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GENERALIZED COMPOSITION OPERATOR FROM BLOCH–TYPE SPACES TO MIXED–NORM SPACE ON THE UNIT BALL

LI ZHANG AND ZE-HUA ZHOU

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Abstract. Let $H(\mathbb{B})$ be the space of all holomorphic functions on the unit ball $\mathbb{B}$ in $\mathbb{C}^N$, and $S(\mathbb{B})$ the collection of all holomorphic self-maps of $\mathbb{B}$. Let $\varphi \in S(\mathbb{B})$ and $g \in H(\mathbb{B})$ with $g(0) = 0$, the generalized composition operator is defined by

$$C_\varphi^g(f)(z) = \int_0^1 \Re f(\varphi(tz))g(tz)\frac{dt}{t},$$

Here, we characterize the boundedness and compactness of the generalized composition operator acting from Bloch-type spaces $\mathscr{B}_\omega$ and $\mathscr{B}_{\omega,0}$ to mixed-norm space $H(p,q,\phi)$ on the unit ball $\mathbb{B}$.

1. Introduction

Let $H(\mathbb{B})$ be the class of all holomorphic functions on $\mathbb{B}$, $S(\mathbb{B})$ the collection of all holomorphic self-maps of $\mathbb{B}$, where $\mathbb{B}$ is the unit ball in the n-dimensional complex space $\mathbb{C}^n$. Let $d\sigma$ be the normalized rotation invariant measure on the boundary $S = \partial\mathbb{B}$ of $\mathbb{B}$.

For $f \in H(\mathbb{B})$, let

$$\Re f(z) = \langle \nabla f, \bar{z} \rangle = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z)$$

be the radial derivative of f, $\nabla f = (\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n})$. If $f \in H(\mathbb{B})$ with the Taylor expansion

$$f(z) = \sum_{|\beta| \geqslant 0} a_\beta z^\beta,$$

then

$$\Re f(z) = \sum_{|\beta| \geqslant 0} |\beta| a_\beta z^\beta,$$

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where $|\beta| = \beta_1 + ... + \beta_n$, and $z^\beta = z_1^{\beta_1}...z_n^{\beta_n}$.

A positive continuous function μ on $[0,1)$ is called normal (see [20]), if there exist positive numbers s and t, $0 < s < t$, and $\delta \in [0,1)$ such that

(i) $\frac{\mu(r)}{(1-r)^s}$ is decreasing on $[\delta,1)$, $\lim\limits_{r\to 1}\frac{\mu(r)}{(1-r)^s} \to 0$;

(ii) $\frac{\mu(r)}{(1-r)^t}$ is increasing on $[\delta,1)$, $\lim\limits_{r\to 1}\frac{\mu(r)}{(1-r)^t} \to \infty$.

A normal function ω can be extended on whole $\mathbb{B}$ by $\omega(z) = \omega(|z|)$. We recall that the Bloch-type space $\mathscr{B}_\omega$ consists of all $f \in H(\mathbb{B})$ such that

$$\mathscr{B}_\omega(f) = \sup_{z\in\mathbb{B}} \omega(z)|\Re f(z)| < \infty. \tag{1.1}$$

The expression $\mathscr{B}_\omega(f)$ defines a semi-norm while the natural norm is given by

$$\|f\|_{\mathscr{B}_\omega} = |f(0)| + \mathscr{B}_\omega(f).$$

This norm makes $\mathscr{B}_\omega$ into a Banach space.

Let $\mathscr{B}_{\omega,0}$ denote the subspace of $\mathscr{B}_\omega$ consisting of those $f \in \mathscr{B}_\omega$ for which

$$\lim_{|z|\to 1} \omega(z)|\Re f(z)| = 0.$$

This space is called the little Bloch-type space.

In [21], the author showed that $\|f\|_\omega$ is equivalent to $|f(0)| + \sup\limits_{z\in\mathbb{B}} \omega(z)|\nabla f(z)|$, and $f \in \mathscr{B}_{\omega,0}$ if and only if $\lim\limits_{|z|\to 1} \omega(z)|\nabla f(z)| = 0$. For particular case where $\omega(r) = (1-r^2)^\alpha$ see, for example, paper [2].

It is well-known that $\mathscr{B}_{\omega,0}$ is a closed subspace of $\mathscr{B}_\omega$. When $\omega(r) = (1-r^2)^\alpha$, the induced spaces $\mathscr{B}_\omega$ and $\mathscr{B}_{\omega,0}$ are the α-Bloch space $\mathscr{B}^\alpha$ and the little α-Bloch space $\mathscr{B}_0^\alpha$. In particular, for $\alpha = 1$, $\mathscr{B}^1$ and $\mathscr{B}_0^1$ are the classical Bloch space $\mathscr{B}$ and the classical little Bloch space $\mathscr{B}_0$.

From now on if we say that a function $\phi : \mathbb{B} \to [0,\infty)$ is normal we will also assume that it is radial on $\mathbb{B}$, that is, $\phi(z) = \phi(|z|), z \in \mathbb{B}$. In the rest of this paper we always assume that ω and ϕ are normal on $[0,1)$.

For $0 < p,q < \infty$ and ϕ normal, the mixed-norm space $H(p,q,\phi)$ in $\mathbb{B}$, consists of all functions $f \in H(\mathbb{B})$ such that

$$\|f\|_{H(p,q,\phi)}^p = \int_0^1 M_q^p(f,r)\frac{\phi^p(r)}{1-r}dr < \infty, \tag{1.2}$$

where

$$M_q(f,r) = \left(\int_S |f(r\xi)|^q d\sigma(\xi)\right)^{1/q}.$$

For $p = q$ and $\phi(r) = (1-r^2)^{(\alpha+1)/p}, \alpha > -1$, the mixed-norm space is equivalent with the weighted Bergman space A_α^p. In particular, $H(p,q,\phi)$ is equivalent to the Bergman space L_a^p if $0 < p = q < \infty$ and $\phi(r) = (1-r)^{1/p}$.

Every $\varphi \in S(\mathbb{B})$ induces a composition operator C_φ defined by $C_\varphi f = f \circ \varphi$ for $f \in H(\mathbb{B})$, $z \in \mathbb{B}$. It is of interest to provide function-theoretic characterizations for when φ induces a bounded or compact composition operator on various spaces. For some results on composition operators, see, e.g. [1] and the references therein.

Let $g \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$, where $\mathbb{D}$ is the unit disk of $\mathbb{C}$. The generalized composition operator is defined by

$$(C_\varphi^g f)(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi$$

for $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$. When $g = \varphi'$, we see that this operator is essentially composition operator, since the following difference $C_\varphi^g - C_\varphi$ is a constant. Therefore, C_φ^g is a generalization of the composition operator C_φ. The generalized composition operator was introduced in [4] and [9]. For related results and operators, see, e.g., [5, 6, 7] and the references therein.

Let $\varphi \in S(\mathbb{B})$ and $g \in H(\mathbb{B})$ with $g(0) = 0$. The following operator, so called, generalized composition operator on the unit ball

$$(C_\varphi^g f)(z) = \int_0^1 \Re f(\varphi(tz))g(tz)\frac{dt}{t}, \quad f \in H(\mathbb{B}),\ z \in \mathbb{B}.$$

was recently introduced by S. Stević and X. Zhu and studied in [8, 11, 12, 15, 16, 17, 18, 19, 22, 23, 24, 25]. S. Stević [9] characterized the equivalent conditions about the boundedness and compactness of C_φ^g acting from logarithmic Bloch-type space to the mixed-norm space on the unit ball and obtained some conditions about C_φ^g acting from the mixed-norm space to weighted Bloch space on the disk or unit ball in [10, 12], respectively. For a natural counterpart of operator C_φ^g see operator P_φ^g defined in [13].

The purpose of this paper is to characterize the boundedness and compactness of the generalized composition operator C_φ^g from the ω-Bloch spaces to the mixed-norms pace $H(p,q,\phi)$.

Throughout the rest of this paper, C will denote a positive constant, the exact value of which will vary from one appearance to the next. The notation $A \asymp B$ means that there is a positive constant C such that $B/C \leqslant A \leqslant CB$.

2. Auxiliary results

Several auxiliary results, which are used in the proofs of the main results, are quoted in this section.

The proof of the following lemma was given by Stević in [11, 15].

LEMMA 2.1. *Assume that $\varphi \in S(\mathbb{B})$, and $g \in H(\mathbb{B})$ with $g(0) = 0$. Then for every $f \in H(\mathbb{B})$ it holds*

$$\Re C_\varphi^g(f)(z) = \Re f(\varphi(z))g(z).$$

LEMMA 2.2. ([3, Theorem 2]) *Assume that* $0 < p,q < \infty$, ϕ *is normal and* $m \in N$. *Then the following asymptotic relationship holds*

$$\int_0^1 M_q(f,r)\frac{\phi^p(r)}{1-r}dr \asymp \sum_{j=0}^{m-1}|grad_j f(0)| + \left(\int_0^1 M_q^p(\Re^m f,r)(1-r)^{mp}\frac{\phi^p(r)}{1-r}dr\right)^{1/p} \tag{2.1}$$

for every $f \in H(\mathbb{B})$, *where* $|grad_j f| = \sum\limits_{|\alpha|=j}|\frac{\partial^\alpha f}{\partial z^\alpha}|$.

The following lemma can be proved in a standard way (see, for example, Proposition 3.11 in [1]), we omit its proof.

LEMMA 2.3. *Assume that* $\varphi \in S(\mathbb{B})$, *and* $g \in H(\mathbb{B})$ *with* $g(0) = 0$. *Then* $C_\varphi^g : \mathscr{B}_\omega \to H(p,q,\phi)$ *is compact if and only if* $C_\varphi^g : \mathscr{B}_\omega \to H(p,q,\phi)$ *is bounded and for any bounded sequence* $\{f_k\}_{k\in\mathbb{N}}$ *in* $\mathscr{B}_\omega$ *which converges to zero uniformly on compact subsets of* $\mathbb{B}$ *as* $k \to \infty$, *we have* $\|C_\varphi^g f_k\|_{H(p,q,\phi)} \to 0$ *as* $k \to \infty$.

Lemma 2.3 also holds if space $\mathscr{B}_\omega$ is replaced by $\mathscr{B}_{\omega,0}$.

The following lemma was obtained by Stević in [17].

LEMMA 2.4. ([17, Lemma 6]) *Let* ω *be a normal weight, then there are* $N \in \mathbb{N}$, $r_0 \in (0,1)$ *and functions* $f_1,\cdots,f_N \in \mathscr{B}_\omega$ *such that*

$$|\Re f_1(z)| + |\Re f_2(z)| + \cdots + |\Re f_N(z)| \geqslant \frac{1}{\omega(z)}$$

for $r_0 \leqslant |z| < 1$.

The next lemma is well-known.

LEMMA 2.5. *For* $0 < p < \infty$, *there is a positive constant* C_p *depending on* p *and* n, *such that* $\left(\sum\limits_{i=1}^n x_i\right)^p \leqslant C_p\left(\sum\limits_{i=1}^n x_i^p\right)$, *when* $x_i \in (0,\infty), i \in \{1,2,\cdots,n\}$.

3. Main results

In this section, we formulate and prove our main results. We use and modify some ideas from papers [14, 16, 17].

THEOREM 3.1. *Assume that* $\varphi \in S(\mathbb{B})$, *and* $g \in H(\mathbb{B})$ *with* $g(0) = 0$. *Then the following statements are equivalent.*

(a) $C_\varphi^g : \mathscr{B}_\omega \to H(p,q,\phi)$ *is bounded.*

(b) $C_\varphi^g : \mathscr{B}_{\omega,0} \to H(p,q,\phi)$ *is bounded.*

(c)

$$\int_0^1\left(\int_S\left(\frac{|\varphi(r\xi)g(r\xi)|}{\omega(\varphi(r\xi))}\right)^q d\sigma(\xi)\right)^{p/q}\frac{\phi^p(r)}{(1-r)^{1-p}}dr < \infty. \tag{3.1}$$

Proof. First note that the implication (a)$\Rightarrow$(b) is trivial.

(b)$\Rightarrow$(c). Assume that (b) holds. Then for $f \in \mathscr{B}_{\omega,0}$, by Lemma 2.1 and Lemma 2.2 with $m=1$, we obtain

$$\begin{aligned}
\|C_\varphi^g f\|_{H(p,q,\phi)}^p &= \int_0^1 M_q^p(C_\varphi^g f, r)\frac{\phi^p(r)}{1-r}dr \\
&\asymp \int_0^1 M_q^p(\Re(C_\varphi^g f), r)\frac{\phi^p(r)}{(1-r)^{1-p}}dr \\
&= \int_0^1 M_q^p(g\Re f\circ\varphi, r)\frac{\phi^p(r)}{(1-r)^{1-p}}dr \\
&= \int_0^1 \left(\int_S |\Re f(\varphi(r\xi))g(r\xi)|^q d\sigma(\xi)\right)^{p/q}\frac{\phi^p(r)}{(1-r)^{1-p}}dr.
\end{aligned}$$

Then by Lemma 2.4, using the dilation functions

$$(f_j)_l(z) = f_j(lz) \in \mathscr{B}_{\omega,0}, \quad j = 1,\cdots,N,$$

where $l > r_0 > 0$. When $|lz| > r_0$, we have $\sum_{j=1}^{N} |(\Re f_j)_l(z)| > \frac{1}{\omega(lz)}$, thus according to Lemma 2.5, we have

$$\begin{aligned}
\infty &> \sum_{j=1}^N \|C_\varphi^g f_j\|_{H(p,q,\phi)}^p \geqslant \sum_{j=1}^N \|C_\varphi^g((f_j)_l)\|_{H(p,q,\phi)}^p \\
&\asymp \sum_{j=1}^N \int_0^1 \left(\int_S |(\Re f_j)_l(\varphi(r\xi))g(r\xi)|^q d\sigma(\xi)\right)^{p/q}\frac{\phi^p(r)}{(1-r)^{1-p}}dr \\
&= \int_0^1 \left[\sum_{j=1}^N \left(\int_S |(\Re f_j)_l(\varphi(r\xi))g(r\xi)|^q d\sigma(\xi)\right)^{p/q}\right]\frac{\phi^p(r)}{(1-r)^{1-p}}dr \\
&\geqslant C\int_0^1 \left(\sum_{j=1}^N \int_S |(\Re f_j)_l(\varphi(r\xi))g(r\xi)|^q d\sigma(\xi)\right)^{p/q}\frac{\phi^p(r)}{(1-r)^{1-p}}dr \\
&= C\int_0^1 \left(\int_S \sum_{j=1}^N |(\Re f_j)_l(\varphi(r\xi))|^q |g(r\xi)|^q d\sigma(\xi)\right)^{p/q}\frac{\phi^p(r)}{(1-r)^{1-p}}dr \\
&\geqslant C\int_0^1 \left(\int_S (\sum_{j=1}^N |(\Re f_j)_l(\varphi(r\xi))|)^q |g(r\xi)|^q d\sigma(\xi)\right)^{p/q}\frac{\phi^p(r)}{(1-r)^{1-p}}dr \\
&\geqslant C\int_0^1 \left(\int_{E_1} (\sum_{j=1}^N |(\Re f_j)_l(\varphi(r\xi))|)^q |g(r\xi)|^q d\sigma(\xi)\right)^{p/q}\frac{\phi^p(r)}{(1-r)^{1-p}}dr \\
&\geqslant C\int_0^1 \left(\int_{E_1} \left(\frac{|g(r\xi)|}{\omega(l\varphi(r\xi))}\right)^q d\sigma(\xi)\right)^{p/q}\frac{\phi^p(r)}{(1-r)^{1-p}}dr \\
&\geqslant C\int_0^1 \left(\int_{E_1} \left(\frac{|l\varphi(r\xi)g(r\xi)|}{\omega(l\varphi(r\xi))}\right)^q d\sigma(\xi)\right)^{p/q}\frac{\phi^p(r)}{(1-r)^{1-p}}dr,
\end{aligned}$$

where $E_1 = \{\xi \in S : r_0 < |l\varphi(r\xi)| < 1\}$. From the inequality above and by the monotone convergence theorem we get

$$\int_0^1 \left(\int_{|\varphi(r\xi)|>r_0} \left(\frac{|\varphi(r\xi)g(r\xi)|}{\omega(\varphi(r\xi))}\right)^q d\sigma(\xi)\right)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr < \infty.$$

For the test functions $g_j(z) = z_j \in \mathscr{B}_{\omega,0}$, $j = 1, \cdots, n$. By Lemma 2.5 and $|z|^2 \leqslant (|z_1| + \cdots + |z_n|)^2$, we obtain

$$\begin{aligned}
&\int_0^1 \left(\int_{|\varphi(r\xi)|\leqslant r_0} \left(\frac{|\varphi(r\xi)g(r\xi)|}{\omega(\varphi(r\xi))}\right)^q d\sigma(\xi)\right)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr \\
&\leqslant \int_0^1 \left(\int_{|\varphi(r\xi)|\leqslant r_0} \left(\sum_{j=1}^n \frac{|\varphi_j(r\xi)g(r\xi)|}{\omega(\varphi(r\xi))}\right)^q d\sigma(\xi)\right)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr \\
&\leqslant C \sum_{j=1}^n \int_0^1 \left(\int_{|\varphi(r\xi)|\leqslant r_0} \left(\frac{|\varphi_j(r\xi)g(r\xi)|}{\omega(\varphi(r\xi))}\right)^q d\sigma(\xi)\right)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr \\
&\leqslant C \max_{0\leqslant t\leqslant r_0} \frac{1}{\omega(t)} \sum_{j=1}^n \int_0^1 \left(\int_{|\varphi(r\xi)|\leqslant r_0} |\varphi_j(r\xi)g(r\xi)|^q d\sigma(\xi)\right)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr \\
&\leqslant C \sum_{j=1}^n \|C_\varphi^g f_j\|_{H(p,q,\phi)}.
\end{aligned}$$

So, it follows from Lemma 2.5 that

$$\begin{aligned}
&\int_0^1 \left(\int_S \left(\frac{|\varphi(r\xi)g(r\xi)|}{\omega(\varphi(r\xi))}\right)^q d\sigma(\xi)\right)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr \\
&= \int_0^1 \left[\left(\int_{|\varphi(r\xi)|\leqslant r_0} + \int_{|\varphi(r\xi)|>r_0}\right) \left(\frac{|\varphi(r\xi)g(r\xi)|}{\omega(\varphi(r\xi))}\right)^q d\sigma(\xi)\right]^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr \\
&\leqslant C \int_0^1 \left(\int_{|\varphi(r\xi)|\leqslant r_0} \left(\frac{|\varphi(r\xi)g(r\xi)|}{\omega(\varphi(r\xi))}\right)^q d\sigma(\xi)\right)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr \\
&\quad + C \int_0^1 \left(\int_{|\varphi(r\xi)|\leqslant r_0} \left(\frac{|\varphi(r\xi)g(r\xi)|}{\omega(\varphi(r\xi))}\right)^q d\sigma(\xi)\right)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr < \infty.
\end{aligned}$$

Thus the condition (c) holds.

Next we prove (c)$\Rightarrow$(a). Assume (c) holds. Then for any $f \in \mathscr{B}_\omega$, by the former calculation and the equivalence of norm on $\mathscr{B}_\omega$, we obtain

$$\begin{aligned}
&\|C_\varphi^g f\|_{H(p,q,\phi)}^p \\
&\asymp \int_0^1 \left(\int_S |\Re f(\varphi(r\xi))g(r\xi)|^q d\sigma(\xi)\right)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr \\
&\leqslant C \int_0^1 \left(\int_S \left(\frac{|\varphi(r\xi)g(r\xi)|}{\omega(\varphi(r\xi))}\right)^q (\omega(\varphi(r\xi))|\nabla f(\varphi(r\xi))|)^q d\sigma(\xi)\right)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr \\
&\leqslant C\|f\|_\omega^p.
\end{aligned}$$

Then we get (a). This completes the proof of this theorem. □

Next we prove the necessary and sufficient conditions for the compactness of C_φ^g.

THEOREM 3.2. *Assume that* $\varphi \in S(\mathbb{B})$, *and* $g \in H(\mathbb{B})$ *with* $g(0)=0$. *Then the following statements are equivalent.*

(d) $C_\varphi^g : \mathscr{B}_\omega \to H(p,q,\phi)$ *is compact.*

(e) $C_\varphi^g : \mathscr{B}_{\omega,0} \to H(p,q,\phi)$ *is compact.*

(f) (3.1) *holds.*

Proof. The implication (d) $\Rightarrow$(e) is obvious.

(e)$\Rightarrow$(f). Note that the compactness of $C_\varphi^g : \mathscr{B}_{\omega,0} \to H(p,q,\phi)$ implies the boundedness of $C_\varphi^g : \mathscr{B}_{\omega,0} \to H(p,q,\phi)$, the desired result (3.1) follows by Theorem 3.1.

Next we prove (f)$\Rightarrow$(d). Assume that $\{f_k\}_{k\in\mathbb{N}}$ is a bounded sequence in $\mathscr{B}_\omega$, say by L, and $f_k \to 0$ uniformly on compact subsets of $\mathbb{B}$ as $k\to\infty$.

From (3.1) we have that for any $\varepsilon>0$, there exists a constant $\delta\in(0,1)$, such that

$$\int_\delta^1 \left(\int_S \Big(\frac{|\varphi(r\xi)g(r\xi)|}{\omega(\varphi(r\xi))}\Big)^q d\sigma(\xi)\right)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}}dr < \varepsilon^p. \tag{3.2}$$

Let $\delta\mathbb{B}=\{w\in\mathbb{B}, |w|\leqslant\delta\}$. By Lemma 2.1 and Lemma 2.2 with $m=1$ we obtain

$$\begin{aligned}
\|C_\varphi^g f_k\|_{H(p,q,\phi)}^p &= \int_0^1 M_q^p(C_\varphi^g f_k, r)\frac{\phi^p(r)}{1-r}dr \\
&\asymp \int_0^1 \left(\int_S |\Re f_k(\varphi(r\xi))g(r\xi)|^q d\sigma(\xi)\right)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}}dr \\
&= \int_0^\delta \left(\int_S |\Re f_k(\varphi(r\xi))g(r\xi)|^q d\sigma(\xi)\right)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}}dr \\
&\quad + \int_\delta^1 \left(\int_S |\Re f_k(\varphi(r\xi))g(r\xi)|^q d\sigma(\xi)\right)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}}dr \\
&= J_1 + J_2.
\end{aligned}$$

First, we estimate J_1, by (3.1), we have

$$\int_0^\delta \left(\int_S \Big(\frac{|\varphi(r\xi)g(r\xi)|}{\omega(\varphi(r\xi))}\Big)^q d\sigma(\xi)\right)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}}dr < C.$$

then

$$\begin{aligned}
J_1 := &\int_0^\delta \left(\int_S |\Re f_k(\varphi(r\xi))g(r\xi)|^q d\sigma(\xi)\right)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}}dr \\
\leqslant &\max_{|w|\leqslant\delta} \omega^p(\varphi(w)) \sup_{w\in\delta\mathbb{B}} |\nabla f_k(\varphi(w))|^p \\
&\times \int_0^\delta \left(\int_S \Big(\frac{|\varphi(r\xi)g(r\xi)|}{\omega(\varphi(r\xi))}\Big)^q d\sigma(\xi)\right)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}}dr \\
\leqslant &C \sup_{w\in\delta\mathbb{B}} |\nabla f_k(\varphi(w))|^p.
\end{aligned}$$

Since $\overline{\varphi(\delta\mathbb{B})}$ is a compact subset of $\mathbb{B}$, and $f_k \to 0$ uniformly on the compact subset of $\mathbb{B}$ as $k \to \infty$, so $\frac{\partial f}{\partial z_k} \to 0$ uniformly on the compact subset of $\mathbb{B}$ as $k \to \infty$, we have

$$\lim_{k\to\infty} \sup_{w\in\delta\mathbb{B}} |\nabla f_k(\varphi(w))|^p = 0,$$

then

$$\lim_{k\to\infty} J_1 \leqslant C \lim_{k\to\infty} \sup_{w\in\delta\mathbb{B}} |\nabla f_k(\varphi(w))|^p = 0. \tag{3.3}$$

On the other hand by (3.2) and $\sup\limits_{k\in N} \|f_k\|_\omega \leqslant L$, we have

$$\begin{aligned} J_2 := &\int_\delta^1 \left(\int_S |\Re f_k(\varphi(r\xi))g(r\xi)|^q d\sigma(\xi) \right)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr \\ &\leqslant C \int_\delta^1 \left(\int_S \left(\frac{|\varphi(r\xi)g(r\xi)|}{\omega(\varphi(r\xi))} \right)^q d\sigma(\xi) \right)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr \left(\sup_{z\in\mathbb{B}} \omega(z)|\nabla f(z)| \right)^p \\ &\leqslant C\varepsilon^p \|f_k\|_\omega^p \leqslant CL^p\varepsilon^p. \end{aligned}$$

Then

$$\lim_{k\to\infty} \|C_\varphi^g f_k\|_{H(p,q,\phi)}^p \asymp \lim_{k\to\infty} (J_1 + J_2) \leqslant CL^p\varepsilon^p. \tag{3.4}$$

Since ε is an arbitrary positive number, from (3.4) we obtain

$$\lim_{k\to\infty} \|C_\varphi^g f_k\|_{H(p,q,\phi)}^p = 0.$$

Using Lemma 2.3 we get (f). The proof of this theorem is complete. □

4. Other two operators

If we use the radial derivative of some function k to instead of g in operators C_φ^g, or let $\varphi(z) = z$, we can get the following three operators:

$$(L_\varphi^k f)(z) = \int_0^1 \Re f(\varphi(tz))\Re k(tz)\frac{dt}{t}, \ \ f \in H(\mathbb{B}_N),\ z \in \mathbb{B}_N,$$

$$(L^g f)(z) = \int_0^1 \Re f(tz)g(tz)\frac{dt}{t}, \ \ f \in H(\mathbb{B}_N),\ z \in \mathbb{B}_N,$$

and

$$(L^k f)(z) = \int_0^1 \Re f(tz)\Re k(tz)\frac{dt}{t}, \ \ f \in H(\mathbb{B}_N),\ z \in \mathbb{B}_N,$$

S. Stević characterized the boundedness and compactness of L^g acting from logarithmic Bloch-type spaces to mixed-norm spaces on the unit ball in [14]. According to the previous sections, we can obtain some results about these operators at once. We state them as follows:

THEOREM 4.1. *Assume that $\varphi \in S(\mathbb{B})$, and $k \in H(\mathbb{B})$. Then the following statements are equivalent.*

(a) $L_\varphi^k : \mathscr{B}_\omega \to H(p,q,\phi)$ *is bounded.*

(b) $L_\varphi^k : \mathscr{B}_{\omega,0} \to H(p,q,\phi)$ *is bounded.*

(c) $L_\varphi^k : \mathscr{B}_\omega \to H(p,q,\phi)$ *is compact.*

(d) $L_\varphi^k : \mathscr{B}_{\omega,0} \to H(p,q,\phi)$ *is compact.*

(e)

$$\int_0^1 \left(\int_S \left(\frac{|\varphi(r\xi)\Re k(r\xi)|}{\omega(\varphi(r\xi))} \right)^q d\sigma(\xi) \right)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr < \infty.$$

THEOREM 4.2. *Assume that $g \in H(\mathbb{B})$ with $g(0) = 0$. Then the following statements are equivalent.*

(a) $L^g : \mathscr{B}_\omega \to H(p,q,\phi)$ *is bounded.*

(b) $L^g : \mathscr{B}_{\omega,0} \to H(p,q,\phi)$ *is bounded.*

(c) $L^g : \mathscr{B}_\omega \to H(p,q,\phi)$ *is compact.*

(d) $L^g : \mathscr{B}_{\omega,0} \to H(p,q,\phi)$ *is compact.*

(e)

$$\int_0^1 \left(\int_S |g(r\xi)|^q d\sigma(\xi) \right)^{p/q} \frac{r^p \phi^p(r)}{\omega^p(r)(1-r)^{1-p}} dr < \infty.$$

THEOREM 4.3. *Assume that $k \in H(\mathbb{B})$. Then the following statements are equivalent.*

(a) $L^k : \mathscr{B}_\omega \to H(p,q,\phi)$ *is bounded.*

(b) $L^k : \mathscr{B}_{\omega,0} \to H(p,q,\phi)$ *is bounded.*

(c) $L^k : \mathscr{B}_\omega \to H(p,q,\phi)$ *is compact.*

(d) $L^k : \mathscr{B}_{\omega,0} \to H(p,q,\phi)$ *is compact.*

(e)

$$\int_0^1 \left(\int_S |\Re k(r\xi)|^q d\sigma(\xi) \right)^{p/q} \frac{r^p \phi^p(r)}{\omega^p(r)(1-r)^{1-p}} dr < \infty.$$

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Li Zhang
Department of Mathematics
Tianjin University
Tianjin 300072, P. R. China
e-mail: zhangli0977@126.com

Ze-Hua Zhou
Department of Mathematics
Tianjin University
Tianjin 300072, P. R. China
e-mail: zehuazhou2003@yahoo.com.cn