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A REMARK ON SOME ACCURATE ESTIMATES OF π

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Abstract. In this paper we correct and generalize some inequalities related to an accurate asymptotic series of π. A recurrence for the coefficients of this asymptotic series involving the Euler numbers is established as well.

1. Introduction

It is well-known that the constant π can be expressed as an infinite product due to Wallis [6]

$$\frac{\pi}{2} = \frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{6\cdot 6}{5\cdot 7}\cdots = \prod_{k\geqslant 1}\frac{(2k)(2k)}{(2k-1)(2k+1)}.$$

If we put

$$s_0 := 1,\quad s_n := \prod_{k=1}^{n}\frac{(2k)(2k)}{(2k-1)(2k+1)},\quad n\geqslant 1,$$

then the Wallis product can be written as $\lim_{n\to+\infty} s_n = \pi/2$. However, the convergence of s_n is very slow, so it is not suitable for approximating π. Based on the following asymptotic expansion of Fields [2]

$$\frac{\Gamma(x+a)}{\Gamma(x+b)} \sim (x+a-\rho)^{a-b}\sum_{k\geqslant 0}\frac{\Gamma(b-a+2k)\,B_{2k}^{(2\rho)}(\rho)}{\Gamma(b-a)\,(2k)!\,(x+a-\rho)^{2k}},\quad 2\rho = a-b+1,$$

as $x\to+\infty$, Mortici [4] claims to have established the result

$$4\frac{2n+1}{4n+1}s_n\left(\sum_{k=0}^{4}\binom{4k}{2k}\frac{B_{2k}^{(1/2)}(1/4)}{(4n+1)^{2k}}\right)^2 < \pi < 4\frac{2n+1}{4n+1}s_n\left(\sum_{k=0}^{5}\binom{4k}{2k}\frac{B_{2k}^{(1/2)}(1/4)}{(4n+1)^{2k}}\right)^2$$

for $n\geqslant 1$. Here Γ denotes the Gamma function and $B_k^{(\kappa)}(\lambda)$ is the kth Nörlund polynomial, defined by

$$\left(\frac{x}{e^x-1}\right)^{\kappa} e^{\lambda x} = \sum_{k\geqslant 0} B_k^{(\kappa)}(\lambda)\frac{x^k}{k!},\quad |x|<2\pi.$$

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Numerical calculation easily shows these bounds to be incorrect, *e.g.*, for $n = 1$. The failure comes from the misunderstanding of the monotonicity properties of some sequences in his proof. The correct inequalities are

$$4\frac{2n+1}{4n+1}s_n\left(\sum_{k=0}^{5}\binom{4k}{2k}\frac{B_{2k}^{(1/2)}(1/4)}{(4n+1)^{2k}}\right)^2 < \pi < 4\frac{2n+1}{4n+1}s_n\left(\sum_{k=0}^{4}\binom{4k}{2k}\frac{B_{2k}^{(1/2)}(1/4)}{(4n+1)^{2k}}\right)^2$$

for $n \geqslant 1$, in the reverse order than Mortici expected. This will follow from a more general formula that we will derive as a consequence of a theorem by Frenzen. We will also give a recurrence for the coefficients $\binom{4k}{2k}B_{2k}^{(1/2)}(1/4)$.

2. The main results

THEOREM 2.1. *For all integers $n, N \geqslant 0$, the following bounds hold*

$$4\frac{2n+1}{4n+1}s_n\left(\sum_{k=0}^{2N+1}\binom{4k}{2k}\frac{B_{2k}^{(1/2)}(1/4)}{(4n+1)^{2k}}\right)^2 < \pi < 4\frac{2n+1}{4n+1}s_n\left(\sum_{k=0}^{2N}\binom{4k}{2k}\frac{B_{2k}^{(1/2)}(1/4)}{(4n+1)^{2k}}\right)^2.$$

If $N = 2$, we get back the corrected version of Mortici's bounds, even for $n = 0$. The table below shows the accuracy of the lower and upper bounds for various values of n and N. All listed digits of the approximations are correct, expect for those underlined.

n	N	lower bound	upper bound
10	2	3.1415926535897628070423307	3.1415926535909571508605035
10	5	3.1415926535897932384591445	3.1415926535897932384918964
15	2	3.1415926535897929746560479	3.1415926535898154462691564
15	5	3.1415926535897932384626431	3.1415926535897932384626483
20	2	3.1415926535897932296210783	3.1415926535897945480896845
20	5	3.1415926535897932384626433	3.1415926535897932384626433

The Euler numbers can be defined by the exponential generating function

$$\frac{2}{e^x + e^{-x}} = \sum_{k \geqslant 0} E_k \frac{x^k}{k!}, \quad |x| < \frac{\pi}{2}.$$

The coefficients $\binom{4k}{2k}B_{2k}^{(1/2)}(1/4)$ in Theorem 2.1 have a nice relation with these numbers.

THEOREM 2.2. *The sequence $\binom{4k}{2k}B_{2k}^{(1/2)}(1/4)$ satisfies the recurrence relation*

$$\binom{0}{0}B_0^{(1/2)}\left(\frac{1}{4}\right) = 1$$

and

$$\binom{4k+4}{2k+2}B_{2k+2}^{(1/2)}\left(\frac{1}{4}\right) = \frac{1}{4k+4}\sum_{j=0}^{k}\binom{4j}{2j}B_{2j}^{(1/2)}\left(\frac{1}{4}\right)E_{2k-2j+2} \tag{2.1}$$

for $k \geqslant 0$.

3. The Proof of the Theorems

Proof of Theorem 2.1. Frenzen [3] showed that if $x+\min[a,(a+b-1)/2]>0$ and $0<a-b+1<1$, then

$$\frac{\Gamma(x+a)}{\Gamma(x+b)}=(x+a-\rho)^{a-b}\left(\sum_{k=0}^{M-1}\frac{\Gamma(b-a+2k)\,B_{2k}^{(2\rho)}(\rho)}{\Gamma(b-a)\,(2k)!\,(x+a-\rho)^{2k}}+R_M\right)$$

where

$$R_M:=\theta_M\frac{\Gamma(b-a+2M)\,B_{2M}^{(2\rho)}(\rho)}{\Gamma(b-a)\,(2M)!\,(x+a-\rho)^{2M}}$$

and $0<\theta_M<1$, $M\geqslant 0$. If we put $x=n, a=1/2, b=1$ and $M=2N+1$ into this expression we obtain

$$\frac{\Gamma(n+1/2)}{\Gamma(n+1)}=\left(n+\frac{1}{4}\right)^{-1/2}\left(\sum_{k=0}^{2N}\binom{4k}{2k}\frac{B_{2k}^{(1/2)}(1/4)}{(4n+1)^{2k}}+\theta_{2N+1}\binom{8N+4}{4N+2}\frac{B_{4N+2}^{(1/2)}(1/4)}{(4n+1)^{4N+2}}\right) \tag{3.1}$$

for every integers $n,N\geqslant 0$. From the recurrence (see, *e.g.*, [2])

$$B_{2k+2}^{(2\kappa)}(\kappa)=(-2\kappa)\sum_{j=0}^{k}\frac{1}{2j+2}\binom{2k+1}{2j+1}B_{2j+2}B_{2k-2j}^{(2\kappa)}(\kappa)$$

we have

$$B_{2k+2}^{(1/2)}\left(\frac{1}{4}\right)=-\frac{1}{2}\sum_{j=0}^{k}\frac{1}{2j+2}\binom{2k+1}{2j+1}B_{2j+2}B_{2k-2j}^{(1/2)}\left(\frac{1}{4}\right). \tag{3.2}$$

Here $B_j=B_j^{(1)}(0)$ denotes the jth Bernoulli number. For the first two terms we find $B_0^{(1/2)}(1/4)=1>0$, $B_2^{(1/2)}(1/4)=-1/24<0$. Using the fact that

$$B_k<0,\ \ k\equiv 0 \bmod 4,$$
$$B_k>0,\ \ k\equiv 2 \bmod 4$$

and the recurrence (3.2), it follows by induction that

$$B_k^{(1/2)}(1/4)>0,\ \ k\equiv 0 \bmod 4,$$
$$B_k^{(1/2)}(1/4)<0,\ \ k\equiv 2 \bmod 4.$$

And hence by (3.1)

$$\sum_{k=0}^{2N+1}\binom{4k}{2k}\frac{B_{2k}^{(1/2)}(1/4)}{(4n+1)^{2k}}<\left(n+\frac{1}{4}\right)^{1/2}\frac{\Gamma(n+1/2)}{\Gamma(n+1)}<\sum_{k=0}^{2N}\binom{4k}{2k}\frac{B_{2k}^{(1/2)}(1/4)}{(4n+1)^{2k}}$$

for every integers $n,N\geqslant 0$. By taking squares, then multiplying through by $4\frac{2n+1}{4n+1}s_n$ and noting that

$$4\frac{2n+1}{4n+1}s_n\left(n+\frac{1}{4}\right)\frac{\Gamma^2(n+1/2)}{\Gamma^2(n+1)}=(2n+1)\prod_{k=1}^{n}\frac{(2k)(2k)}{(2k-1)(2k+1)}\frac{\Gamma^2(n+1/2)}{\Gamma^2(n+1)}$$

$$=\frac{(2n)!!^2}{(2n-1)!!^2}\frac{\frac{(2n-1)!!^2}{2^{2n}}\pi}{n!^2}=\pi,$$

we conclude the proof of the theorem. □

Proof of Theorem 2.2. It is known that if $0\leqslant h\leqslant 1$ and $x\to+\infty$,

$$\log\Gamma(x+h)\sim\left(x+h-\frac{1}{2}\right)\log x-x+\frac{1}{2}\log(2\pi)-\sum_{k\geqslant 1}\frac{(-1)^k B_{k+1}(h)}{k(k+1)x^k}$$

where $B_k(\lambda)=B_k^{(1)}(\lambda)$ is the kth Bernoulli polynomial [5, p. 141]. The substitution $x=n+1/4$ and $h=1/4$ gives

$$\log\Gamma\left(n+\frac{1}{2}\right)\sim n\log\left(n+\frac{1}{4}\right)-\left(n+\frac{1}{4}\right)+\frac{1}{2}\log(2\pi)-\sum_{k\geqslant 1}\frac{(-1)^k 4^k B_{k+1}(1/4)}{k(k+1)(4n+1)^k}. \tag{3.3}$$

If we put $x=n+1/4$ and $h=3/4$, we obtain

$$\log\Gamma(n+1)\sim\left(n+\frac{1}{2}\right)\log\left(n+\frac{1}{4}\right)-\left(n+\frac{1}{4}\right)+\frac{1}{2}\log(2\pi)-\sum_{k\geqslant 1}\frac{(-1)^k 4^k B_{k+1}(3/4)}{k(k+1)(4n+1)^k}. \tag{3.4}$$

By subtracting (3.4) from (3.3) we get

$$\log\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}\sim-\frac{1}{2}\log\left(n+\frac{1}{4}\right)+\sum_{k\geqslant 1}\frac{(-1)^k 4^k}{k(k+1)(4n+1)^k}\left(B_{k+1}\left(\frac{3}{4}\right)-B_{k+1}\left(\frac{1}{4}\right)\right)$$

$$=-\frac{1}{2}\log\left(n+\frac{1}{4}\right)+\sum_{k\geqslant 1}\frac{(-1)^k 4^k}{k(k+1)(4n+1)^k}\left((-1)^{k+1}B_{k+1}\left(\frac{1}{4}\right)-B_{k+1}\left(\frac{1}{4}\right)\right)$$

$$=-\frac{1}{2}\log\left(n+\frac{1}{4}\right)-\sum_{k\geqslant 1}\frac{4^{2k}}{k(2k+1)(4n+1)^{2k}}B_{2k+1}\left(\frac{1}{4}\right)$$

$$=-\frac{1}{2}\log\left(n+\frac{1}{4}\right)+\sum_{k\geqslant 1}\frac{E_{2k}}{4k(4n+1)^{2k}}.$$

Here we have used the identities

$$B_k(\lambda)=(-1)^k B_k(1-\lambda),$$

$$B_{2k+1}\left(\frac{1}{4}\right)=-\frac{2k+1}{4^{2k+1}}E_{2k}$$

(see, *e.g.*, [1, pp. 804–806]). Taking the exponential of each side we have

$$\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n+1\right)} \sim \left(n+\frac{1}{4}\right)^{-1/2} \exp\left(\sum_{k\geqslant 1}\frac{E_{2k}}{4k\left(4n+1\right)^{2k}}\right) \sim \left(n+\frac{1}{4}\right)^{-1/2} \sum_{k\geqslant 0}\frac{b_k}{\left(4n+1\right)^{2k}}$$

for some rational coefficients b_k. On the other hand from (3.1) the limit case $N \to +\infty$ gives

$$\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n+1\right)} \sim \left(n+\frac{1}{4}\right)^{-1/2} \sum_{k\geqslant 0}\binom{4k}{2k}\frac{B_{2k}^{(1/2)}\left(1/4\right)}{\left(4n+1\right)^{2k}}.$$

From the uniqueness theorem on asymptotic series it follows that $b_k = \binom{4k}{2k}B_{2k}^{(1/2)}\left(1/4\right)$ for all $k \geqslant 0$, and hence we have derived the formal generating function

$$\exp\left(\sum_{k\geqslant 1}\frac{E_{2k}}{4k}x^k\right) = \sum_{k\geqslant 0}\binom{4k}{2k}B_{2k}^{(1/2)}\left(\frac{1}{4}\right)x^k.$$

Differentiating both sides with respect to x yields

$$\left(\sum_{k\geqslant 1}\frac{E_{2k}}{4}x^{k-1}\right)\exp\left(\sum_{k\geqslant 1}\frac{E_{2k}}{4k}x^k\right) = \sum_{k\geqslant 1}\binom{4k}{2k}kB_{2k}^{(1/2)}\left(\frac{1}{4}\right)x^{k-1}$$

$$\left(\sum_{k\geqslant 0}\frac{E_{2k+2}}{4}x^k\right)\left(\sum_{k\geqslant 0}\binom{4k}{2k}B_{2k}^{(1/2)}\left(\frac{1}{4}\right)x^k\right) = \sum_{k\geqslant 0}\binom{4k+4}{2k+2}\left(k+1\right)B_{2k+2}^{(1/2)}\left(\frac{1}{4}\right)x^k.$$

Expanding the product and comparing the coefficients of the powers of x we deduce (2.1). □

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