Journal of
Mathematical
Inequalities
Volume 6, Number 3 (2012), 493–500 doi:10.7153/jmi-06-46

REFINEMENTS AND SHARPNESS OF SOME NEW HUYGENS TYPE INEQUALITIES

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(*Communicated by E. Neuman*)

Abstract. In the article, some Huygens inequalities involving trigonometric and hyperbolic functions are refined and sharpened.

1. Introduction

The famous Huygens inequality[7] for the sine and tangent functions states that for $x \in \left(0, \frac{\pi}{2}\right)$

$$2\sin x + \tan x > 3x. \tag{1.1}$$

The hyperbolic counterpart of (1.1) was established in [6] as follows: For $x > 0$

$$2\sinh x + \tanh x > 3x. \tag{1.2}$$

The inequalities (1.1) and (1.2) were respectively refined in [6, Theorem 2.6] as

$$2\frac{\sin x}{x} + \frac{\tan x}{x} > 2\frac{x}{\sin x} + \frac{x}{\tan x} > 3, \quad 0 < x < \frac{\pi}{2}, \tag{1.3}$$

and

$$2\frac{\sinh x}{x} + \frac{\tanh x}{x} > 2\frac{x}{\sinh x} + \frac{x}{\tanh x} > 3, \quad x \neq 0. \tag{1.4}$$

In [4] the inequality (1.2) was improved as

$$2\frac{\sinh x}{x} + \frac{\tanh x}{x} > 3 + \frac{3}{20}x^4 - \frac{3}{56}x^6, \quad x > 0. \tag{1.5}$$

In [9], Wilker proved

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2. \tag{1.6}$$

and proposed that there exists a largest constant c such that

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + cx^3 \tan x, \tag{1.7}$$

Mathematics subject classification (2010): Primary 26D05; Secondary 33B10.
Keywords and phrases: Refinement, sharpening, Huygens inequality, Wilker inequality, trigonometric function, hyperbolic function.

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holds for $0 < x < \frac{\pi}{2}$.

In [8], the best constant c in (1.7) was found and it was proved that

$$2+\frac{8}{45}x^3\tan x > \left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x} > 2+\left(\frac{2}{\pi}\right)^4 x^3\tan x, \tag{1.8}$$

for $0 < x < \frac{\pi}{2}$. The constants $\frac{8}{45}$ and $\left(\frac{2}{\pi}\right)^4$ in the inequality (1.8) are the best possible.

Recently the inequalities (1.3) and (1.4) were respectively refined in [5] as

$$2\frac{\sin x}{x}+\frac{\tan x}{x} > \frac{\sin x}{x}+2\frac{\tan(x/2)}{x/2} > 2\frac{x}{\sin x}+\frac{x}{\tan x} > 3. \tag{1.9}$$

and

$$2\frac{\sinh x}{x}+\frac{\tanh x}{x} > \frac{\sinh x}{x}+2\frac{\tanh(x/2)}{x/2} > 2\frac{x}{\sinh x}+\frac{x}{\tanh x} > 3. \tag{1.10}$$

Inspired by (1.8), Jiang et al. [15] first proved

$$3+\frac{1}{60}x^3\sin x < 2\frac{x}{\sin x}+\frac{x}{\tan x} < 3+\frac{8\pi-24}{\pi^3}x^3\sin x. \tag{1.11}$$

for $0 < |x| < \frac{\pi}{2}$. The constants $\frac{1}{60}$ and $\frac{8\pi-24}{\pi^3}$ in (1.11) are the best possible.

Recently, Chen and Sándor [14] proved that

$$3+\frac{3}{20}x^3\tan x < 2\left(\frac{\sin x}{x}\right)+\frac{\tan x}{x} < 3+\left(\frac{2}{\pi}\right)^4 x^3\tan x.$$

for $0 < |x| < \frac{\pi}{2}$. The constants $\frac{3}{20}$ and $\left(\frac{2}{\pi}\right)^4$ are the best possible.

The aim of this paper is to refine and sharpen some of the above-mentioned Huygens type inequalities in (1.9) and (1.10).

2. Some Lemmas

In order to attain our aim, we need several lemmas below.

LEMMA 2.1. *The Bernoulli numbers B_{2n} for $n \in \mathbb{N}$ have the property*

$$(-1)^{n-1}B_{2n} = |B_{2n}|, \tag{2.1}$$

where the Bernoulli numbers B_i for $i \geqslant 0$ are defined by

$$\frac{x}{e^x-1} = \sum_{i=0}^{\infty}\frac{B_i}{n!}x^i = 1-\frac{x}{2}+\sum_{i=1}^{\infty}B_{2i}\frac{x^{2i}}{(2i)!}, \quad |x| < 2\pi. \tag{2.2}$$

Proof. In [2, p. 16 and p. 56], it is listed that for $q \geqslant 1$

$$\zeta(2q) = (-1)^{q-1}\frac{(2\pi)^{2q}}{(2q)!}\frac{B_{2q}}{2}, \tag{2.3}$$

where ζ is the Riemann zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}.$$

In [16, p.18, theorem 3.4], the following formula was given

$$\sum_{n=1}^{\infty}\frac{1}{n^{2q}} = \frac{2^{2q-1}\pi^{2q}|B_{2q}|}{(2q)!}. \tag{2.4}$$

From (2.3) and (2.4), the formula (2.1) follows. □

LEMMA 2.2. [12, 13] *Let B_{2n} be the even-indexed Bernoulli numbers. Then*

$$\frac{2(2n)!}{(2\pi)^{2n}}\frac{1}{1-2^{-2n}} < |B_{2n}| < \frac{2(2n)!}{(2\pi)^{2n}}\frac{1}{1-2^{1-2n}}, n = 1,2,3,\cdots.$$

LEMMA 2.3. *For $0 < |x| < \pi$, we have*

$$\frac{x}{\sin x} = 1 + \sum_{n=1}^{\infty}\frac{2\left(2^{2n-1}-1\right)|B_{2n}|}{(2n)!}x^{2n}. \tag{2.5}$$

Proof. This is an easy consequence of combining the equality

$$\frac{1}{\sin x} = \csc x = \frac{1}{x} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}2\left(2^{2n-1}-1\right)B_{2n}}{(2n)!}x^{2n-1}, \quad |x| < \pi. \tag{2.6}$$

see [1, p. 75, 4.3.68], with Lemma 2.1. □

LEMMA 2.4. ([1, p. 75, 4.3.70]) *For $0 < |x| < \pi$,*

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty}\frac{2^{2n}|B_{2n}|}{(2n)!}x^{2n-1}. \tag{2.7}$$

LEMMA 2.5. *For $0 < |x| < \pi$,*

$$\frac{1}{\sin^2 x} = \frac{1}{x^2} + \sum_{n=1}^{\infty}\frac{2^{2n}(2n-1)|B_{2n}|}{(2n)!}x^{2(n-1)}. \tag{2.8}$$

Proof. Since

$$\frac{1}{\sin^2 x} = \csc^2 x = -\frac{\mathrm{d}}{\mathrm{d}x}(\cot x),$$

the formula (2.8) follows from differentiating (2.7). □

LEMMA 2.6. *For* $0 < |x| < \pi$,

$$\frac{\cos x}{\sin^2 x} = \frac{1}{x^2} - \sum_{n=1}^{\infty} \frac{2(2n-1)\left(2^{2n-1}-1\right)|B_{2n}|}{(2n)!} x^{2(n-1)}. \tag{2.9}$$

Proof. This follows from differentiating on both sides of (2.6) and using (2.1). □

LEMMA 2.7. [17, 3, 11] *Let* a_n *and* $b_n (n = 0, 1, 2, \cdots)$ *be real numbers, and let the power series* $A(t) = \sum_{n=0}^{\infty} a_n t^n$ *and* $B(t) = \sum_{n=0}^{\infty} b_n t^n$ *be convergent for* $|t| < R$. *If* $b_n > 0$ *for* $n = 0, 1, 2, \cdots$, *and if* $\frac{a_n}{b_n}$ *is strictly increasing (or decreasing) for* $n = 0, 1, 2, \cdots$, *then the function* $\frac{A(t)}{B(t)}$ *is strictly increasing (or decreasing) on* $(0, R)$.

3. Main results

Now we are in a position to state and prove our main results.

THEOREM 1. *For* $0 < |x| < \frac{\pi}{2}$, *we have*

$$3 + \frac{1}{40} x^3 \sin x < \frac{\sin x}{x} + 2\frac{\tan x/2}{x/2} < 3 + \frac{80 - 24\pi}{\pi^4} x^3 \sin x. \tag{3.1}$$

The constants $\frac{1}{40}$ *and* $\frac{80-24\pi}{\pi^4}$ *in* (3.1) *are the best possible.*

Proof. Let

$$\begin{aligned} f(x) =& \frac{\frac{\sin x}{x} + 2\frac{\tan x/2}{x/2} - 3}{x^3 \sin x} \\ =& \frac{\sin^2 x + 4(1 - \cos x) - 3x \sin x}{x^4 \sin^2 x} \\ =& \frac{1}{x^4}\left(1 + \frac{4}{\sin^2 x} - \frac{4\cos x}{\sin^2 x} - \frac{3x}{\sin x}\right) \end{aligned}$$

for $x \in \left(0, \frac{\pi}{2}\right)$. By virtue of (2.5), (2.8), and (2.9), we have

$$\begin{aligned} f(x) =& \frac{1}{x^4}\Bigg[1 + \frac{4}{x^2} + \sum_{n=1}^{\infty} \frac{4(2n-1)2^{2n}}{(2n)!}|B_{2n}|x^{2n-2} \\ & - \frac{4}{x^2} + \sum_{n=1}^{\infty} \frac{8\left(2^{2n-1}-1\right)(2n-1)}{(2n)!}|B_{2n}|x^{2n-2} \\ & - 3 - \sum_{n=1}^{\infty} \frac{6(2^{2n}-1)}{(2n)!}|B_{2n}|x^{2n}\Bigg] \\ =& \frac{1}{x^4}\left[\sum_{n=1}^{\infty} \frac{8(2n-1)(2^{2n}-1)}{(2n)!}|B_{2n}|x^{2n-2} - \sum_{n=1}^{\infty} \frac{6(2^{2n-1}-1)}{(2n)!}|B_{2n}|x^{2n} - 2\right] \end{aligned}$$

$$=\frac{1}{x^4}\left[\sum_{n=2}^{\infty}\frac{8(2n-1)(2^{2n}-1)}{(2n)!}|B_{2n}|x^{2n-2}-\sum_{n=1}^{\infty}\frac{6(2^{2n-1}-1)}{(2n)!}|B_{2n}|x^{2n}\right]$$
$$=\frac{1}{x^4}\left[\sum_{n=1}^{\infty}\frac{8(2n+1)(2^{2n+2}-1)}{(2n)!}|B_{2n+2}|x^{2n}-\sum_{n=1}^{\infty}\frac{6(2^{2n-1}-1)}{(2n)!}|B_{2n}|x^{2n}\right]$$
$$=\sum_{n=2}^{\infty}\left[\frac{8(2n+1)(2^{2n+2}-1)}{(2n+2)!}|B_{2n+2}|-\frac{6(2^{2n-1}-1)}{(2n)!}|B_{2n}|\right]x^{2n-4}.$$

Let $a_n=\frac{8(2n+1)(2^{2n+2}-1)}{(2n+2)!}|B_{2n+2}|-\frac{6(2^{2n-1}-1)}{(2n)!}|B_{2n}|$ for $n\geqslant 2$.

By a simple computation, we have $a_2=\frac{1}{40}$.

Furthermore, when $n\geqslant 3$, From Lemma 2.2 one can get

$$\begin{aligned}a_n=&\frac{8(2n+1)(2^{2n+2}-1)}{(2n+2)!}|B_{2n+2}|-\frac{6(2^{2n-1}-1)}{(2n)!}|B_{2n}|\\ >&\frac{8(2n+1)(2^{2n+2}-1)}{(2n+2)!}\frac{2(2n+2)!}{(2\pi)^{2n+2}}\frac{1}{1-2^{-2n-2}}\\ &-\frac{6(2^{2n-1}-1)}{(2n)!}\frac{2(2n)!}{(2\pi)^{2n}}\frac{1}{1-2^{1-2n}}\\ =&\frac{2}{\pi^{2n}}\left[\frac{8(2n+1)}{\pi^2}-3\right]>0.\end{aligned}$$

So the function $f(x)$ is strictly increasing on $\left(0,\frac{\pi}{2}\right)$. Moreover, it is easy to obtain

$$\lim_{x\to 0^+}f(x)=a_2=\frac{1}{40}\quad\text{and}\quad\lim_{x\to(\pi/2)^-}f(x)=\frac{80-24\pi}{\pi^4}.$$

The proof of Theorem 1 is complete. □

REMARK 3.1. Since $f(x)$ is an odd function we conclude that Theorem 1 holds for all x which satisfy $0<|x|<\frac{\pi}{2}$.

THEOREM 2. *For $x\neq 0$, we have*

$$3+\frac{3}{20}x^3\tanh x<2\frac{\sinh x}{x}+\frac{\tanh x}{x}<3+\frac{3}{20}x^3\sinh x. \tag{3.2}$$

The constant $\frac{3}{20}$ is the best possible.

Proof. Without loss of generality, we assume that $x>0$.

We firstly prove the first inequality of (3.2).

Consider the function $F(x)$ defined by

$$F(x)=\frac{\frac{2\sinh x}{x}+\frac{\tanh x}{x}-3}{x^3\tanh x}=\frac{\sinh 2x+\sinh x-3x\cosh x}{x^4\sinh x}.$$

and let

$$f(x) = \sinh 2x + \sinh x - 3x\cosh x \quad \text{and} \quad g(x) = x^4 \sinh x.$$

From the power series expansions

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \quad \text{and} \quad \cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, \tag{3.3}$$

it follows that

$$\begin{aligned} f(x) &= \sinh 2x + \sinh x - 3x\cosh x \\ &= \sum_{n=0}^{\infty} \frac{2^{2n+1}x^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} - \sum_{n=0}^{\infty} \frac{3x^{2n+1}}{(2n)!} \\ &= \sum_{n=2}^{\infty} \frac{\left(2^{2n+1} - 6n - 2\right)x^{2n+1}}{(2n+1)!} \\ &\triangleq \sum_{n=2}^{\infty} a_n x^{2n+1} \end{aligned}$$

and

$$\begin{aligned} g(x) &= \sum_{n=0}^{\infty} \frac{x^{2n+5}}{(2n+1)!} \\ &= \sum_{n=2}^{\infty} \frac{4n(n-1)\left(4n^2-1\right)x^{2n+1}}{(2n+1)!} \\ &\triangleq \sum_{n=2}^{\infty} b_n x^{2n+1}. \end{aligned}$$

It is easy to see that the quotient

$$c_n = \frac{a_n}{b_n} = \frac{2^{2n+1} - 6n - 2}{4n(n-1)\left(4n^2-1\right)}$$

satisfies

$$c_{n+1} - c_n = \frac{\left(6n^2 - 17n + 1\right)4^n + 18n^2 + 23n - 1}{2n(2n+3)\left(4n^2-1\right)\left(n^2-1\right)} > 0$$

for $n \geqslant 2$. This means that the sequence c_n is increasing. By Lemma 2.7, the function $F(x)$ is increasing on $(0,\infty)$. Moreover, it is not difficult to obtain $\lim_{x\to 0^+} F(x) = c_2 = \frac{3}{20}$. Therefore, the first inequality in (3.2) holds.

Finally, we prove the second inequality of (3.2).

Define a function $G(x)$ by

$$G(x) = \frac{\frac{2\sinh x}{x} + \frac{\tanh x}{x} - 3}{x^3 \sinh x} = \frac{\sinh 2x + \sinh x - 3x\cosh x}{x^4 \sinh x \cosh x}$$

And let

$$f(x) = \sinh 2x + \sinh x - 3x\cosh x \quad \text{and} \quad g(x) = x^4 \sinh x \cosh x.$$

By using (3.3) one get

$$\begin{aligned} f(x) &= \sinh 2x + \sinh x - 3x\cosh x \\ &= \sum_{n=0}^{\infty} \frac{2^{2n+1}x^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} - \sum_{n=0}^{\infty} \frac{3x^{2n+1}}{(2n)!} \\ &= \sum_{n=2}^{\infty} \frac{\left(2^{2n+1} - 6n - 2\right)x^{2n+1}}{(2n+1)!} \\ &\triangleq \sum_{n=2}^{\infty} a_n x^{2n+1} \end{aligned}$$

and

$$\begin{aligned} g(x) &= \sum_{n=0}^{\infty} \frac{2^{2n}x^{2n+5}}{(2n+1)!} \\ &= \sum_{n=2}^{\infty} \frac{4n(n-1)\left(4n^2-1\right)2^{2n-4}x^{2n+1}}{(2n+1)!} \\ &\triangleq \sum_{n=2}^{\infty} b_n x^{2n+1}. \end{aligned}$$

Let

$$c_n = \frac{a_n}{b_n} = \frac{2^{2n+1} - 6n - 2}{4n(n-1)\left(4n^2-1\right)2^{2n-4}}$$

satisfies $c_2 = \frac{3}{20}$. Furthermore, when $n \geqslant 2$, by a simple computation, we have

$$c_{n+1} - c_n = -\frac{2[8(4n+1)4^n - (18n^3 + 69n^2 + 65n + 8)]}{n(2n+3)\left(4n^2-1\right)(n^2-1)4^n},$$

for $n \geqslant 2$.

Since

$$\begin{aligned} &8(4n+1)4^n - (18n^3 + 69n^2 + 65n + 8) \\ &> 32n^2(4n+1) - (18n^3 + 69n^2 + 65n + 8) \\ &= 110n^3 - 37n^2 - 65n - 8 \\ &= 110n(n-2)^2 + 403n(n-2) + 301(n-2) + 594 > 0. \end{aligned}$$

This means that the sequence c_n is decreasing. By Lemma 2.7, the function $G(x)$ is decreasing on $(0, +\infty)$. Moreover, it is not difficult to obtain $\lim_{x \to 0^+} G(x) = c_2 = \frac{3}{20}$.

This completes the proof of Theorem 2 . □

REMARK 3.2. Since $F(x)$ and $G(x)$ both are odd function, we conclude that Theorem 2 holds for all $x \neq 0$.

Acknowledgements

We would like to express our cordial thanks to the Professor E. Neuman for giving useful advice and comments to the first version of the paper. This work was supported by the Project of ShanDong Province Higher Educational Science and Technology Program under grant No. J11LA57.

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(Received January 23, 2012)

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