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SHARP BOUNDS FOR THE NEUMAN–SÁNDOR MEAN IN TERMS OF GENERALIZED LOGARITHMIC MEAN

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Abstract. In this paper, we find the largest value α and least value β such that the double inequality $L_\alpha(a,b) < M(a,b) < L_\beta(a,b)$ holds for all $a,b > 0$ with $a \neq b$. Here, $M(a,b)$ and $L_p(a,b)$ are the Neuman-Sándor and p-th generalized logarithmic means of a and b, respectively.

1. Introduction

For $p \in \mathbb{R}$ the p-th generalized logarithmic mean $L_p(a,b)$ [1] and Neuman-Sándor mean $M(a,b)$ [2] of two positive numbers a and b are defined by

$$L_p(a,b) = \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, & a \neq b, p \neq -1, p \neq 0, \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}, & a \neq b, p = 0, \\ \frac{b-a}{\log b - \log a}, & a \neq b, p = -1, \\ a, & a = b \end{cases} \tag{1.1}$$

and

$$M(a,b) = \begin{cases} \frac{a-b}{2\sinh^{-1}\left(\frac{a-b}{a+b}\right)}, & a \neq b, \\ a, & a = b, \end{cases} \tag{1.2}$$

respectively.

It is well-known that $L_p(a,b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a,b > 0$ with $a \neq b$. Recently, the generalized logarithmic and Neuman-Sándor means have been the subject of intensive research. In particular, many remarkable inequalities for the generalized logarithmic mean can be found in the literature [3-33].

The power mean $M_r(a,b)$ of order r of two positive numbers a and b is defined by

$$M_r(a,b) = \begin{cases} \left(\frac{a^r+b^r}{2}\right)^{1/r}, & r \neq 0, \\ \sqrt{ab}, & r = 0. \end{cases}$$

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The main properties for $M_r(a,b)$ are given in [34]. In particular, the function $r \mapsto M_r(a,b)$ $(a \neq b)$ is continuous and strictly increasing on $\mathbb{R}$.

Let $H(a,b) = 2ab/(a+b)$, $G(a,b) = \sqrt{ab}$, $L(a,b) = (b-a)/(\log b - \log a)$, $I(a,b) = 1/e(b^b/a^a)^{1/(b-a)}$, $P(a,b) = (a-b)/[4\arctan(\sqrt{a/b}) - \pi]$, $T(a,b) = (a-b)/[2\arctan(\frac{a-b}{a+b})]$, $A(a,b) = (a+b)/2$, and $S(a,b) = \sqrt{(a^2+b^2)/2}$ be the harmonic, geometric, logarithmic, identric, first Seiffert, second Seiffert, arithmetic, and root-square means of a and b with $a \neq b$, respectively. Then it is known that the inequalities

$$\begin{aligned} H(a,b) &= M_{-1}(a,b) < G(a,b) = M_0(a,b) = L_{-2}(a,b) < L(a,b) = L_{-1}(a,b) \\ &< P(a,b) < I(a,b) = L_0(a,b) < A(a,b) = L_1(a,b) = M_1(a,b) < T(a,b) \\ &< S(a,b) = M_2(a,b) \end{aligned}$$

hold for all $a,b > 0$ with $a \neq b$.

Pittenger [35] proved that the double inequality

$$M_{r_1}(a,b) \leqslant L_p(a,b) \leqslant M_{r_2}(a,b) \tag{1.3}$$

holds for all $a,b > 0$ with

$$r_1 = \begin{cases} \min\{\frac{p+2}{3}, \frac{p\log 2}{\log(p+1)}\}, & p > -1, p \neq 0, \\ \frac{2}{3}, & p = 0, \\ \min\{\frac{p+2}{3}, 0\}, & p \leqslant -1, \end{cases}$$

$$r_2 = \begin{cases} \max\{\frac{p+2}{3}, \frac{p\log 2}{\log(p+1)}\}, & p > -1, p \neq 0, \\ \log 2, & p = 0, \\ \max\{\frac{p+2}{3}, 0\}, & p \leqslant -1. \end{cases}$$

Here r_1 and r_2 are sharp and inequality (1.3) becomes equality if and only if $a = b$ or $p = 1, -2$ or $-1/2$.

The following sharp bounds for H, $(G+H)/2$, and $(A+H)/2$ in terms of generalized logarithmic mean were given in [21]:

$$H(a,b) < L_{-5}(a,b),\ (G(a,b) + H(a,b))/2 > L_{-7/2}(a,b),$$

$$(A(a,b) + H(a,b))/2 > L_{-2}(a,b)$$

for all $a,b > 0$ with $a \neq b$.

Long and Chu [36] found the best possible parameters $\lambda = \lambda(\alpha)$ and $\mu = \mu(\alpha)$ such that the double inequality

$$L_\lambda(a,b) < \alpha A(a,b) + (1-\alpha)G(a,b) < L_\mu(a,b)$$

holds for any $\alpha \in (0,1/2) \cup (1/2,1)$ and all $a,b > 0$ with $a \neq b$.

In [37] the authors answered the question: for any $\alpha, \beta, \gamma \in (0,1)$ with $\alpha+\beta+\gamma=1$, what are the greatest value p and the least value q, such that the double inequality

$$L_p(a,b) < A^{\alpha}(a,b)G^{\beta}(a,b)H^{\gamma}(a,b) < L_q(a,b)$$

holds for all $a,b>0$ with $a \neq b$?

Neuman and Sándor [2, 38] established that

$$P(a,b) < A(a,b) < M(a,b) < T(a,b),$$

$$P(a,b)M(a,b) < A^2(a,b),$$

$$A(a,b)T(a,b) < M^2(a,b) < (A^2(a,b)+T^2(a,b))/2$$

for all $a,b>0$ with $a \neq b$.

Let $0 < a,b \leqslant 1/2$ with $a \neq b$, $a' = 1-a$ and $b' = 1-b$. Then the following Ky Fan inequalities

$$\frac{G(a,b)}{G(a',b')} < \frac{L(a,b)}{L(a',b')} < \frac{P(a,b)}{P(a',b')} < \frac{A(a,b)}{A(a',b')} < \frac{M(a,b)}{M(a',b')} < \frac{T(a,b)}{T(a',b')}$$

were presented in [2].

It is the aim of this paper to find the best possible generalized logarithmic mean bounds for the Neuman-Sándor Mean $M(a,b)$.

2. Main result

In order to establish our main result we need the following Lemma 2.1.

LEMMA 2.1. *The equation*

$$(p+1)^{1/p} = 2\log(1+\sqrt{2}) = 2\sinh^{-1}(1)$$

has an unique solution $p = p_0 = 1.8435\ldots$.

Proof. Let

$$g(p) = \begin{cases} (p+1)^{1/p}, & p \in (-1,0)\cup(0,+\infty), \\ e, & p=0. \end{cases} \tag{2.1}$$

Then it is not difficult to verify that the function g is continuous and strictly decreasing from $(-1,+\infty)$ onto $(1,+\infty)$. Therefore, Lemma 2.1 follows easily from the continuity and monotonicity of g together with the facts that $g(1.8435) = 1.762751\ldots > 2\log(1+\sqrt{2}) = 1.762747\ldots$ and $g(1.8436) = 1.762730\ldots < 2\log(1+\sqrt{2})$. $\square$

THEOREM 2.2. *The double inequality*

$$L_{p_0}(a,b) < M(a,b) < L_2(a,b)$$

holds for all $a,b>0$ *with* $a\neq b$, *where* $p_0 = 1.8435\ldots$ *is the unique solution of the equation* $(p+1)^{1/p} = 2\log(1+\sqrt{2})$, *and* $L_{p_0}(a,b)$ *and* $L_2(a,b)$ *are the best possible lower and upper generalized logarithmic mean bounds for the Neuman-Sándor mean* $M(a,b)$, *respectively.*

Proof. From (1.1) and (1.2) we clearly see that both $M(a,b)$ and $L_p(a,b)$ are symmetric and homogenous of degree 1. Without loss of generality, we assume that $b=1$ and $a=x>1$.

Firstly, we prove that inequality $L_{p_0}(x,1) < M(x,1)$ holds for all $x>1$. From (1.1) and (1.2) one has

$$\begin{aligned} &\log L_{p_0}(x,1) - \log M(x,1) \\ &= \frac{1}{p_0}\log\frac{x^{p_0+1}-1}{(p_0+1)(x-1)} - \log\frac{x-1}{2\sinh^{-1}(\frac{x-1}{x+1})}. \end{aligned} \tag{2.2}$$

Let

$$f(x) = \frac{1}{p_0}\log\frac{x^{p_0+1}-1}{(p_0+1)(x-1)} - \log\frac{x-1}{2\sinh^{-1}(\frac{x-1}{x+1})}. \tag{2.3}$$

Then simple computations and Lemma 2.1 lead to

$$\lim_{x\to 1^+} f(x) = 0, \tag{2.4}$$

$$\lim_{x\to +\infty} f(x) = \log\left[\frac{2\log(1+\sqrt{2})}{(p_0+1)^{1/p_0}}\right] = 0, \tag{2.5}$$

$$f'(x) = \frac{(p_0+1)(x^{p_0}-1)f_1(x)}{p_0(x-1)(x^{p_0+1}-1)\sinh^{-1}(\frac{x-1}{x+1})}, \tag{2.6}$$

where

$$f_1(x) = \frac{\sqrt{2}p_0(x-1)(x^{p_0+1}-1)}{(p_0+1)(x+1)(x^{p_0}-1)\sqrt{1+x^2}} - \sinh^{-1}\left(\frac{x-1}{x+1}\right),$$

$$\lim_{x\to 1^+} f_1(x) = 0, \tag{2.7}$$

$$\lim_{x\to +\infty} f_1(x) = \frac{\sqrt{2}p_0}{p_0+1} - \sinh^{-1}(1) = 0.0354\ldots > 0, \tag{2.8}$$

$$f_1'(x) = \frac{\sqrt{2}f_2(x)}{(p_0+1)(x+1)^2(x^{p_0}-1)^2(1+x^2)^{3/2}}, \tag{2.9}$$

where

$$\begin{aligned} f_2(x) = {} & (p_0-1)x^{2p_0+3} - x^{2p_0+2} + (p_0-1)x^{2p_0+1} - (2p_0+1)x^{2p_0} \\ & - p_0^2x^{p_0+4} + (p_0^2+p_0+2)x^{p_0+3} + (2-p_0)x^{p_0+2} + (2-p_0)x^{p_0+1} \end{aligned}$$

$$
\begin{aligned}
&+(p_0^2+p_0+2)x^{p_0}-p_0^2x^{p_0-1}-(2p_0+1)x^3+(p_0-1)x^2\\
&-x+p_0-1,
\end{aligned}
$$

$$f_2(1)=0, \tag{2.10}$$

$$\lim_{x\to+\infty} f_2(x)=+\infty, \tag{2.11}$$

$$
\begin{aligned}
f_2'(x)=&\,(p_0-1)(2p_0+3)x^{2p_0+2}-2(p_0+1)x^{2p_0+1}+(p_0-1)(2p_0+1)x^{2p_0}\\
&-2p_0(2p_0+1)x^{2p_0-1}-p_0^2(p_0+4)x^{p_0+3}+(p_0^2+p_0+2)(p_0+3)x^{p_0+2}\\
&+(2-p_0)(p_0+2)x^{p_0+1}+(2-p_0)(p_0+1)x^{p_0}+p_0(p_0^2+p_0+2)x^{p_0-1}\\
&-p_0^2(p_0-1)x^{p_0-2}-3(2p_0+1)x^2+2(p_0-1)x-1,
\end{aligned}
$$

$$f_2'(1)=0, \tag{2.12}$$

$$\lim_{x\to+\infty} f_2'(x)=+\infty, \tag{2.13}$$

$$
\begin{aligned}
f_2''(x)=&\,2(p_0-1)(2p_0+3)(p_0+1)x^{2p_0+1}-2(p_0+1)(2p_0+1)x^{2p_0}\\
&+2p_0(p_0-1)(2p_0+1)x^{2p_0-1}-2p_0(2p_0+1)(2p_0-1)x^{2p_0-2}\\
&-p_0^2(p_0+4)(p_0+3)x^{p_0+2}+(p_0^2+p_0+2)(p_0+3)(p_0+2)x^{p_0+1}\\
&+(2-p_0)(p_0+2)(p_0+1)x^{p_0}+p_0(2-p_0)(p_0+1)x^{p_0-1}\\
&+p_0(p_0^2+p_0+2)(p_0-1)x^{p_0-2}-p_0^2(p_0-1)(p_0-2)x^{p_0-3}\\
&-6(2p_0+1)x+2(p_0-1),
\end{aligned}
$$

$$f_2''(1)=0, \tag{2.14}$$

$$\lim_{x\to+\infty} f_2''(x)=+\infty, \tag{2.15}$$

$$
\begin{aligned}
f_2'''(x)=&\,2(p_0-1)(2p_0+3)(p_0+1)(2p_0+1)x^{2p_0}\\
&-4p_0(p_0+1)(2p_0+1)x^{2p_0-1}\\
&+2p_0(p_0-1)(2p_0+1)(2p_0-1)x^{2p_0-2}\\
&-4p_0(2p_0+1)(2p_0-1)(p_0-1)x^{2p_0-3}\\
&-p_0^2(p_0+4)(p_0+3)(p_0+2)x^{p_0+1}\\
&+(p_0^2+p_0+2)(p_0+3)(p_0+2)(p_0+1)x^{p_0}\\
&+p_0(2-p_0)(p_0+2)(p_0+1)x^{p_0-1}\\
&+p_0(2-p_0)(p_0+1)(p_0-1)x^{p_0-2}\\
&+p_0(p_0^2+p_0+2)(p_0-1)(p_0-2)x^{p_0-3}\\
&-p_0^2(p_0-1)(p_0-2)(p_0-3)x^{p_0-4}-6(2p_0+1),
\end{aligned}
$$

$$f_2'''(1)=0, \tag{2.16}$$

$$\lim_{x\to+\infty} f_2'''(x)=+\infty, \tag{2.17}$$

$$f_2^{(4)}(x)=p_0x^{p_0-5}f_3(x), \tag{2.18}$$

where

$$
\begin{aligned}
f_3(x)=&\,4(p_0-1)(2p_0+3)(p_0+1)(2p_0+1)x^{p_0+4}\\
&-4(p_0+1)(2p_0+1)(2p_0-1)x^{p_0+3}
\end{aligned}
$$

$$
\begin{aligned}
&+4(p_0-1)^2(2p_0+1)(2p_0-1)x^{p_0+2}\\
&-4(2p_0+1)(2p_0-1)(p_0-1)(2p_0-3)x^{p_0+1}\\
&-p_0(p_0+4)(p_0+3)(p_0+2)(p_0+1)x^5\\
&+(p_0^2+p_0+2)(p_0+3)(p_0+2)(p_0+1)x^4\\
&+(2-p_0)(p_0+2)(p_0+1)(p_0-1)x^3\\
&-(p_0+1)(p_0-1)(p_0-2)^2x^2\\
&+(p_0^2+p_0+2)(p_0-1)(p_0-2)(p_0-3)x\\
&-p_0(p_0-1)(p_0-2)(p_0-3)(p_0-4),
\end{aligned}
$$

$$f_3(1) = 24(p_0-2)(p_0+1) < 0, \tag{2.19}$$

$$\lim_{x\to+\infty} f_3(x) = +\infty, \tag{2.20}$$

$$
\begin{aligned}
f_3'(x) = {}&4(p_0-1)(2p_0+3)(p_0+1)(2p_0+1)(p_0+4)x^{p_0+3}\\
&-4(p_0+1)(2p_0+1)(2p_0-1)(p_0+3)x^{p_0+2}\\
&+4(p_0-1)^2(2p_0+1)(2p_0-1)(p_0+2)x^{p_0+1}\\
&-4(2p_0+1)(2p_0-1)(p_0-1)(2p_0-3)(p_0+1)x^{p_0}\\
&-5p_0(p_0+4)(p_0+3)(p_0+2)(p_0+1)x^4\\
&+4(p_0^2+p_0+2)(p_0+3)(p_0+2)(p_0+1)x^3\\
&+3(2-p_0)(p_0+2)(p_0+1)(p_0-1)x^2\\
&-2(p_0+1)(p_0-1)(p_0-2)^2x\\
&+(p_0^2+p_0+2)(p_0-1)(p_0-2)(p_0-3),
\end{aligned}
$$

$$f_3'(1) = 12p_0(p_0-2)(p_0+1)(8p_0+5) < 0, \tag{2.21}$$

$$\lim_{x\to+\infty} f_3'(x) = +\infty, \tag{2.22}$$

$$f_3''(x) = 2(p_0+1)f_4(x), \tag{2.23}$$

where

$$
\begin{aligned}
f_4(x) = {}&2(p_0-1)(2p_0+3)(2p_0+1)(p_0+4)(p_0+3)x^{p_0+2}\\
&-2(2p_0+1)(2p_0-1)(p_0+3)(p_0+2)x^{p_0+1}\\
&+2(p_0-1)^2(2p_0+1)(2p_0-1)(p_0+2)x^{p_0}\\
&-2p_0(2p_0+1)(2p_0-1)(2p_0-3)(p_0-1)x^{p_0-1}\\
&-10p_0(p_0+4)(p_0+3)(p_0+2)x^3\\
&+6(p_0^2+p_0+2)(p_0+3)(p_0+2)x^2\\
&+3(2-p_0)(p_0+2)(p_0-1)x-(p_0-1)(p_0-2)^2,
\end{aligned}
$$

$$
\begin{aligned}
f_4(1) &= 2(8p_0^5+18p_0^4+29p_0^3-102p_0^2-148p_0+3)\\
&= -113.1306\ldots < 0,
\end{aligned}
\tag{2.24}
$$

$$\lim_{x\to+\infty} f_4(x) = +\infty, \tag{2.25}$$

$$f_4'(x) = 2(p_0-1)(2p_0+3)(2p_0+1)(p_0+4)(p_0+3)(p_0+2)x^{p_0+1}$$

$$
\begin{aligned}
&-2(2p_0+1)(2p_0-1)(p_0+3)(p_0+2)(p_0+1)x^{p_0}\\
&+2p_0(p_0-1)^2(2p_0+1)(2p_0-1)(p_0+2)x^{p_0-1}\\
&-2p_0(2p_0+1)(2p_0-1)(2p_0-3)(p_0-1)^2x^{p_0-2}\\
&-30p_0(p_0+4)(p_0+3)(p_0+2)x^2\\
&+12(p_0^2+p_0+2)(p_0+3)(p_0+2)x\\
&+3(2-p_0)(p_0+2)(p_0-1),
\end{aligned}
$$

$$
\begin{aligned}
f_4'(1) &= p_0(96p_0^4+146p_0^3+43p_0^2-801p_0-900)\\
&= -381.6533\ldots < 0, \qquad (2.26)\\
\lim_{x\to+\infty} f_4'(x) &= +\infty, \qquad (2.27)\\
f_4''(x) &= 2(p_0-1)(2p_0+3)(2p_0+1)(p_0+4)(p_0+3)(p_0+2)(p_0+1)x^{p_0}\\
&\quad -2p_0(2p_0+1)(2p_0-1)(p_0+3)(p_0+2)(p_0+1)x^{p_0-1}\\
&\quad +2p_0(p_0-1)^3(2p_0+1)(2p_0-1)(p_0+2)x^{p_0-2}\\
&\quad +2p_0(2p_0+1)(2p_0-1)(2p_0-3)(p_0-1)^2(2-p_0)x^{p_0-3}\\
&\quad -60p_0(p_0+4)(p_0+3)(p_0+2)x\\
&\quad +12(p_0^2+p_0+2)(p_0+3)(p_0+2)\\
&> 2(p_0-1)(2p_0+3)(2p_0+1)(p_0+4)(p_0+3)(p_0+2)(p_0+1)x^{p_0}\\
&\quad -2p_0(2p_0+1)(2p_0-1)(p_0+3)(p_0+2)(p_0+1)x^{p_0}\\
&\quad -60p_0(p_0+4)(p_0+3)(p_0+2)x^{p_0}\\
&= 2(p_0+3)(p_0+2)(4p_0^5+20p_0^4+27p_0^3-41p_0^2-154p_0-12)\\
&= 1864.7110\ldots > 0 \qquad (2.28)
\end{aligned}
$$

for $x>1$.

Inequality (2.28) implies that f_4' is strictly increasing in $[1,+\infty)$. Then inequality (2.26) and equation (2.27) together with the monotonicity of f_4' lead to the conclusion that there exists $x_1>1$, such that f_4 is strictly decreasing in $[1,x_1]$ and strictly increasing in $[x_1,+\infty)$.

From inequality (2.24) and equation (2.25) together with the piecewise monotonicity of f_4 we clearly see that there exists $x_2>x_1>1$, such that $f_4<0$ in $[1,x_2)$ and $f_4>0$ in $(x_2,+\infty)$. Then equation (2.23) implies that f_3' is strictly decreasing in $[1,x_2]$ and strictly increasing in $[x_2,+\infty)$.

Inequality (2.21) and equation (2.22) together with the piecewise monotonicity of f_3' show that there exists $x_3>x_2>1$, such that f_3 is strictly decreasing in $[1,x_3]$ and strictly increasing in $[x_3,+\infty)$.

From (2.18)–(2.20) and the piecewise monotonicity of f_3 we clearly see that there exists $x_4>x_3>1$, such that f_2''' is strictly decreasing in $[1,x_4]$ and strictly increasing in $[x_4,+\infty)$.

It follows from equations (2.16) and (2.17) together with the piecewise monotonicity of f_2''' that there exists $x_5>x_4>1$, such that f_2'' is strictly decreasing in $[1,x_5]$ and strictly increasing in $[x_5,+\infty)$.

Equations (2.14) and (2.15) together with the piecewise monotonicity of f_2'' lead to the conclusion that there exists $x_6 > x_5 > 1$, such that f_2' is strictly decreasing in $[1, x_6]$ and strictly increasing in $[x_6, +\infty)$.

From equations (2.12) and (2.13) together with the piecewise monotonicity of f_2' we clearly see that there exists $x_7 > x_6 > 1$, such that f_2 is strictly decreasing in $[1, x_7]$ and strictly increasing in $[x_7, +\infty)$.

It follows from equations (2.9)–(2.11) and the piecewise monotonicity of f_2 that there exists $x_8 > x_7 > 1$, such that f_1 is strictly decreasing in $(1, x_8]$ and strictly increasing in $[x_8, +\infty)$.

From (2.6)–(2.8) and the piecewise monotonicity of f_1 we conclude that there exists $x_9 > x_8 > 1$, such that f is strictly decreasing in $(1, x_9]$ and strictly increasing in $[x_9, +\infty)$.

Therefore, $L_{p_0}(x,1) < M(x,1)$ for $x > 1$ follows from equations (2.2)–(2.5) and the piecewise monotonicity of f.

Secondly, we prove that inequality $L_2(x,1) > M(x,1)$ holds for all $x > 1$.

From (1.1) and (1.2), we have

$$\begin{aligned} &\log L_2(x,1) - \log M(x,1) \\ &= \frac{1}{2}\log\frac{x^2+x+1}{3} - \log(x-1) + \log\left[2\sinh^{-1}\left(\frac{x-1}{x+1}\right)\right]. \end{aligned} \tag{2.29}$$

Let

$$F(x) = \frac{1}{2}\log\frac{x^2+x+1}{3} - \log(x-1) + \log\left[2\sinh^{-1}\left(\frac{x-1}{x+1}\right)\right]. \tag{2.30}$$

Then simple computations lead to

$$\lim_{x\to 1^+} F(x) = 0, \tag{2.31}$$

$$F'(x) = \frac{(x+1)G(x)}{2(x^3-1)\sinh^{-1}(\frac{x-1}{x+1})}, \tag{2.32}$$

where

$$G(x) = \frac{2\sqrt{2}(x^3-1)}{(x+1)^2\sqrt{1+x^2}} - 3\sinh^{-1}\left(\frac{x-1}{x+1}\right),$$

$$G(1) = 0, \tag{2.33}$$

$$G'(x) = \frac{\sqrt{2}(x-1)^4}{(x+1)^3(1+x^2)^{3/2}} > 0 \tag{2.34}$$

for $x > 1$.

Equation (2.33) and inequality (2.34) imply that

$$G(x) > 0 \tag{2.35}$$

for $x > 1$. Then equation (2.32) and inequality (2.35) lead to the conclusion that F is strictly increasing in $(1, +\infty)$.

Therefore, $L_2(x,1) > M(x,1)$ for $x > 1$ follows from equations (2.29)–(2.31) and the monotonicity of F.

Next, we prove that $L_2(a,b)$ is the best possible upper generalized logarithmic mean bound for the Neuman-Sándor mean $M(a,b)$.

For any $0 < \varepsilon < 2$ and $x > 0$, from (1.1) and (1.2) one has

$$L_{2-\varepsilon}(1+x,1) - M(1+x,1) = \left[\frac{(1+x)^{3-\varepsilon}-1}{(3-\varepsilon)x}\right]^{1/(2-\varepsilon)} - \frac{x}{2\sinh^{-1}\left(\frac{x}{2+x}\right)}. \tag{2.36}$$

Letting $x \to 0$ and making use of Taylor expansion, we get

$$\begin{aligned}
&\left[\frac{(1+x)^{3-\varepsilon}-1}{(3-\varepsilon)x}\right]^{1/(2-\varepsilon)} - \frac{x}{2\sinh^{-1}\left(\frac{x}{2+x}\right)} \\
&= \left[1 + \frac{2-\varepsilon}{2}x + \frac{(2-\varepsilon)(1-\varepsilon)}{6}x^2 + o(x^2)\right]^{1/(2-\varepsilon)} \\
&\quad - \frac{x}{x - \frac{1}{2}x^2 + \frac{5}{24}x^3 + o(x^3)} \\
&= \left[1 + \frac{1}{2}x + \frac{1-\varepsilon}{24}x^2 + o(x^2)\right] - \left[1 + \frac{1}{2}x + \frac{1}{24}x^2 + o(x^2)\right] \\
&= -\frac{\varepsilon}{24}x^2 + o(x^2).
\end{aligned} \tag{2.37}$$

Equations (2.36) and (2.37) imply that for any $0 < \varepsilon < 2$ there exists $\delta = \delta(\varepsilon) > 0$, such that $L_{2-\varepsilon}(1+x,1) < M(1+x,1)$ for $x \in (0,\delta)$.

Finally, we prove that $L_{p_0}(a,b)$ is the best possible lower generalized logarithmic mean bound for the Neuman-Sándor mean $M(a,b)$.

For any $\varepsilon > 0$ and $x > 1$, from (1.1) and (1.2) together with Lemma 2.1 one has

$$\begin{aligned}
&\lim_{x\to+\infty} \log\left[\frac{L_{p_0+\varepsilon}(x,1)}{M(x,1)}\right] \\
&= \lim_{x\to+\infty}\left[\frac{1}{p_0+\varepsilon}\log\frac{x^{p_0+\varepsilon+1}-1}{(p_0+\varepsilon+1)(x-1)} - \log\frac{x-1}{2\sinh^{-1}\left(\frac{x-1}{x+1}\right)}\right] \\
&= \log\left[2\sinh^{-1}(1)\right] - \frac{1}{p_0+\varepsilon}\log(p_0+\varepsilon+1) \\
&= \frac{1}{p_0}\log(p_0+1) - \frac{1}{p_0+\varepsilon}\log(p_0+\varepsilon+1) \\
&> 0.
\end{aligned} \tag{2.38}$$

Inequality (2.38) implies that for any $\varepsilon > 0$ there exists $X = X(\varepsilon) > 1$, such that $L_{p_0+\varepsilon}(x,1) > M(x,1)$ for $x \in (X,+\infty)$. $\square$

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