\omega'}\|_1}$.

We remark that $\|\cdot\|_1$ means 1-norm, that is, $\|x\|_1=\sum_i|x_i|$ for $x=(x_1,\ldots,x_n)$. In order to prove Theorem 3.1, we use the following result.

THEOREM 3.A. ([21]) *Let $A_1,\ldots,A_n\in P_m(\mathbb{C})$ and $\omega=(w_1,\ldots,w_n)$ be a probability vector. Then*

$$w_1\log A_1+\cdots+w_n\log A_n\leqslant 0\quad\textit{implies}\quad\mathfrak{G}_\delta(\omega;A_1,\ldots,A_n)\leqslant I.$$

Proof of Theorem 3.1. Let $X = \mathfrak{G}_\delta(\omega; A_1,\ldots,A_n) \leqslant I$. Then for each $p_1,\ldots,p_n \in [1,2]$, by Theorem 2.A and Hansen's inequality [14],

$$0 = \frac{1}{\|\widehat{\omega'}\|_1}\sum w_i \log X^{\frac{1}{2}} A_i^{-1} X^{\frac{1}{2}} = \frac{1}{\|\widehat{\omega'}\|_1}\sum \frac{w_i}{p_i}\log(X^{\frac{1}{2}} A_i^{-1} X^{\frac{1}{2}})^{p_i}$$
$$\leqslant \frac{1}{\|\widehat{\omega'}\|_1}\sum \frac{w_i}{p_i}\log X^{\frac{1}{2}} A_i^{-p_i} X^{\frac{1}{2}},$$

that is, $\sum \frac{\frac{w_i}{p_i}}{\|\widehat{\omega'}\|_1}\log X^{\frac{-1}{2}} A_i^{p_i} X^{\frac{-1}{2}} \leqslant 0$. By applying Theorem 3.A,

$$\mathfrak{G}_\delta(\omega'; X^{\frac{-1}{2}} A_1^{p_1} X^{\frac{-1}{2}},\ldots,X^{\frac{-1}{2}} A_n^{p_n} X^{\frac{-1}{2}}) \leqslant I$$

where $\widehat{\omega'} = (\frac{w_1}{p_1},\ldots,\frac{w_n}{p_n})$ and $\omega' = \frac{\widehat{\omega'}}{\|\widehat{\omega'}\|_1}$. Therefore we have that

$$X \leqslant I \quad \text{implies} \quad \mathfrak{G}_\delta(\omega'; A_1^{p_1},\ldots,A_n^{p_n}) \leqslant X \leqslant I \ \text{ for } p_1,\ldots,p_n \in [1,2]. \tag{3.1}$$

Put $Y = \mathfrak{G}_\delta(\omega'; A_1^{p_1},\ldots,A_n^{p_n}) \leqslant I$. Then by (3.1), we get

$$\mathfrak{G}_\delta(\omega''; A_1^{p_1 p_1'},\ldots,A_n^{p_n p_n'}) \leqslant Y \leqslant X \leqslant I$$

for $p_1',\ldots,p_n' \in [1,2]$, where $\widehat{\omega''} = (\frac{w_1}{p_1 p_1'},\ldots,\frac{w_n}{p_n p_n'})$ and $\omega'' = \frac{\widehat{\omega''}}{\|\widehat{\omega''}\|_1}$. Therefore, by putting $q_i = p_i p_i'$ for $i = 1,\ldots,n$, we have that

$$X \leqslant I \quad \text{implies} \quad \mathfrak{G}_\delta(\omega''; A_1^{q_1},\ldots,A_n^{q_n}) \leqslant X \leqslant I \ \text{ for } q_1,\ldots,q_n \in [1,4], \tag{3.2}$$

where $\widehat{\omega''} = (\frac{w_1}{q_1},\ldots,\frac{w_n}{q_n})$ and $\omega'' = \frac{\widehat{\omega''}}{\|\widehat{\omega''}\|_1}$.

By repeating the same way from (3.1) to (3.2), we have the conclusion. □

Theorem 3.1 also implies generalized Ando-Hiai inequality [9] since $\mathfrak{G}_\delta(\omega; A, B) = A \sharp_\alpha B$ for $\omega = (1-\alpha, \alpha)$ and $\omega' = \left(\frac{\frac{1-\alpha}{r}}{\frac{1-\alpha}{r}+\frac{\alpha}{s}}, \frac{\frac{\alpha}{s}}{\frac{1-\alpha}{r}+\frac{\alpha}{s}}\right) = \left(\frac{(1-\alpha)s}{(1-\alpha)s+\alpha r}, \frac{\alpha r}{(1-\alpha)s+\alpha r}\right)$.

THEOREM 3.B. (Generalized Ando-Hiai inequality [9]) *Let $A, B > 0$. If $A \sharp_\alpha B \leqslant I$ for $\alpha \in (0,1)$, then*

$$A^r \sharp_{\frac{\alpha r}{(1-\alpha)s+\alpha r}} B^s \leqslant A \sharp_\alpha B \leqslant I \ \textit{for } s \geqslant 1 \textit{ and } r \geqslant 1.$$

The following Theorem 3.2 is a variant from Theorem 3.1.

THEOREM 3.2. *Let $A_1,\ldots,A_n \in P_m(\mathbb{C})$ and $\omega = (w_1,\ldots,w_n)$ be a probability vector. For each $i = 1,\ldots,n$ and $q \in \mathbb{R}$, if*

$$\mathfrak{G}_\delta(\omega; A_1^{p_1},\ldots,A_i^{p_i},\ldots,A_n^{p_n}) \leqslant A_i^q \quad \textit{for } p_1,\ldots,p_n \in \mathbb{R} \textit{ with } p_i > q,$$

then

$$\begin{aligned}&\mathfrak{G}_\delta(\omega';A_1^{p_1},\ldots,A_{i-1}^{p_{i-1}},A_i^{p'_i},A_{i+1}^{p_{i+1}},\ldots,A_n^{p_n})\\ \leqslant\ &\mathfrak{G}_\delta(\omega;A_1^{p_1},\ldots,A_{i-1}^{p_{i-1}},A_i^{p_i},A_{i+1}^{p_{i+1}},\ldots,A_n^{p_n})\\ \leqslant\ &A_i^q\end{aligned}$$

for $p'_i \geqslant p_i$, *where* $\widehat{\omega'} = (w_1,\ldots,w_{i-1},\frac{p_i-q}{p'_i-q}w_i,w_{i+1},\ldots,w_n)$ *and* $\omega' = \frac{\widehat{\omega'}}{\|\widehat{\omega'}\|_1}$.

Proof. We may assume $i=1$ by permutation invariance of $\mathfrak{G}_\delta$.
For $p_1,\ldots,p_n \in \mathbb{R}$ with $p_1 \geqslant q$, $\mathfrak{G}_\delta(\omega;A_1^{p_1},A_2^{p_2},\ldots,A_n^{p_n}) \leqslant A_1^q$ if and only if

$$\mathfrak{G}_\delta(\omega;A_1^{p_1-q},A_1^{\frac{-q}{2}}A_2^{p_2}A_1^{\frac{-q}{2}},\ldots,A_1^{\frac{-q}{2}}A_n^{p_n}A_1^{\frac{-q}{2}}) \leqslant I.$$

By applying Theorem 3.1,

$$\begin{aligned}&\mathfrak{G}_\delta(\omega';A_1^{p'_1-q},A_1^{\frac{-q}{2}}A_2^{p_2}A_1^{\frac{-q}{2}},\ldots,A_1^{\frac{-q}{2}}A_n^{p_n}A_1^{\frac{-q}{2}})\\ \leqslant\ &\mathfrak{G}_\delta(\omega;A_1^{p_1-q},A_1^{\frac{-q}{2}}A_2^{p_2}A_1^{\frac{-q}{2}},\ldots,A_1^{\frac{-q}{2}}A_n^{p_n}A_1^{\frac{-q}{2}})\\ \leqslant\ &I,\end{aligned}$$

holds for $\frac{p'_1-q}{p_1-q} \geqslant 1$, where $\widehat{\omega'} = (\frac{p_1-q}{p'_1-q}w_1,w_2,\ldots,w_n)$. Therefore

$$\mathfrak{G}_\delta(\omega';A_1^{p'_1},A_2^{p_2},\ldots,A_n^{p_n}) \leqslant \mathfrak{G}_\delta(\omega;A_1^{p_1},A_2^{p_2},\ldots,A_n^{p_n}) \leqslant A_1^q$$

holds for $p'_1 \geqslant p_1$. □

Next, we show our main result. The following Theorem 3.3 is a parallel result to (i) $\Longrightarrow$ (iii) in Theorem 1.E. In the next section, we shall recognize that Theorem 3.3 is a generalization of Theorem 1.A.

THEOREM 3.3. *Let* $A_1,\ldots,A_n \in P_m(\mathbb{C})$ *and* $w_1,\ldots,w_n > 0$. *If*

$$A_i^{q_i} \geqslant A_n^{q_n} > 0 \tag{3.3}$$

and

$$\begin{aligned}&\frac{w_1}{p_1-q_1}\log A_n^{\frac{-q_n}{2}}A_1^{p_1}A_n^{\frac{-q_n}{2}}+\cdots\\ &\quad+\frac{w_{n-1}}{p_{n-1}-q_{n-1}}\log A_n^{\frac{-q_n}{2}}A_{n-1}^{p_{n-1}}A_n^{\frac{-q_n}{2}}+\frac{w_n}{p_n-q_n}\log A_n^{p_n-q_n} \leqslant 0\end{aligned} \tag{3.4}$$

hold for $q_i \in \mathbb{R}$, $p_i > q_i$ *and* $i = 1,\ldots,n$, *then*

$$\mathfrak{G}_\delta(\omega';A_1^{p'_1},\ldots,A_n^{p'_n}) \leqslant \mathfrak{G}_\delta(\omega;A_1^{p_1},\ldots,A_n^{p_n}) \leqslant A_n^{q_n} \quad \text{for all } p'_i \geqslant p_i \text{ and } i=1,\ldots,n,$$

where $\widehat{\omega} = \left(\frac{w_1}{p_1-q_1},\ldots,\frac{w_n}{p_n-q_n}\right)$, $\widehat{\omega'} = \left(\frac{w_1}{p'_1-q_1},\ldots,\frac{w_n}{p'_n-q_n}\right)$, $\omega = \frac{\widehat{\omega}}{\|\widehat{\omega}\|_1}$ *and* $\omega' = \frac{\widehat{\omega'}}{\|\widehat{\omega'}\|_1}$.

Proof. Applying Theorem 3.A to (3.4), we have

$$\mathfrak{G}_\delta(\omega; A_n^{\frac{-q_n}{2}} A_1^{p_1} A_n^{\frac{-q_n}{2}}, \dots, A_n^{\frac{-q_n}{2}} A_{n-1}^{p_{n-1}} A_n^{\frac{-q_n}{2}}, A_n^{p_n-q_n}) \leqslant I,$$

so that by (3.3),

$$X_0 \equiv \mathfrak{G}_\delta(\omega; A_1^{p_1}, \dots, A_{n-1}^{p_{n-1}}, A_n^{p_n}) \leqslant A_n^{q_n} \leqslant A_1^{q_1}. \tag{3.5}$$

By applying Theorem 3.2 to (3.5) and by (3.3),

$$X_1 \equiv \mathfrak{G}_\delta(\omega_1; A_1^{p_1'}, A_2^{p_2}, \dots, A_n^{p_n}) \leqslant X_0 \leqslant A_n^{q_n} \leqslant A_2^{q_2} \tag{3.6}$$

for $p_1' \geqslant p_1$, where $\widehat{\omega}_1 = \left(\frac{w_1}{p_1'-q_1}, \frac{w_2}{p_2-q_2}, \dots, \frac{w_n}{p_n-q_n}\right)$ and $\omega_1 = \frac{\widehat{\omega}_1}{\|\widehat{\omega}_1\|_1}$. By applying Theorem 3.2 to (3.6) and by (3.3),

$$X_2 \equiv \mathfrak{G}_\delta(\omega_2; A_1^{p_1'}, A_2^{p_2'}, A_3^{p_3}, \dots, A_n^{p_n}) \leqslant X_1 \leqslant X_0 \leqslant A_n^{q_n} \leqslant A_3^{q_3}$$

for $p_1' \geqslant p_1$ and $p_2' \geqslant p_2$, where $\widehat{\omega}_2 = \left(\frac{w_1}{p_1'-q_1}, \frac{w_2}{p_2'-q_2}, \frac{w_3}{p_3-q_3}, \dots, \frac{w_n}{p_n-q_n}\right)$ and $\omega_2 = \frac{\widehat{\omega}_2}{\|\widehat{\omega}_2\|_1}$. By repeating this argument, we can get

$$X_n \equiv \mathfrak{G}_\delta(\omega'; A_1^{p_1'}, \dots, A_n^{p_n'}) \leqslant X_{n-1} \leqslant X_0 \leqslant A_n^{q_n}$$

for $p_i' \geqslant p_i$ for $i = 1, \dots, n$, where $\widehat{\omega'} = \widehat{\omega}_n = \left(\frac{w_1}{p_1'-q_1}, \dots, \frac{w_n}{p_n'-q_n}\right)$. □

REMARK. (i) in Theorem 1.E, that is, $\log A_1 + \cdots + \log A_n \leqslant 0$ holds if and only if

$$\frac{1}{p_1}\log A_1^{p_1} + \cdots + \frac{1}{p_n}\log A_n^{p_n} \leqslant 0 \quad \text{for every } p_i > 0 \text{ and } i = 1, \dots, n.$$

Therefore we recognize that Theorem 3.3 implies (i) $\Longrightarrow$ (iii) in Theorem 1.E by putting $q_1 = \cdots = q_n = 0$ and $w_1 = \cdots = w_n = 1$ since

$$\frac{\frac{1}{p_i}}{\|\widehat{\omega}\|_1} = \frac{\frac{1}{p_i}}{\frac{1}{p_1} + \cdots + \frac{1}{p_n}} = \frac{p_{\neq i}}{\sum_j p_{\neq j}} \quad \text{for } i = 1, \dots, n$$

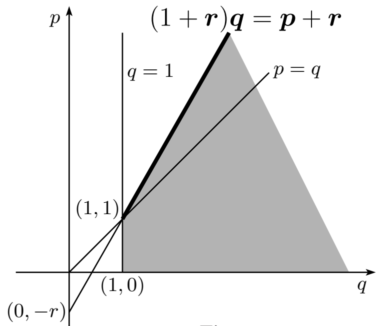
ensures $\omega = \frac{\widehat{\omega}}{\|\widehat{\omega}\|_1} = \left(\frac{\frac{1}{p_1}}{\|\widehat{\omega}\|_1}, \dots, \frac{\frac{1}{p_n}}{\|\widehat{\omega}\|_1}\right) = \left(\frac{p_{\neq 1}}{\sum_i p_{\neq i}}, \dots, \frac{p_{\neq n}}{\sum_i p_{\neq i}}\right).$

4. Furuta inequality

Furuta inequality [10] (see also [6, 11, 13, 17, 20]) has the following original form.

THEOREM 4.A.
(Original form of Furuta inequality [10])

If $A \geqslant B \geqslant 0$*, then for each* $r \geqslant 0$*,*
(i) $\quad (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$
and
(ii) $\quad (A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$
hold for $p \geqslant 0$ *and* $q \geqslant 1$ *with* $(1+r)q \geqslant p+r$.



Figure

We remark that Theorem 4.A implies Löwner-Heinz theorem "$A \geqslant B \geqslant 0$ *ensures* $A^{\alpha} \geqslant B^{\alpha}$ *for any* $\alpha \in [0,1]$" by putting $r = 0$. By Löwner-Heinz theorem, we recognize that the essence of Theorem 4.A is the case that $p \geqslant 1$ and $q = \frac{p+r}{1+r}$ (cf. Theorem 1.A). We can interpret Theorem 1.A as a consequence of monotonicity of an operator function.

THEOREM 4.B. ([7]) *Let* $A \geqslant B \geqslant 0$ *with* $A > 0$*. Then*

$$f(p,r) = A^{\frac{-r}{2}}(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}}A^{\frac{-r}{2}} = A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \tag{4.1}$$

is decreasing for $p \geqslant 1$ *and* $r \geqslant 0$.

In fact, Theorem 4.B ensures Theorem 1.A since $A \geqslant B \geqslant 0$ with $A > 0$ implies $f(p,r) \leqslant f(1,0) = B \leqslant A$ for $p \geqslant 1$ and $r \geqslant 0$.

REMARK. Similarly to Theorem 4.B, we can easily get monotonicity of $\mathfrak{G}_{\delta}(\omega;A_1^{p_1},\ldots,A_n^{p_n})$ corresponding to Theorems 3.1, 3.2 and 3.3, respectively.
It is well known that we have a variant from Theorem 1.A by replacing A,B with A^q,B^q and p,r with $\frac{p}{q},\frac{r}{q}$ in Theorem 1.A respectively.

THEOREM 4.C. ([8]) *Let* $A > 0$*,* $B \geqslant 0$ *and* $q > 0$*. Then*

$$A^q \geqslant B^q \quad \textit{implies} \quad A^{-r} \sharp_{\frac{q+r}{p+r}} B^p \leqslant B^q \leqslant A^q \ \textit{for}\ p \geqslant q\ \textit{and}\ r \geqslant 0.$$

Here we show that Theorem 3.3 is a generalization of Furuta inequality via weighted Riemannian mean of n-matrices. Precisely, we show that Theorem 3.3 ensures the following Theorem 4.1 and Theorem 4.1 is a generalization of Theorem 4.C.

THEOREM 4.1. *Let* $A_1,\ldots,A_n \in P_m(\mathbb{C})$ *and* $q > 0$*. Then* $A_i^q \geqslant A_n^q > 0$ *for* $i = 1,\ldots,n-1$ *implies*

$$\mathfrak{G}_{\delta}(\omega;A_1^{-p_1},\ldots,A_{n-1}^{-p_{n-1}},A_n^{p_n}) \leqslant A_n^q \leqslant A_i^q \tag{4.2}$$

for all $p_i \geqslant 0$, $i = 1, \dots, n-1$ *and* $p_n > q$, *where* $\widehat{\omega} = \left(\frac{1}{p_1+q}, \dots, \frac{1}{p_{n-1}+q}, \frac{n-1}{p_n-q}\right)$ *and* $\omega = \frac{\widehat{\omega}}{\|\widehat{\omega}\|_1}$.

Proof. Assume that $A_i^q \geqslant A_n^q > 0$ for $q > 0$ and $i = 1, \dots, n-1$. Then $A_i^q \geqslant A_n^q > 0$ implies $\log A_i \geqslant \log A_n$. By (i) $\Longrightarrow$ (iii) in Theorem 1.C, $\log A_i \geqslant \log A_n$ implies $A_i^{-p_i} \sharp_{\frac{p_i}{q+p_i}} A_n^q \leqslant I$ for all $p_i \geqslant 0$. This is equivalent to $A_n^{-q} \sharp_{\frac{q}{q+p_i}} A_i^{p_i} \geqslant I$, that is, $(A_n^{\frac{q}{2}} A_i^{p_i} A_n^{\frac{q}{2}})^{\frac{q}{p_i+q}} \geqslant A_n^q$. By taking logarithm, we have $\frac{1}{p_i+q} \log A_n^{\frac{q}{2}} A_i^{p_i} A_n^{\frac{q}{2}} \geqslant \frac{1}{p_n-q} \log A_n^{p_n-q}$, that is,

$$\frac{1}{p_i+q} \log A_n^{\frac{-q}{2}} (A_i^{-1})^{p_i} A_n^{\frac{-q}{2}} + \frac{1}{p_n-q} \log A_n^{p_n-q} \leqslant 0 \tag{4.3}$$

for all $p_i \geqslant 0$, $i = 1, \dots, n-1$ and $p_n > q$. Summing up (4.3) for $i = 1, \dots, n-1$, we have

$$\begin{aligned} &\frac{1}{p_1+q} \log A_n^{\frac{-q}{2}} (A_1^{-1})^{p_1} A_n^{\frac{-q}{2}} + \cdots \\ &\quad + \frac{1}{p_{n-1}+q} \log A_n^{\frac{-q}{2}} (A_{n-1}^{-1})^{p_{n-1}} A_n^{\frac{-q}{2}} + \frac{n-1}{p_n-q} \log A_n^{p_n-q} \leqslant 0. \end{aligned} \tag{4.4}$$

By applying Theorem 3.3 to $(A_i^{-1})^{-q} \geqslant A_n^q > 0$ and (4.4), we can obtain

$$\mathfrak{G}_\delta(\omega; A_1^{-p_1}, \dots, A_{n-1}^{-p_{n-1}}, A_n^{p_n}) \leqslant A_n^q \leqslant A_i^q$$

for all $p_i \geqslant 0 > -q$, $i = 1, \dots, n-1$ and $p_n > q$. $\square$

Proof of Theorem 4.C. Put $n = 2$, $p_1 = r$ and $p_2 = p$ in Theorem 4.1. Then $\widehat{\omega} = \left(\frac{1}{r+q}, \frac{1}{p-q}\right)$ and $\omega = \left(\frac{p-q}{p+r}, \frac{q+r}{p+r}\right)$. Therefore we obtain the desired result. $\square$

5. Remarks on (3.4) in Theorem 3.3

Here we discuss the following inequality (3.4) in Theorem 3.3.

$$\begin{aligned} &\frac{w_1}{p_1-q_1} \log A_n^{\frac{-q_n}{2}} A_1^{p_1} A_n^{\frac{-q_n}{2}} + \cdots \\ &\quad + \frac{w_{n-1}}{p_{n-1}-q_{n-1}} \log A_n^{\frac{-q_n}{2}} A_{n-1}^{p_{n-1}} A_n^{\frac{-q_n}{2}} + \frac{w_n}{p_n-q_n} \log A_n^{p_n-q_n} \leqslant 0. \end{aligned} \tag{3.4}$$

Firstly, we obtain monotonicity of left hand side of (3.4).

PROPOSITION 5.1. *Let* $A_1, \dots, A_n \in P_m(\mathbb{C})$, $q_1, \dots, q_n \in \mathbb{R}$ *and* $w_1, \dots, w_n \geqslant 0$. *If* $A_i^{q_i} \geqslant A_n^{q_n} > 0$ *for* $i = 1, \dots, n-1$, *then*

$$\begin{aligned} F(p_1, \dots, p_{n-1}) = &\frac{w_1}{p_1-q_1} \log A_n^{\frac{-q_n}{2}} A_1^{p_1} A_n^{\frac{-q_n}{2}} + \cdots \\ &+ \frac{w_{n-1}}{p_{n-1}-q_{n-1}} \log A_n^{\frac{-q_n}{2}} A_{n-1}^{p_{n-1}} A_n^{\frac{-q_n}{2}} + \frac{w_n}{p_n-q_n} \log A_n^{p_n-q_n} \end{aligned}$$

is decreasing for $p_1 > q_1, \dots, p_{n-1} > q_{n-1}$.

Proposition 5.1 is immediately shown by the following Proposition 5.2.

PROPOSITION 5.2. *Let* $A,B>0$ *and* $q,r\in\mathbb{R}$. *If* $A^q \geqslant B^r > 0$, *then*

$$F(p)=\frac{1}{p-q}\log B^{\frac{-r}{2}}A^pB^{\frac{-r}{2}} \quad \textit{is decreasing for } p>q.$$

Proof. By Hansen's inequality [14], we easily obtain that $T^*T\geqslant I$ ensures

$$(T^*ST)^\alpha \leqslant T^*S^\alpha T \ \text{ for } S\geqslant 0 \text{ and } \alpha\in[0,1]. \tag{5.1}$$

Put $T=A^{\frac{q}{2}}B^{\frac{-r}{2}}$ and $S=A^{p'-q}$. Then by (5.1),

$$\begin{aligned} F(p') &= \frac{1}{p'-q}\log B^{\frac{-r}{2}}A^{p'}B^{\frac{-r}{2}} = \log(B^{\frac{-r}{2}}A^{\frac{q}{2}}A^{p'-q}A^{\frac{q}{2}}B^{\frac{-r}{2}})^{\frac{p-q}{p'-q}\cdot\frac{1}{p-q}} \\ &\leqslant \log(B^{\frac{-r}{2}}A^{\frac{q}{2}}A^{p-q}A^{\frac{q}{2}}B^{\frac{-r}{2}})^{\frac{1}{p-q}} = \frac{1}{p-q}\log B^{\frac{-r}{2}}A^pB^{\frac{-r}{2}} = F(p) \end{aligned}$$

for $p'\geqslant p>q$. □

Put $p_i=q_i+\alpha$ for $\alpha>0$ and $i=1,\dots,n$ in (3.4). Then

$$\frac{w_1}{\alpha}\log A_n^{\frac{-q_n}{2}}A_1^{q_1+\alpha}A_n^{\frac{-q_n}{2}}+\cdots+\frac{w_{n-1}}{\alpha}\log A_n^{\frac{-q_n}{2}}A_{n-1}^{q_{n-1}+\alpha}A_n^{\frac{-q_n}{2}}+\frac{w_n}{\alpha}\log A_n^\alpha\leqslant 0,$$

that is,

$$w_1\log A_n^{\frac{-q_n}{2}}A_1^{q_1+\alpha}A_n^{\frac{-q_n}{2}}+\cdots+w_{n-1}\log A_n^{\frac{-q_n}{2}}A_{n-1}^{q_{n-1}+\alpha}A_n^{\frac{-q_n}{2}}+w_n\log A_n^\alpha\leqslant 0, \tag{5.2}$$

Let $\alpha\to+0$ in (5.2). Then we have

$$w_1\log A_n^{\frac{-q_n}{2}}A_1^{q_1}A_n^{\frac{-q_n}{2}}+\cdots+w_{n-1}\log A_n^{\frac{-q_n}{2}}A_{n-1}^{q_{n-1}}A_n^{\frac{-q_n}{2}}\leqslant 0. \tag{5.3}$$

We have the following proposition on (5.3).

PROPOSITION 5.3. *Let* $A_1,\dots,A_n\in P_m(\mathbb{C})$ *and* $w_1,\dots,w_{n-1}>0$. *If*

$$A_i^{q_i}\geqslant A_n^{q_n}>0 \tag{5.4}$$

and

$$w_1\log A_n^{\frac{-q_n}{2}}A_1^{q_1}A_n^{\frac{-q_n}{2}}+\cdots+w_{n-1}\log A_n^{\frac{-q_n}{2}}A_{n-1}^{q_{n-1}}A_n^{\frac{-q_n}{2}}\leqslant 0 \tag{5.3}$$

hold for $q_i\in\mathbb{R}$ *and* $i=1,\dots,n$, *then* $A_i^{q_i}=A_n^{q_n}$ *for* $i=1,\dots,n-1$.

Proof. (5.4) is equivalent to

$$\log A_n^{\frac{-q_n}{2}}A_i^{q_i}A_n^{\frac{-q_n}{2}}\geqslant 0 \quad \text{for } i=1,\dots,n-1,$$

so we get $\log A_n^{\frac{-q_n}{2}}A_i^{q_i}A_n^{\frac{-q_n}{2}}=0$, that is, $A_i^{q_i}=A_n^{q_n}$ by (5.3). □

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REFERENCES

[1] T. ANDO AND F. HIAI, *Log majorization and complementary Golden-Thompson type inequalities*, Linear Algebra Appl., **197, 198** (1994), 113–131.
[2] T. ANDO, C. K. LI AND R. MATHIAS, *Geometric means*, Linear Algebra Appl., **385** (2004), 305–334.
[3] R. BHATIA, *Positive definite matrices*, Princeton Series in Applied Mathematics, Princeton University Press, Princeton, NJ, 2007.
[4] R. BHATIA AND J. HOLBROOK, *Riemannian geometry and matrix geometric means*, Linear Algebra Appl., **413** (2006), 594–618.
[5] D. A. BINI, B. MEINI AND F. POLONI, *An effective matrix geometric mean satisfying the Ando-Li-Mathias properties*, Math. Comp., **79** (2010), 437–452.
[6] M. FUJII, *Furuta's inequality and its mean theoretic approach*, J. Operator Theory, **23** (1990), 67–72.
[7] M. FUJII, T. FURUTA AND E. KAMEI, *Furuta's inequality and its application to Ando's theorem*, Linear Algebra Appl., **179** (1993), 161–169.
[8] M. FUJII, J. F. JIANG AND E. KAMEI, *A characterization of orders defined by $A^\delta \geqslant B^\delta$ via Furuta inequality*, Math. Japon., **45** (1997), 519–525.
[9] M. FUJII AND E. KAMEI, *Ando-Hiai inequality and Furuta inequality*, Linear Algebra Appl., **416** (2006), 541–545.
[10] T. FURUTA, *$A \geqslant B \geqslant 0$ assures $(B^rA^pB^r)^{1/q} \geqslant B^{(p+2r)/q}$ for $r \geqslant 0,\ p \geqslant 0,\ q \geqslant 1$ with $(1+2r)q \geqslant p+2r$*, Proc. Amer. Math. Soc., **101** (1987), 85–88.
[11] T. FURUTA, *An elementary proof of an order preserving inequality*, Proc. Japan Acad. Ser. A Math. Sci., **65** (1989), 126.
[12] T. FURUTA, *Applications of order preserving operator inequalities*, Oper. Theory Adv. Appl., **59** (1992), 180–190.
[13] T. FURUTA, *Invitation to Linear Operators*, Taylor & Francis, London, 2001.
[14] F. HANSEN, *An operator inequality*, Math. Ann. **246** (1979/80), 249–250.
[15] S. IZUMINO AND N. NAKAMURA, *Weighted geometric means of positive operators*, Kyungpook Math. J., **50** (2010), 213–228.
[16] C. JUNG, H. LEE, Y. LIM AND T. YAMAZAKI, *Weighted geometric mean of n-operators with n-parameters*, Linear Algebra Appl. **432** (2010), 1515–1530.
[17] E. KAMEI, *A satellite to Furuta's inequality*, Math. Japon., **33** (1988), 883–886.
[18] J. D. LAWSON AND Y. LIM, *Monotonic properties of the least squares mean*, Math. Ann., **351** (2011), 267–279.
[19] M. MOAKHER, *A differential geometric approach to the geometric mean of symmetric positive-definite matrices*, SIAM J. Matrix Anal. Appl., **26** (2005), 735–747.
[20] K. TANAHASHI, *Best possibility of the Furuta inequality*, Proc. Amer. Math. Soc., **124** (1996), 141–146.
[21] T. YAMAZAKI, *The Riemannian mean and matrix inequalities related to the Ando-Hiai inequality and chaotic order*, to appear in Oper. Matrices.

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