Journal of Mathematical Inequalities
Volume 6, Number 4 (2012), 557–565 doi:10.7153/jmi-06-53

HOMOMORPHISMS AND DERIVATIONS ON UNITAL C^*–ALGEBRAS RELATED TO CAUCHY–JENSEN FUNCTIONAL INEQUALITY

HARK-MAHN KIM, MADJID ESHAGHI GORDJI,
ABBAS JAVADIAN AND ICK-SOON CHANG

(*Communicated by N. Elezović*)

Abstract. In this paper, we investigate homomorphisms from unital C^*–algebras to unital Banach algebras and derivations from unital C^*–algebras to Banach A–modules related to a Cauchy–Jensen functional inequality.

1. Introduction

The authors in the reference [2] have proved that if a mapping f satisfies the following functional inequality

$$\|f(x)+f(y)+f(z)\| \leqslant \left\|kf\left(\frac{x+y+z}{k}\right)\right\|, \qquad k \geqq 3 \tag{1}$$

in non-Archimedean Banach spaces, then f is additive. During the last decades, a number of papers have been published on the stability of functional inequalities and several stability problems associated with functional inequalities have been investigated by a number of mathematicians, see [1, 3, 4, 8, 9, 10, 11, 15, 16, 17, 18] and references therein.

In this paper, we expand the functional inequality to the following generalized Cauchy–Jensen functional inequality

$$\left\|\sum_{i=1}^{l} f(x_i)\right\| \leqslant \left\|mf\left(\frac{\sum_{i=1}^{l} x_i}{m}\right)\right\|, \tag{2}$$

where $l \geqslant 3, m \geqslant 1$ are fixed integers such that $l > m$. It is easy to see that if a mapping f satisfies the generalized Cauchy–Jensen inequality (2), then f is additive. In fact, if a mapping f satisfies the generalized Cauchy–Jensen inequality (2), then by setting $x_i = 0$ for all $i = 1, \cdots, l$, we arrive at $|l|\|f(0)\| \leqslant |m|\|f(0)\|$, which yields $f(0) = 0$. Thus, letting $x_i = 0$ for all $i = 4, \cdots, l$, if $l \geqslant 4$, we reduce (2) to (1) with $k = m$ and hence f is additive.

Mathematics subject classification (2010): 39B72, 46H30, 46B06.
Keywords and phrases: Hyers–Ulam stability, unital C^*–algebras, unitary group, homomorphisms, derivations.

Paper JMI-06-53

Now, by using the generalized Cauchy–Jensen inequality (2), we are going to investigate the homomorphisms and derivations on unital C^*-algebra. Throughout this paper, let A be a unital C^*-algebra, $U(A)$ be the set of unitary elements in A. Let B be a unital Banach algebra and let $Inv(B)$ be the set of invertible elements of B. Let $l \geqslant 3, m \geqslant 1$ be integers. Moreover, we assume that $n_0 \in \mathbb{N}$ is a positive integer and suppose that $\mathbb{T}^1_{\frac{1}{n_o}} := \{e^{i\theta}; \quad 0 \leqslant \theta \leqslant \frac{2\pi}{n_o}\}$.

2. Homomorphisms

In this section, we establish the homomorphisms from unital C^*-algebras to unital Banach algebras. We start our work with the following lemma [5].

LEMMA 2.1. *Assume that a mapping $f : A \to B$ is additive and for each fixed $x \in A$ $f(tx) = tf(x)$ for all $t \in \mathbb{T}^1_{\frac{1}{n_o}}$. Then f is $\mathbb{C}$-linear.*

Now, we introduce our main theorem on homomorphisms from unital C^*-algebras to unital Banach algebras.

THEOREM 2.2. *Assume that a mapping $f : A \to B$ with $f(0) = 0$ satisfies*

$$\lim_{k\to\infty} \frac{f((l-1)^k 1_A)}{(l-1)^k} \in Inv(B),$$

and

$$f((l-1)^k ux) = f((l-1)^k u)f(x) \tag{3}$$

for all $u \in U(A), x \in A$, and all $k \in \mathbb{N}$. Suppose that f satisfies the functional inequality

$$\Big\| \sum_{i=1}^{l} f(x_i) + f(tx) - tf(x) \Big\| \leqslant \Big\| mf\Big(\frac{\sum_{i=1}^{l} x_i}{m}\Big) \Big\| + \varphi(x_1, \cdots, x_l, x) \tag{4}$$

for all $x_1, \cdots, x_l, x \in A$ and all $t \in \mathbb{T}^1_{\frac{1}{n_o}}$, and that there exists a constant L with $0 < L < 1$ for which the function $\varphi : A^{l+1} \to \mathbb{R}^+ := [0, \infty)$ satisfies

$$\varphi\Big((l-1)(x_1, \cdots, x_l, x)\Big) \leqslant L \cdot (l-1)\varphi\big(x_1, \cdots, x_l, x\big) \tag{5}$$

for all $x_1, \cdots, x_l, x \in A$. Then the mapping $f : A \to B$ is a homomorphism.

Proof. Put $x = 0$ in (4) to get

$$\Big\| \sum_{i=1}^{l} f(x_i) \Big\| \leqslant \Big\| mf\Big(\frac{\sum_{i=1}^{l} x_i}{m}\Big) \Big\| + \varphi(x_1, \cdots, x_l, 0) \tag{6}$$

for all $x_1, \cdots, x_l, \in A$.

Put $(x_1,\cdots,x_l,x) := (\underbrace{x,-x,\cdots,x,-x}_{2\lceil \frac{l}{2}\rceil},0,0)$ in (6) where $\lceil\cdot\rceil$ denotes the Gaussian notation. We have an approximate oddness condition

$$\|f(x)+f(-x)\| \leqslant \frac{1}{\lceil \frac{l}{2}\rceil}\varphi(\underbrace{x,-x,\cdots,x,-x}_{2\lceil \frac{l}{2}\rceil},0,0) \tag{7}$$

for all $x\in A$. Replacing $(x_1,\cdots,x_l,0) := (-x,\cdots,-x,(l-1)x,0)$ in (6), one leads to

$$\|(l-1)f(-x)+f((l-1)x)\| \leqslant \varphi(-x,\cdots,-x,(l-1)x,0) \tag{8}$$

for all $x\in A$. Associating (7) with (8) yields

$$\begin{aligned}\left\|f(x)-\frac{f((l-1)x)}{(l-1)}\right\| \leqslant \Psi(x) := &\frac{1}{\lceil \frac{l}{2}\rceil}\varphi(\underbrace{x,-x,\cdots,x,-x}_{2\lceil \frac{l}{2}\rceil},0,0) \\ &+\frac{1}{l-1}\varphi(-x,\cdots,-x,(l-1)x,0)\end{aligned} \tag{9}$$

for all $x\in A$. Thus, it follows from (9) and (5) that for all nonnegative integers k and j with $j>k\geqslant 0$ and $x\in A$

$$\begin{aligned}\left\|\frac{f((l-1)^k x)}{(l-1)^k}-\frac{f((l-1)^{k+j}x)}{(l-1)^{k+j}}\right\| &\leqslant \sum_{i=k}^{k+j-1}\left\|\frac{f((l-1)^i x)}{(l-1)^i}-\frac{f((l-1)^{i+1}x)}{(l-1)^{i+1}}\right\| \\ &\leqslant \sum_{i=k}^{k+j-1}\frac{1}{(l-1)^i}\Psi\Big((l-1)^i x\Big) \\ &\leqslant \sum_{i=k}^{k+j-1} L^i\Psi(x),\end{aligned}$$

which tends to zero as $k\to\infty$. Hence the sequence $\left\{\frac{f((l-1)^k x)}{(l-1)^k}\right\}$ is Cauchy for all $x\in A$, and so we can define a function $h_1 : A\to B$ by

$$h_1(x) = \lim_{k\to\infty}\frac{f((l-1)^k x)}{(l-1)^k}, \quad x\in A.$$

It follows from (6) and (5) that

$$\begin{aligned}\frac{1}{(l-1)^k}\left\|\sum_{i=1}^{l} f\Big((l-1)^k x_i\Big)\right\| &\leqslant \frac{1}{(l-1)^k}\left\|mf\Big(\frac{1}{m}\sum_{i=1}^{l}(l-1)^k x_i\Big)\right\| \\ &\quad +\frac{1}{(l-1)^k}\varphi\Big((l-1)^k x_1,\cdots,(l-1)^k x_l,0\Big) \\ &\leqslant \frac{1}{(l-1)^k}\left\|mf\Big(\frac{1}{m}\sum_{i=1}^{l}(l-1)^k x_i\Big)\right\| \\ &\quad +L^k\varphi(x_1,\cdots,x_l,x,0)\end{aligned}$$

for all $k \in \mathbb{N}$ and all $x_1, \cdots, x_l \in A$. Taking $k \to \infty$ in the last relation, we see that

$$\left\| \sum_{i=1}^{l} h_1(x_i) \right\| \leqslant \left\| m h_1\left(\frac{\sum_{i=1}^{l} x_i}{m} \right) \right\|$$

for all $x_1, \cdots, x_l \in A$. This implies that the mapping h_1 is additive. It follows from (3) that

$$h_1(ux) = \lim_{k\to\infty} \frac{f((l-1)^k ux)}{(l-1)^k} = \lim_{k\to\infty} \frac{f((l-1)^k u)}{(l-1)^k} f(x) = h_1(u)f(x) \tag{10}$$

for all $u \in U(A)$ and all $x \in A$.

On the other hand, since h_1 is additive, it follows from (10) that

$$h_1(ux) = \frac{h_1(u((l-1)^k x))}{(l-1)^k} = h_1(u)\frac{f((l-1)^k x)}{(l-1)^k}$$

for all $u \in U(A)$ and all $x \in A$. By letting $k \to \infty$ in the last inequality above, we obtain

$$h_1(ux) = h_1(u)h_1(x) \tag{11}$$

for all $u \in U(A)$ and all $x \in A$. By putting $u := 1_A$ in (10) and (11), we conclude that

$$h_1(x) = h_1(1_A x) = h_1(1_A)f(x) = h_1(1_A)h_1(x)$$

for all $x \in A$. Then since $h_1(1_A) \in Inv(B)$ by hypothesis, we have

$$f(x) = h_1(x)$$

for all $x \in A$, and so the mapping f is additive. Put $x_1 = x_2 = \cdots = x_l = 0$ in (4) to get

$$\|f(tx) - tf(x)\| \leqslant \varphi(0,0,\cdots,0,x)$$

for all $x \in A$ and all $t \in \mathbb{T}^1_{\frac{1}{n_o}}$, we can show that $f(tx) = tf(x)$ for all $x \in A$ and all $t \in \mathbb{T}^1_{\frac{1}{n_o}}$. Then, it follows that the additive mapping f is $\mathbb{C}-$linear [5].

Now, let $x \in A$ be an arbitrary element. Then by Theorem 4.1.7 of [12], x is a finite linear combination of unitary elements, i.e., $x = \sum_{j=1}^{n} c_j u_j$, $(c_j \in \mathbb{C}, u_j \in U(A))$. Since f is $\mathbb{C}-$linear, it follows from (11) that

$$\begin{aligned} f(xa) &= f((\sum_{j=1}^{n} c_j u_j)a) = \sum_{j=1}^{n} c_j f(u_j a) = \sum_{j=1}^{n} c_j h_1(u_j a) = \sum_{j=1}^{n} c_j h_1(u_j) h_1(a) \\ &= \sum_{j=1}^{n} c_j f(u_j) f(a) = f(\sum_{j=1}^{n} c_j u_j) f(a) = f(x)f(a) \end{aligned}$$

for all $a \in A$. This means that f is a homomorphism from A into B. $\square$

REMARK 2.1. We note that if, in addition, $|l+1-t| > m$ and $L(l-1) < 1$ in Theorem 2.2, then $f(0) = 0$. Indeed, apply the functional inequality (5) for $x = x_1 = \cdots = x_l = 0$ to get

$$\varphi(0,\cdots,0) \leqslant L(l-1)\varphi(0,\cdots,0),$$

which implies $\varphi(0,\cdots,0) = 0$ because of $L(l-1) < 1$ and $l \geqslant 3$. Then, it follows from (4) that

$$|l+1-t|\|f(0)\| \leqslant m\|f(0)\|,$$

and therefore, we get $f(0) = 0$.

REMARK 2.2. Suppose that a mapping $f : A \to B$ with $f(0) = 0$ satisfies

$$\lim_{k\to\infty} \frac{f((l-1)^k 1_A)}{(l-1)^k} \in Inv(B)$$

and the functional inequalities (3) jointly with (4) for which the function $\varphi : A^{l+1} \to \mathbb{R}^+$ satisfies

$$\sum_{i=0}^{\infty} \frac{\varphi\big((l-1)^i(x_1,\cdots,x_l,x)\big)}{(l-1)^i} < \infty$$

for all $x_1,\cdots,x_l,x \in A$ instead of the condition (5). Then it follows from a similar argument to Theorem 2.2 that the mapping f is a homomorphism from A into B.

COROLLARY 2.3. *Let $0 < r < 1$ and $\theta > 0$. If a mapping $f : A \to B$ with $f(0) = 0$ satisfies the equation (3),*

$$\lim_{k\to\infty} (l-1)^{-k} f((l-1)^k 1_A) \in Inv(B),$$

and the following functional inequality

$$\Big\|\sum_{i=1}^{l} f(x_i) + f(tx) - tf(x)\Big\| \leqslant \Big\|mf\Big(\frac{\sum_{i=1}^{l} x_i}{m}\Big)\Big\| + \theta\Big(\sum_{i=1}^{l} \|x_i\|^r + \|x\|^r\Big)$$

for all $x_1,\cdots,x_l,x \in A$ and all $t \in \mathbb{T}^1_{\frac{1}{n_o}}$, then the mapping $f : A \to B$ is a homomorphism.

COROLLARY 2.4. *Let $\theta > 0$. If a mapping $f : A \to B$ with $f(0) = 0$ satisfies the equation (3),*

$$\lim_{k\to\infty} (l-1)^{-k} f((l-1)^k 1_A) \in Inv(B),$$

and the following functional inequality

$$\Big\|\sum_{i=1}^{l} f(x_i) + f(tx) - tf(x)\Big\| \leqslant \Big\|mf\Big(\frac{\sum_{i=1}^{l} x_i}{m}\Big)\Big\| + \theta$$

for all $x_1,\cdots,x_l,x \in A$ and all $t \in \mathbb{T}^1_{\frac{1}{n_o}}$. Then $f : A \to B$ is a homomorphism.

The following theorem is an alternative result of Theorem 2.2.

THEOREM 2.5. *Assume that a mapping* $f: A \to B$ *satisfies*

$$\lim_{k\to\infty}(l-1)^k f((l-1)^{-k}1_A) \in Inv(B)$$

and

$$f((l-1)^{-k}ux) = f((l-1)^{-k}u)f(x) \tag{12}$$

for all $u \in U(A), x \in A$*, and all* $k \in \mathbb{N}$*. Suppose that* f *satisfies the functional inequality (4) and that there exists a constant* L *with* $0 < L < 1$ *for which the function* $\varphi : A^{l+1} \to \mathbb{R}^+$ *satisfies*

$$(l-1)\varphi\big(x_1,\cdots,x_l,x\big) \leqslant L\varphi\big((l-1)(x_1,\cdots,x_l,x)\big) \tag{13}$$

for all $x_1,\cdots,x_l,x \in A$*. Then the mapping* f *is a homomorphism from* A *into* B*.*

Proof. It follows from the inequality (9) that

$$\begin{aligned}&\left\|(l-1)^k f\Big(\frac{x}{(l-1)^k}\Big) - (l-1)^{k+j} f\Big(\frac{x}{(l-1)^{k+j}}\Big)\right\| \\ &\leqslant \sum_{i=k}^{k+j-1}(l-1)^{i+1}\Psi\Big(\frac{x}{(l-1)^{i+1}}\Big) \leqslant \sum_{i=k}^{k+j-1} L^{i+1}\Psi(x),\end{aligned}$$

which tends to zero as $k \to \infty$.

The remaining part of the proof is similar to the corresponding part of the proof of Theorem 2.2. □

REMARK 2.3. We note that $f(0) = 0$ in Theorem 2.5 and in the following Remark 2.4 because the conditions (13) or (14) yields

$$\varphi(0,\cdots,0) \leqslant \frac{L}{l-1}\varphi(0,\cdots,0), \quad \text{or} \quad \sum_{i=0}^{\infty}(l-1)^i\varphi(0,\cdots,0) < \infty,$$

and $\varphi(0,\cdots,0) = 0$, and so $f(0) = 0$ in the sequel.

REMARK 2.4. Suppose that a mapping $f: A \to B$ satisfies the equation (12),

$$\lim_{k\to\infty}(l-1)^k f((l-1)^{-k}1_A) \in Inv(B),$$

and the functional inequality (6) for which the function $\varphi : A^{l+1} \to \mathbb{R}^+$ satisfies

$$\sum_{i=0}^{\infty}(l-1)^i\varphi\Big(\frac{1}{(l-1)^i}(x_1,\cdots,x_l,x)\Big) < \infty \tag{14}$$

for all $x_1,\cdots,x_l,x \in A$ instead of the condition (13). Then f is a homomorphism from A into B.

COROLLARY 2.6. *Let $r > 1$ and $\theta > 0$. If a mapping $f : A \to B$ satisfies the equation (12),*

$$\lim_{k\to\infty}(l-1)^k f((l-1)^{-k}1_A) \in Inv(B),$$

and the following functional inequality

$$\Big\|\sum_{i=1}^{l} f(x_i)+f(tx)-tf(x)\Big\| \leqslant \Big\|mf\Big(\frac{\sum_{i=1}^{l}x_i}{m}\Big)\Big\| + \theta\Big(\sum_{i=1}^{l}\|x_i\|^r+\|x\|^r\Big)$$

for all $x_1,\cdots,x_l,x \in A$ and all $t \in \mathbb{T}^1_{\frac{1}{n_o}}$. Then the mapping $f : A \to B$ is a homomorphism.

3. Derivations

In this section, we assume that X is a Banach $A-$module. We use the results of Section 2 to investigate the derivations from A into X.

THEOREM 3.1. *Assume that a mapping $f : A \to X$ with $f(0) = f(1_A) = 0$ satisfies*

$$f((l-1)^k ux) = f((l-1)^k u)x + (l-1)^k uf(x) \tag{15}$$

for all $u \in U(A), x \in A$, and all $k \in \mathbb{N}$. Suppose that f satisfies the functional inequality (4) and that there exists a constant L with $0 < L < 1$ for which the function $\varphi : A^{l+1} \to \mathbb{R}^+$ satisfies (5). Then $f : A \to X$ is a derivation.

Proof. It is easy to show that $X \oplus_1 A$ is a unital Banach algebra equipped with the following ℓ_1-norm

$$\|(x,a)\| = \|x\| + \|a\|, \qquad (a \in A, x \in X),$$

and the product

$$(x_1,a_1)(x_2,a_2) = (x_1\cdot a_2 + a_1\cdot x_2, a_1a_2), \qquad (a_1,a_2 \in A, x_1,x_2 \in X).$$

We refer the readers to [6, 7] for details. We define a mapping $\varphi_f : A \to X \oplus_1 A$ by $a \mapsto (f(a),a)$. Then it is easy to show that $\varphi_f(1_A) = (0,1_A) = 1_{X\oplus_1 A} \in Inv(X \oplus_1 A)$.

It follows from (15) that

$$\begin{aligned}\varphi_f((l-1)^k ux) &= (f((l-1)^k ux), (l-1)^k ux)\\ &= (f((l-1)^k u)x + (l-1)^k uf(x), (l-1)^k ux)\\ &= (f((l-1)^k u), (l-1)^k u)(f(x),x)\\ &= \varphi_f((l-1)^k u)\varphi_f(x)\end{aligned}$$

for all $u \in U(A), x \in A$, and all $k \in \mathbb{N}$. Thus, the mapping $\varphi_f : A \to X \oplus_1 A$ satisfies (3).

By using (4), we have

$$\begin{aligned}\Big\|\sum_{i=1}^{l}\varphi_f(x_i)+\varphi_f(tx)-t\varphi_f(x)\Big\| &= \Big\|\sum_{i=1}^{l}(f(x_i),x_i)+(f(tx),tx)-t(f(x),x)\Big\| \\ &= \Big\|\sum_{i=1}^{l}f(x_i)+f(tx)-tf(x)\Big\|+\Big\|\sum_{i=1}^{l}(x_i)\Big\| \\ &\leqslant \Big\|mf\Big(\frac{\sum_{i=1}^{l}x_i}{m}\Big)\Big\|+\Big\|\sum_{i=1}^{l}(x_i)\Big\|+\varphi(x_1,\cdots,x_l,x) \\ &= \Big\|m\varphi_f\Big(\frac{\sum_{i=1}^{l}x_i}{m}\Big)\Big\|+\varphi(x_1,\cdots,x_l,x)\end{aligned}$$

for all $x_1,\cdots,x_l,x\in A$ and all $t\in\mathbb{T}^1_{\frac{1}{n_o}}$. This means that $\varphi_f:A\to X\oplus_1 A$ satisfies the functional inequality (4). Therefore, by Theorem 2.2, the mapping $\varphi_f:A\to X\oplus_1 A$ is a homomorphism from A into $X\oplus_1 A$.

On the other hand, it is easy to see that f is a derivation from A into X if and only if $\varphi_f:A\to X\oplus_1 A$ is a homomorphism from A into $X\oplus_1 A$ (see [7]). Thus $f:A\to X$ is a derivation. □

By the same reasoning as above and by using Theorem 2.5, we can prove the following theorem.

THEOREM 3.2. *Assume that a mapping $f:A\to X$ with $f(0)=f(1_A)=0$ satisfies*

$$f((l-1)^{-k}ux)=f((l-1)^{-k}u)x+(l-1)^{-k}uf(x) \tag{16}$$

for all $u\in U(A)$, $x\in A$, and all $k\in\mathbb{N}$. Suppose that f satisfies the functional inequality (4) and that there exists a constant L with $0<L<1$ for which the function $\varphi:A^{l+1}\to\mathbb{R}^+$ satisfying (13). Then the mapping $f:A\to X$ is a derivation.

Acknowledgement

The authors would like to thank the referees for giving useful suggestions which led to the improvement of this manuscript. This work was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2011–0002614 and No. 2011-0004878). Correspondence should be addressed to Madjid Eshaghi Gordji and Ick-Soon Chang, madjid.eshaghi@gmail.com and ischang@mokwon.ac.kr.

REFERENCES

[1] Y. CHO, C. PARK, TH. M. RASSIAS AND R. SAADATI, *Inner product spaces and functional equations*, J. Comput. Anal. Appl. **13** (2011), 296–304.

[2] Y. CHO, C. PARK AND R. SAADATI, *Functional inequalities in non-Archimedean Banach spaces*, Appl. Math. Lett. **23** (2010), 1238-1242.

[3] Y. CHO, C. PARK AND R. SAADATI, *Fuzzy functional inequalities*, J. Comput. Anal. Appl. **13**, 2 (2011), 305–320.

[4] S. S. DRAGOMIR, Y. CHO AND J. KIM, *Subadditivity of some functionals associated to Jensen's inequality with applications*, Taiwan. J. Math. **15** (2011), 1815–1828.
[5] M. ESHAGHI GORDJI, *Nearly involutions on Banach algebras; A fixed point approach*, to appear in Fixed Point Theory (FPT).
[6] M. ESHAGHI GORDJI AND M. FILALI, *Arens regularity of module actions*, Studia Math. **181**, 3 (2007), 237–254.
[7] M. ESHAGHI GORDJI AND M. S. MOSLEHIAN, *A trick for investigation of approximate derivations*, Math. Commun. **15**, 1 (2010), 99–105.
[8] W. FECHNER, *Stability of a functional inequality associated with the Jordan-von Neumann functional equation*, Aequationes Math. **71** (2006), 149–161.
[9] W. FECHNER, *On some composite functional inequalities*, Aequationes Math. **79**, 3 (2010), 307–314.
[10] ZX. GAO, HX. CAO, WT. ZHENG, L. XU, *Generalized Hyers–Ulam–Rassias stability of functional inequalities and functional equations*, J. Math. Inequal. **3**, 1 (2009), 63–77.
[11] A. GILÁNYI, *On a problem by K. Nikodem*, Math. Inequal. Appl. **5** (2002), 707–710.
[12] R. V. KADISON, J. R. RINGROSE, *Fundamentals of the Theory of Operator Algebras, Elementary Theory*, Academic Press, New York, 1983.
[13] H. KIM, S. KANG, AND I. CHANG, *On functional inequalities originating from module Jordan left derivations,* J. Inequal. Appl. Volume **2008** (2008), Article ID 278505, 9 pages.
[14] J. LEE, C. PARK, AND D. SHIN, *On the stability of generalized additive functional inequalities in Banach spaces,* J. Inequal. Appl. Volume **2008** (2008), Article ID 210626, 13 pages.
[15] J. LEE, C. PARK, D. SHIN, *Stability of an additive functional inequality in proper CQ^*-algebras*, Bull. Korean Math. Soc. **48**, 4 (2011), 853–871.
[16] C. PARK AND A. NAJATI, *Homomorphisms and derivations in C^*-algebras*, Abstract and Appl. Anal. Volume **2007**, Article ID 80630, 12 pages.
[17] C. PARK, *Fuzzy stability of additive functional inequalities with the fixed point alternative*, J. Inequal. Appl. **2009**, Art. ID 410576, 17 pages.
[18] C. PARK, *Fixed points and stability of the Cauchy-Jensen functional inequality in fuzzy Banach algebras*, Appl. Math. Lett. **24**, 12 (2011), 2024–2029.

(Received December 12, 2011)

Hark-Mahn Kim
Department of Mathematics
Chungnam National University
79 Daehangno, Yuseong-gu
Daejeon 305-764, Korea
e-mail: hmkim@@cnu.ac.kr

Madjid Eshaghi Gordji
Department of Mathematics
Semnan University
P. O. Box 35195-363
Semnan, Iran
e-mail: madjid.eshaghi@@gmail.com

Abbas Javadian
Department of Physics
Semnan University
P. O. Box 35195-363
Semnan, Iran
e-mail: abasjavadian@@gmail.com

Ick-Soon Chang
Department of Mathematics
Mokwon University
Mokwon Gil 21, Seo-gu
Daejeon, 302-318
Korea
e-mail: ischang@@mokwon.ac.kr