ON MONOTONICITY OF SOME OPERATOR FUNCTIONS
RELATED TO ORDER PRESERVING OPERATOR INEQUALITIES

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(*Communicated by J. Pečarić*)

Abstract. In this paper, we shall show an equivalence relation between extensions of order preserving operator inequalities and monotonicity of related operator functions.

1. Introduction

A capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive semidefinite (denoted by $0 \leqslant T$) if $0 \leqslant (Tx,x)$ for all $x \in H$ and also an operator T is said to be positive definite (denoted by $0 < T$) if T is positive semidefinite and invertible.

THEOREM 1. [10], [13] *Let* $0 \leqslant p \leqslant 1$. *If* $0 \leqslant B \leqslant A$, *then* $B^p \leqslant A^p$ *holds.*

It is well known that $0 \leqslant B \leqslant A$ does not always ensure $B^p \leqslant A^p$ for $1 < p$ in general. The next result has been obtained from this point of view.

THEOREM 2. [1] *Let* $0 \leqslant p$, $1 \leqslant q$ *and* $0 \leqslant r$ *with* $p+r \leqslant (1+r)q$. *If* $0 \leqslant B \leqslant A$, *then we have the following inequality:*

$$\left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}} \right)^{\frac{1}{q}} \leqslant A^{\frac{p+r}{q}}.$$

It is known that the next Theorem 3 is an equivalence between generalized Furuta inequalities and monotonicity of related operator functions.

THEOREM 3. [6] *The following statements hold and follow from each other:*
(1) *Let* $1 \leqslant p$, $1 \leqslant s$, $0 \leqslant t \leqslant 1$ *and* $t \leqslant r$. *If* $0 \leqslant B \leqslant A$ *with* $0 < A$, *then the following inequality holds:*

$$\left\{ A^{\frac{r}{2}} \left(A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}} \right)^s A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{(p-t)s+r}} \leqslant A^{1-t+r}.$$

(2) *Let* $0 \leqslant t \leqslant q \leqslant 1$, $q \leqslant p$, $1 \leqslant s$ *and* $t \leqslant r$. *If* $0 \leqslant B \leqslant A$ *with* $0 < A$, *then the following inequality holds:*

$$\left\{ A^{\frac{r}{2}} \left(A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}} \right)^s A^{\frac{r}{2}} \right\}^{\frac{q-t+r}{(p-t)s+r}} \leqslant A^{q-t+r}.$$

(3) *Let* $0 \leqslant t \leqslant 1$ *and* $1 \leqslant p$. *If* $0 \leqslant B \leqslant A$ *with* $0 < A$, *then*

$$F(A,B,r,s) = A^{-\frac{r}{2}} \left\{ A^{\frac{r}{2}} \left(A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}} \right)^s A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{(p-t)s+r}} A^{-\frac{r}{2}}$$

is decreasing for $t \leqslant r$ *and* $1 \leqslant s$.

(4) *Let* $0 \leqslant t \leqslant 1$, $0 \leqslant q$ *and* $t \leqslant p$. *If* $0 \leqslant B \leqslant A$ *with* $0 < A$, *then*

$$G(A,B,r,s) = A^{-\frac{r}{2}} \left\{ A^{\frac{r}{2}} \left(A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}} \right)^s A^{\frac{r}{2}} \right\}^{\frac{q-t+r}{(p-t)s+r}} A^{-\frac{r}{2}}$$

is decreasing for $t \leqslant r$ *and* $1 \leqslant s$ *such that* $q - t \leqslant (p-t)s$.

(1) and (3) in Theorem 3 have been proved as a theorem in [3]. (2) in Theorem 3 was shown in [9] and (4) in Theorem 3 was shown in [8] as an extension of [7]. Moreover, (1), (2), (3) and (4) have been proved to be equivalent each other in [6] (see also [11]).

LEMMA 4. [2] *Let A, B be positive definite and let* λ *be a real number. Then*

$$(ABA)^\lambda = AB^{\frac{1}{2}} \left(B^{\frac{1}{2}} A^2 B^{\frac{1}{2}} \right)^{\lambda-1} B^{\frac{1}{2}} A.$$

DEFINITION 1. [12] *Let* n *be a natural number. We set*

$$\alpha(2n) = 1 - t_1 + t_2 - \cdots - t_{2n-1} + t_{2n}$$
$$\psi(2n) = \{ \cdots ((((p_1 - t_1)p_2 + t_2) p_3 - t_3) p_4 + t_4) p_5 - \cdots - t_{2n-1} \} p_{2n} + t_{2n}.$$

Furuta [4] obtained an extension of (1) in Theorem 3 stated before. The next Theorem 5 is Corollary 11 in [12], which is an extension of Theorem 3.3 in [4].

THEOREM 5. [12] *Let* n *be a natural number. Let* $1 \leqslant p_j$ ($j = 1, \cdots, 2n$), $0 \leqslant t_{2k-1} \leqslant 1$ *and* $t_{2k-1} \leqslant t_{2k}$ ($k = 1, \cdots, n$). *If* $0 \leqslant B \leqslant A$ *with* $0 < A$, *then the following inequality holds:*

$$\left\{ A^{\frac{t_{2n}}{2}} \left(A^{-\frac{t_{2n-1}}{2}} \cdots \left(A^{\frac{t_2}{2}} \left(A^{-\frac{t_1}{2}} B^{p_1} A^{-\frac{t_1}{2}} \right)^{p_2} A^{\frac{t_2}{2}} \right)^{p_3} \cdots A^{-\frac{t_{2n-1}}{2}} \right)^{p_{2n}} A^{\frac{t_{2n}}{2}} \right\}^{\frac{\alpha(2n)}{\psi(2n)}}$$

$$\leqslant A^{\alpha(2n)}.$$

2. Results

For the convenience we introduce the following notations.

DEFINITION 2. Let n be a natural number. We set

$$\widetilde{\alpha}(2n) = q - r_1 + r_2 - \cdots - r_{2n-1} + r_{2n}$$
$$\widetilde{\psi}(2n) = \{\cdots((((s_1 - r_1)s_2 + r_2)s_3 - r_3)s_4 + r_4)s_5 - \cdots - r_{2n-1}\}s_{2n} + r_{2n}$$
$$\widetilde{\gamma}(2n-1) = \{\cdots((((s_1 - r_1)s_2 + r_2)s_3 - r_3)s_4 + r_4)s_5 - \cdots + r_{2n-2}\}s_{2n-1} - r_{2n-1}.$$

DEFINITION 3. Let n be a natural number. We set

$$F_{2n}(A,B,t_{2n},p_{2n}) =$$

$$A^{-\frac{t_{2n}}{2}}\left\{A^{\frac{t_{2n}}{2}}\left(A^{-\frac{t_{2n-1}}{2}}\cdots\left(A^{\frac{t_2}{2}}\left(A^{-\frac{t_1}{2}}B^{p_1}A^{-\frac{t_1}{2}}\right)^{p_2}A^{\frac{t_2}{2}}\right)^{p_3}\cdots A^{-\frac{t_{2n-1}}{2}}\right)^{p_{2n}}A^{\frac{t_{2n}}{2}}\right\}^{\frac{\alpha(2n)}{\psi(2n)}}A^{-\frac{t_{2n}}{2}}$$

$$G_{2n}(A,B,r_{2n},s_{2n}) =$$

$$A^{-\frac{r_{2n}}{2}}\left\{A^{\frac{r_{2n}}{2}}\left(A^{-\frac{r_{2n-1}}{2}}\cdots\left(A^{\frac{r_2}{2}}\left(A^{-\frac{r_1}{2}}B^{s_1}A^{-\frac{r_1}{2}}\right)^{s_2}A^{\frac{r_2}{2}}\right)^{s_3}\cdots A^{-\frac{r_{2n-1}}{2}}\right)^{s_{2n}}A^{\frac{r_{2n}}{2}}\right\}^{\frac{\widetilde{\alpha}(2n)}{\widetilde{\psi}(2n)}}A^{-\frac{r_{2n}}{2}}.$$

The purpose of this paper is to show the following Theorem 6, which is an extension of Theorem 3. The method of our proof of Theorem 6 is almost similar to that of Furuta, Hashimoto and Ito [6].

THEOREM 6. *The following statements hold and follow from each other:*
(1) Let n be a natural number. Let $1 \leqslant p_j$ $(j = 1,\cdots,2n)$, $0 \leqslant t_{2k-1} \leqslant 1$ and $t_{2k-1} \leqslant t_{2k}$ $(k = 1,\cdots,n)$. If $0 \leqslant B \leqslant A$ with $0 < A$, then the following inequality holds:

$$\left\{A^{\frac{t_{2n}}{2}}\left(A^{-\frac{t_{2n-1}}{2}}\cdots\left(A^{\frac{t_2}{2}}\left(A^{-\frac{t_1}{2}}B^{p_1}A^{-\frac{t_1}{2}}\right)^{p_2}A^{\frac{t_2}{2}}\right)^{p_3}\cdots A^{-\frac{t_{2n-1}}{2}}\right)^{p_{2n}}A^{\frac{t_{2n}}{2}}\right\}^{\frac{\alpha(2n)}{\psi(2n)}}$$
$$\leqslant A^{\alpha(2n)}.$$

(2) Let n be a natural number. Let $0 \leqslant r_{2k-1} \leqslant q \leqslant 1$ $(k = 1,2,\cdots,n)$, $q \leqslant s_1$, $1 \leqslant s_j$ $(j = 2,3,\cdots,2n)$, $r_{2k-1} \leqslant r_{2k}$ $(k = 1,2,\cdots,n)$. If $0 \leqslant B \leqslant A$ with $0 < A$, then the following inequality holds:

$$\left\{A^{\frac{r_{2n}}{2}}\left(A^{-\frac{r_{2n-1}}{2}}\cdots\left(A^{\frac{r_2}{2}}\left(A^{-\frac{r_1}{2}}B^{s_1}A^{-\frac{r_1}{2}}\right)^{s_2}A^{\frac{r_2}{2}}\right)^{s_3}\cdots A^{-\frac{r_{2n-1}}{2}}\right)^{s_{2n}}A^{\frac{r_{2n}}{2}}\right\}^{\frac{\widetilde{\alpha}(2n)}{\widetilde{\psi}(2n)}}$$
$$\leqslant A^{\widetilde{\alpha}(2n)}.$$

(3) Let n be a natural number. Let $0 \leqslant t_{2k-1} \leqslant 1$ $(k = 1,2,\cdots,n)$, $1 \leqslant p_j$ $(j = 1,\cdots,2n-1)$ and $t_{2i-1} \leqslant t_{2i}$ $(i = 1,\cdots,n-1)$. If $0 \leqslant B \leqslant A$ with $0 < A$, then

$$F_{2n}(A,B,t_{2n},p_{2n}) =$$

$$A^{-\frac{t_{2n}}{2}}\left\{A^{\frac{t_{2n}}{2}}\left(A^{-\frac{t_{2n-1}}{2}}\cdots\left(A^{\frac{t_2}{2}}\left(A^{-\frac{t_1}{2}}B^{p_1}A^{-\frac{t_1}{2}}\right)^{p_2}A^{\frac{t_2}{2}}\right)^{p_3}\cdots A^{-\frac{t_{2n-1}}{2}}\right)^{p_{2n}}A^{\frac{t_{2n}}{2}}\right\}^{\frac{\alpha(2n)}{\psi(2n)}}A^{-\frac{t_{2n}}{2}}$$

is decreasing for $t_{2n-1} \leqslant t_{2n}$ and $1 \leqslant p_{2n}$.

(4) *Let n be a natural number. Let $0 \leqslant r_{2k-1} \leqslant 1$ $(k = 1, 2, \cdots, n)$, $0 \leqslant q$, $r_{2k-1} \leqslant s_1$ $(k = 1, 2, \cdots, n)$, $1 \leqslant s_j$ $(j = 2, 3, \cdots, 2n-1)$ and $r_{2i-1} \leqslant r_{2i}$ $(i = 1, \cdots, n-1)$.*

If $0 \leqslant B \leqslant A$ with $0 < A$, then

$$G_{2n}(A, B, r_{2n}, s_{2n}) =$$

$$A^{-\frac{r_{2n}}{2}} \left\{ A^{\frac{r_{2n}}{2}} \left(A^{-\frac{r_{2n-1}}{2}} \cdots \left(A^{\frac{r_2}{2}} \left(A^{-\frac{r_1}{2}} B^{s_1} A^{-\frac{r_1}{2}} \right)^{s_2} A^{\frac{r_2}{2}} \right)^{s_3} \cdots A^{-\frac{r_{2n-1}}{2}} \right)^{s_{2n}} A^{\frac{r_{2n}}{2}} \right\}^{\frac{\widetilde{\alpha}(2n)}{\widetilde{\psi}(2n)}} A^{-\frac{r_{2n}}{2}}$$

is a decreasing function of $r_{2n-1} \leqslant r_{2n}$ and $1 \leqslant s_{2n}$ such that $\widetilde{\alpha}(2n) \leqslant \widetilde{\psi}(2n)$.

Proof. We may assume that B is invertible without loss of generality. (1) has been already proved in [12]. We have only to prove the equivalence among (1), (2), (3) and (4).

$(1) \Rightarrow (2)$

We may assume that $q \neq 0$, for if $q = 0$, then $r_{2k-1} = 0 (k = 1, 2, \cdots, n)$ and it is just the case of Theorem 3.1 in [5]. One can also obtain the result by using the technique in [14].

Since $q \in (0, 1]$, $B^q \leqslant A^q$.

Put $A_1 = A^q$, $B_1 = B^q$, $p_1 = \frac{s_1}{q}$, $p_j = s_j$ $(j = 2, 3, \cdots, 2n)$ and $t_i = \frac{r_i}{q}$ $(i = 1, 2, \cdots, 2n)$.

Then it is obvious that $0 \leqslant B_1 \leqslant A_1$, $1 \leqslant p_j (j = 1, 2, \cdots, 2n)$, $0 \leqslant t_{2k-1} \leqslant 1$ and $t_{2k-1} \leqslant t_{2k} (k = 1, 2, \cdots, n)$.

Therefore we have the following inequality by (1):

$$\left\{ A_1^{\frac{t_{2n}}{2}} \left(A_1^{-\frac{t_{2n-1}}{2}} \cdots \left(A_1^{\frac{t_2}{2}} \left(A_1^{-\frac{t_1}{2}} B_1^{p_1} A_1^{-\frac{t_1}{2}} \right)^{p_2} A_1^{\frac{t_2}{2}} \right)^{p_3} \cdots A_1^{-\frac{t_{2n-1}}{2}} \right)^{p_{2n}} A_1^{\frac{t_{2n}}{2}} \right\}^{\frac{\alpha(2n)}{\psi(2n)}}$$

$$\leqslant A_1^{\alpha(2n)}.$$

It is easy to see that

$$A_1^{\alpha(2n)} = A^{\widetilde{\alpha}(2n)}, \; A_1^{\frac{t_{2i}}{2}} = A^{\frac{r_{2i}}{2}} \; (i = 1, 2, \cdots, n), \; B_1^{p_1} = B^{s_1}$$

and

$$\frac{\alpha(2n)}{\psi(2n)} = \frac{1 - t_1 + t_2 - \cdots - t_{2n-1} + t_{2n}}{\{ \cdots ((p_1 - t_1)p_2 + t_2) p_3 - \cdots - t_{2n-1}\} p_{2n} + t_{2n}}$$

$$= \frac{1 - \frac{r_1}{q} + \frac{r_2}{q} - \cdots - \frac{r_{2n-1}}{q} + \frac{r_{2n}}{q}}{\left\{ \cdots \left((\frac{s_1}{q} - \frac{r_1}{q})s_2 + \frac{r_2}{q} \right) s_3 - \cdots - \frac{r_{2n-1}}{q} \right\} s_{2n} + \frac{r_{2n}}{q}}$$

$$= \frac{q - r_1 + r_2 - \cdots - r_{2n-1} + r_{2n}}{\{ \cdots ((s_1 - r_1)s_2 + r_2) s_3 - \cdots - r_{2n-1}\} s_{2n} + r_{2n}}$$

$$= \frac{\widetilde{\alpha}(2n)}{\widetilde{\psi}(2n)},$$

that is, we have the following inequality:

$$\left\{ A^{\frac{r_{2n}}{2}} \left(A^{-\frac{r_{2n-1}}{2}} \cdots \left(A^{\frac{r_2}{2}} \left(A^{-\frac{r_1}{2}} B^{s_1} A^{-\frac{r_1}{2}} \right)^{s_2} A^{\frac{r_2}{2}} \right)^{s_3} \cdots A^{-\frac{r_{2n-1}}{2}} \right)^{s_{2n}} A^{\frac{r_{2n}}{2}} \right\}^{\frac{\tilde{\alpha}(2n)}{\tilde{\psi}(2n)}}$$
$$\leqslant A^{\tilde{\alpha}(2n)}.$$

$(2) \Rightarrow (4)$

Put $q_1 = max\{r_{2k-1}; k = 1, 2, \cdots, n\}$. Then we have the following inequality by (2):

$$\left\{ A^{\frac{r_{2n}}{2}} \left(A^{-\frac{r_{2n-1}}{2}} \cdots \left(A^{\frac{r_2}{2}} \left(A^{-\frac{r_1}{2}} B^{s_1} A^{-\frac{r_1}{2}} \right)^{s_2} A^{\frac{r_2}{2}} \right)^{s_3} \cdots A^{-\frac{r_{2n-1}}{2}} \right)^{s_{2n}} A^{\frac{r_{2n}}{2}} \right\}^{\frac{q_1 - r_1 + r_2 - \cdots - r_{2n-1} + r_{2n}}{\psi(2n)}}$$
$$\leqslant A^{q_1 - r_1 + r_2 - \cdots - r_{2n-1} + r_{2n}}.$$

Raise each side of this inequality to the power $\frac{r_{2n}}{q_1 - r_1 + r_2 - \cdots - r_{2n-1} + r_{2n}} \in [0, 1]$, then we have the following inequality by Theorem 1:

$$\left\{ A^{\frac{r_{2n}}{2}} \left(A^{-\frac{r_{2n-1}}{2}} \cdots \left(A^{\frac{r_2}{2}} \left(A^{-\frac{r_1}{2}} B^{s_1} A^{-\frac{r_1}{2}} \right)^{s_2} A^{\frac{r_2}{2}} \right)^{s_3} \cdots A^{-\frac{r_{2n-1}}{2}} \right)^{s_{2n}} A^{\frac{r_{2n}}{2}} \right\}^{\frac{r_{2n}}{\psi(2n)}}$$
$$\leqslant A^{r_{2n}}.$$

Put

$$D = A^{-\frac{r_{2n-1}}{2}} \left(A^{-\frac{r_{2n-2}}{2}} \cdots \left(A^{\frac{r_2}{2}} \left(A^{-\frac{r_1}{2}} B^{s_1} A^{-\frac{r_1}{2}} \right)^{s_2} A^{\frac{r_2}{2}} \right)^{s_3} \cdots A^{-\frac{r_{2n-2}}{2}} \right)^{s_{2n-1}} A^{-\frac{r_{2n-1}}{2}}.$$

Then we have the following inequalities:

$$A^{r_{2n}} \geqslant \left\{ A^{\frac{r_{2n}}{2}} D^{s_{2n}} A^{\frac{r_{2n}}{2}} \right\}^{\frac{r_{2n}}{\psi(2n)}} \cdots (a)$$
$$= A^{\frac{r_{2n}}{2}} D^{\frac{s_{2n}}{2}} \left(D^{\frac{s_{2n}}{2}} A^{r_{2n}} D^{\frac{s_{2n}}{2}} \right)^{\frac{-\tilde{\psi}(2n) + r_{2n}}{\psi(2n)}} D^{\frac{s_{2n}}{2}} A^{\frac{r_{2n}}{2}} \qquad \text{(by Lemma 4)}$$
$$D^{s_{2n}} \leqslant \left(D^{\frac{s_{2n}}{2}} A^{r_{2n}} D^{\frac{s_{2n}}{2}} \right)^{\frac{\tilde{\psi}(2n) - r_{2n}}{\psi(2n)}} \cdots (b).$$

Put $\gamma = \frac{w}{s_{2n}} \in [0, 1]$ for $0 < w \leqslant s_{2n}$.

Raise each side of (b) to the power $\gamma = \frac{w}{s_{2n}} \in [0, 1]$, then we have the following inequality by Theorem 1:

$$D^w \leqslant \left(D^{\frac{s_{2n}}{2}} A^{r_{2n}} D^{\frac{s_{2n}}{2}} \right)^{\frac{(\tilde{\psi}(2n) - r_{2n})\gamma}{\psi(2n)}}.$$

(i) Proof of the result that $G_{2n}(A,B,r_{2n},s_{2n})$ is a decreasing function of s_{2n}.

$$g(s_{2n})$$

$$= \left\{ A^{\frac{r_{2n}}{2}} \left(A^{-\frac{r_{2n-1}}{2}} \cdots \left(A^{\frac{r_2}{2}} \left(A^{-\frac{r_1}{2}} B^{s_1} A^{-\frac{r_1}{2}} \right)^{s_2} A^{\frac{r_2}{2}} \right)^{s_3} \cdots A^{-\frac{r_{2n-1}}{2}} \right)^{s_{2n}} A^{\frac{r_{2n}}{2}} \right\}^{\frac{\widetilde{\alpha}(2n)}{\widetilde{\psi}(2n)}}$$

$$= \left\{ A^{\frac{r_{2n}}{2}} D^{s_{2n}} A^{\frac{r_{2n}}{2}} \right\}^{\frac{\widetilde{\alpha}(2n)}{\widetilde{\psi}(2n)}}$$

$$= \left\{ \left(A^{\frac{r_{2n}}{2}} D^{s_{2n}} A^{\frac{r_{2n}}{2}} \right)^{\frac{\widetilde{\gamma}(2n-1)(s_{2n}+w)+r_{2n}}{\widetilde{\psi}(2n)}} \right\}^{\frac{\widetilde{\alpha}(2n)}{\widetilde{\gamma}(2n-1)(s_{2n}+w)+r_{2n}}}$$

$$= \left\{ A^{\frac{r_{2n}}{2}} D^{\frac{s_{2n}}{2}} \left(D^{\frac{s_{2n}}{2}} A^{r_{2n}} D^{\frac{s_{2n}}{2}} \right)^{\frac{(\widetilde{\psi}(2n)-r_{2n})\gamma}{\widetilde{\psi}(2n)}} D^{\frac{s_{2n}}{2}} A^{\frac{r_{2n}}{2}} \right\}^{\frac{\widetilde{\alpha}(2n)}{\widetilde{\gamma}(2n-1)(s_{2n}+w)+r_{2n}}} \quad \text{(by Lemma 4)}$$

$$\geqslant \left(A^{\frac{r_{2n}}{2}} D^{s_{2n}+w} A^{\frac{r_{2n}}{2}} \right)^{\frac{\widetilde{\alpha}(2n)}{\widetilde{\gamma}(2n-1)(s_{2n}+w)+r_{2n}}}$$

$$= g(s_{2n}+w)$$

and the last inequality holds by Theorem 1 because $\frac{\widetilde{\alpha}(2n)}{\widetilde{\gamma}(2n-1)(s_{2n}+w)+r_{2n}} \in [0,1]$.

(ii) Proof of the result that $G_{2n}(A,B,r_{2n},s_{2n})$ is a decreasing function of r_{2n}. Raise each side of (a) to the power $\frac{u}{r_{2n}} \in [0,1]$ for $0 < u \leqslant r_{2n}$, then we have the following inequality by Theorem 1:

$$A^u \geqslant \left\{ A^{\frac{r_{2n}}{2}} D^{s_{2n}} A^{\frac{r_{2n}}{2}} \right\}^{\frac{u}{\widetilde{\psi}(2n)}}.$$

$$G(A,B,r_{2n},s_{2n})$$

$$= A^{-\frac{r_{2n}}{2}} \left\{ A^{\frac{r_{2n}}{2}} \left(A^{-\frac{r_{2n-1}}{2}} \cdots \left(A^{\frac{r_2}{2}} \left(A^{-\frac{r_1}{2}} B^{s_1} A^{-\frac{r_1}{2}} \right)^{s_2} A^{\frac{r_2}{2}} \right)^{s_3} \cdots A^{-\frac{r_{2n-1}}{2}} \right)^{s_{2n}} A^{\frac{r_{2n}}{2}} \right\}^{\frac{\widetilde{\alpha}(2n)}{\widetilde{\psi}(2n)}} A^{-\frac{r_{2n}}{2}}$$

$$= A^{-\frac{r_{2n}}{2}} \left(A^{\frac{r_{2n}}{2}} D^{s_{2n}} A^{\frac{r_{2n}}{2}} \right)^{\frac{\widetilde{\alpha}(2n)}{\widetilde{\psi}(2n)}} A^{-\frac{r_{2n}}{2}}$$

$$= D^{\frac{s_{2n}}{2}} \left(D^{\frac{s_{2n}}{2}} A^{r_{2n}} D^{\frac{s_{2n}}{2}} \right)^{\frac{\widetilde{\alpha}(2n)-\widetilde{\psi}(2n)}{\widetilde{\psi}(2n)}} D^{\frac{s_{2n}}{2}} \quad \text{(by Lemma 4)}$$

$$= D^{\frac{s_{2n}}{2}} \left\{ \left(D^{\frac{s_{2n}}{2}} A^{r_{2n}} D^{\frac{s_{2n}}{2}} \right)^{\frac{\widetilde{\psi}(2n)+u}{\widetilde{\psi}(2n)}} \right\}^{\frac{\widetilde{\alpha}(2n)-\widetilde{\psi}(2n)}{\widetilde{\psi}(2n)+u}} D^{\frac{s_{2n}}{2}}$$

$$= D^{\frac{s_{2n}}{2}} \left\{ D^{\frac{s_{2n}}{2}} A^{\frac{r_{2n}}{2}} \left(A^{\frac{r_{2n}}{2}} D^{s_{2n}} A^{\frac{r_{2n}}{2}} \right)^{\frac{u}{\widetilde{\psi}(2n)}} A^{\frac{r_{2n}}{2}} D^{\frac{s_{2n}}{2}} \right\}^{\frac{\widetilde{\alpha}(2n)-\widetilde{\psi}(2n)}{\widetilde{\psi}(2n)+u}} D^{\frac{s_{2n}}{2}} \quad \text{(by Lemma 4)}$$

$$\geqslant D^{\frac{s_{2n}}{2}} \left(D^{\frac{s_{2n}}{2}} A^{r_{2n}+u} D^{\frac{s_{2n}}{2}} \right)^{\frac{\widetilde{\alpha}(2n)-\widetilde{\psi}(2n)}{\widetilde{\psi}(2n)+u}} D^{\frac{s_{2n}}{2}}$$

$$= G(A,B,r_{2n}+u,s_{2n}).$$

$\frac{\widetilde{\alpha}(2n)-\widetilde{\psi}(2n)}{\widetilde{\psi}(2n)+u} \in [-1,0]$ since $\widetilde{\alpha}(2n) \leqslant \widetilde{\psi}(2n)$. Therefore the last inequality holds by Theorem 1.

$(4) \Rightarrow (3)$

Put $q = 1$, $s_k = p_k$ and $r_k = t_k (k = 1, 2, \cdots, 2n)$ in (4).

$(3) \Rightarrow (1)$

Since $F_{2k}(A, B, t_{2k}, p_{2k})$ is decreasing for t_{2k} and p_{2k} by (3), we have the following inequality: $F_{2k}(A, B, t_{2k}, p_{2k}) \leqslant F_{2k}(A, B, t_{2k}, 1) \leqslant F_{2k}(A, B, t_{2k-1}, 1)$ holds for any natural number k such that $1 \leqslant k \leqslant n$.

Therefore we have the following inequality:

$$\left\{ A^{\frac{t_{2k}}{2}} \left(A^{-\frac{t_{2k-1}}{2}} \cdots \left(A^{\frac{t_2}{2}} \left(A^{-\frac{t_1}{2}} B^{p_1} A^{-\frac{t_1}{2}} \right)^{p_2} A^{\frac{t_2}{2}} \right)^{p_3} \cdots A^{-\frac{t_{2k-1}}{2}} \right)^{p_{2k}} A^{\frac{t_{2k}}{2}} \right\}^{\frac{\alpha(2k)}{\psi(2k)}}$$

$$\leqslant A^{\frac{t_{2k}-t_{2k-1}}{2}} \left\{ A^{\frac{t_{2k-2}}{2}} \left(A^{-\frac{t_{2k-3}}{2}} \cdots \left(A^{\frac{t_2}{2}} \left(A^{-\frac{t_1}{2}} B^{p_1} A^{-\frac{t_1}{2}} \right)^{p_2} A^{\frac{t_2}{2}} \right)^{p_3} \cdots A^{-\frac{t_{2k-3}}{2}} \right)^{p_{2k-2}} \right.$$

$$\left. \times A^{\frac{t_{2k-2}}{2}} \right\}^{\frac{\alpha(2k-2)}{\psi(2k-2)}} A^{\frac{-t_{2k-1}+t_{2k}}{2}}$$

holds for every natural number k such that $1 \leqslant k \leqslant n$.

So we have the following inequality:

$$\left\{ A^{\frac{t_{2n}}{2}} \left(A^{-\frac{t_{2n-1}}{2}} \cdots \left(A^{\frac{t_2}{2}} \left(A^{-\frac{t_1}{2}} B^{p_1} A^{-\frac{t_1}{2}} \right)^{p_2} A^{\frac{t_2}{2}} \right)^{p_3} \cdots A^{-\frac{t_{2n-1}}{2}} \right)^{p_{2n}} A^{\frac{t_{2n}}{2}} \right\}^{\frac{\alpha(2n)}{\psi(2n)}}$$

$$\leqslant A^{\frac{t_{2n}-t_{2n-1}}{2}} \left\{ A^{\frac{t_{2n-2}}{2}} \cdots \left(A^{\frac{t_2}{2}} \left(A^{-\frac{t_1}{2}} B^{p_1} A^{-\frac{t_1}{2}} \right)^{p_2} A^{\frac{t_2}{2}} \right)^{p_3} \cdots A^{-\frac{t_{2n-2}}{2}} \right\}^{\frac{\alpha(2n-2)}{\psi(2n-2)}} A^{\frac{t_{2n}-t_{2n-1}}{2}}$$

$$\leqslant A^{\frac{t_{2n}-t_{2n-1}+t_{2n-2}-t_{2n-3}}{2}} \left\{ A^{\frac{t_{2n-4}}{2}} \cdots \left(A^{\frac{t_2}{2}} \left(A^{-\frac{t_1}{2}} B^{p_1} A^{-\frac{t_1}{2}} \right)^{p_2} A^{\frac{t_2}{2}} \right)^{p_3} \cdots A^{-\frac{t_{2n-4}}{2}} \right\}^{\frac{\alpha(2n-4)}{\psi(2n-4)}}$$

$$\times A^{\frac{t_{2n}-t_{2n-1}+t_{2n-2}-t_{2n-3}}{2}}$$

$$\vdots$$

$$\leqslant A^{\frac{\alpha(2n)-t_2+t_1-1}{2}} \left\{ A^{\frac{t_2}{2}} \left(A^{-\frac{t_1}{2}} B^{p_1} A^{-\frac{t_1}{2}} \right)^{p_2} A^{\frac{t_2}{2}} \right\}^{\frac{1-t_1+t_2}{(p_1-t_1)p_2+t_2}} A^{\frac{\alpha(2n)-t_2+t_1-1}{2}}$$

$$\leqslant A^{\frac{\alpha(2n)-1}{2}} B A^{\frac{\alpha(2n)-1}{2}}$$

$$\leqslant A^{\alpha(2n)}. \quad \square$$

Although we can show the following Proposition 7 by (3) of Theorem 6, we will give an alternative proof.

PROPOSITION 7. *Let n be a natural number. Let $1 \leqslant p_j(j = 1, 2, \cdots, 2n-1, 2n)$, $t_{2k-1} \in [0,1]$ and $t_{2k-1} \leqslant t_{2k}(k = 1, 2, \cdots, n)$. If $0 \leqslant B \leqslant A$ with $0 < A$, then the fol-*

lowing inequality holds:

$$A^{-\frac{t_{2n}}{2}}\left\{A^{\frac{t_{2n}}{2}}\left(A^{-\frac{t_{2n-1}}{2}}\cdots\left(A^{\frac{t_2}{2}}\left(A^{-\frac{t_1}{2}}B^{p_1}A^{-\frac{t_1}{2}}\right)^{p_2}A^{\frac{t_2}{2}}\right)^{p_3}\cdots A^{-\frac{t_{2n-1}}{2}}\right)^{p_{2n}}A^{\frac{t_{2n}}{2}}\right\}^{\frac{\alpha(2n)}{\psi(2n)}}A^{-\frac{t_{2n}}{2}}$$

$$\leqslant A^{-\frac{t_{2n-1}}{2}}\left\{A^{\frac{t_{2n-2}}{2}}\left(A^{-\frac{t_{2n-3}}{2}}\cdots\left(A^{\frac{t_2}{2}}\left(A^{-\frac{t_1}{2}}B^{p_1}A^{-\frac{t_1}{2}}\right)^{p_2}A^{\frac{t_2}{2}}\right)^{p_3}\cdots A^{-\frac{t_{2n-3}}{2}}\right)^{p_{2n-2}}\right.$$

$$\left.\times A^{\frac{t_{2n-2}}{2}}\right\}^{\frac{\alpha(2n-2)}{\psi(2n-2)}}A^{-\frac{t_{2n-1}}{2}}.$$

Proof. Let $1\leqslant p_j$ $(j=1,2,\cdots,2n-3,2n-2)$, $t_{2k-1}\in[0,1]$ and $t_{2k-1}\leqslant t_{2k}$ $(k=1,2,\cdots,n-1)$.

If $0\leqslant B\leqslant A$ with $0<A$, then we have the following inequality by Theorem 5:

$$\left\{A^{\frac{t_{2n-2}}{2}}\left(A^{-\frac{t_{2n-3}}{2}}\cdots\left(A^{\frac{t_2}{2}}\left(A^{-\frac{t_1}{2}}B^{p_1}A^{-\frac{t_1}{2}}\right)^{p_2}A^{\frac{t_2}{2}}\right)^{p_3}\cdots A^{-\frac{t_{2n-3}}{2}}\right)^{p_{2n-2}}A^{\frac{t_{2n-2}}{2}}\right\}^{\frac{\alpha(2n-2)}{\psi(2n-2)}}$$

$$\leqslant A^{\alpha(2n-2)}.$$

Put

$$A_1 = A^{\alpha(2n-2)} \qquad \text{and}$$

$$B_1 = \left\{A^{\frac{t_{2n-2}}{2}}\left(A^{-\frac{t_{2n-3}}{2}}\cdots\left(A^{\frac{t_2}{2}}\left(A^{-\frac{t_1}{2}}B^{p_1}A^{-\frac{t_1}{2}}\right)^{p_2}A^{\frac{t_2}{2}}\right)^{p_3}\cdots A^{-\frac{t_{2n-3}}{2}}\right)^{p_{2n-2}}A^{\frac{t_{2n-2}}{2}}\right\}^{\frac{\alpha(2n-2)}{\psi(2n-2)}}.$$

Then we have $0\leqslant B_1\leqslant A_1$. Therefore we have the following inequality:

$$A_1^{-\frac{r}{2}}\left\{A_1^{\frac{r}{2}}\left(A_1^{-\frac{t}{2}}B_1^{p}A_1^{-\frac{t}{2}}\right)^{s}A_1^{\frac{r}{2}}\right\}^{\frac{1-t+r}{(p-t)s+r}}A_1^{-\frac{r}{2}}\leqslant A_1^{-\frac{t}{2}}B_1A_1^{-\frac{t}{2}}$$

holds for $t\in[0,1]$, $1\leqslant p,s$ and $t\leqslant r$ by (3) of Theorem 3.

Put $s=p_{2n}$, $p=\frac{\psi(2n-2)}{\alpha(2n-2)}p_{2n-1}$, $t=\frac{t_{2n-1}}{\alpha(2n-2)}$ and $r=\frac{t_{2n}}{\alpha(2n-2)}$.

Then it is obvious that

$$A_1^{\frac{t}{2}}=A^{\frac{t_{2n-1}}{2}},\quad A_1^{\frac{r}{2}}=A^{\frac{t_{2n}}{2}},$$

$$B_1^{p}=\left\{A^{\frac{t_{2n-2}}{2}}\left(A^{-\frac{t_{2n-3}}{2}}\cdots\left(A^{\frac{t_2}{2}}\left(A^{-\frac{t_1}{2}}B^{p_1}A^{-\frac{t_1}{2}}\right)^{p_2}A^{\frac{t_2}{2}}\right)^{p_3}\cdots A^{-\frac{t_{2n-3}}{2}}\right)^{p_{2n-2}}A^{\frac{t_{2n-2}}{2}}\right\}^{p_{2n-1}}$$

and

$$\frac{1-t+r}{(p-t)s+r}=\frac{1-\frac{t_{2n-1}}{\alpha(2n-2)}+\frac{t_{2n}}{\alpha(2n-2)}}{\left(\frac{\psi(2n-2)}{\alpha(2n-2)}p_{2n-1}-\frac{t_{2n-1}}{\alpha(2n-2)}\right)p_{2n}+\frac{t_{2n}}{\alpha(2n-2)}}$$

$$=\frac{\alpha(2n-2)-t_{2n-1}+t_{2n}}{(\psi(2n-2)p_{2n-1}-t_{2n-1})p_{2n}+t_{2n}}$$

$$=\frac{\alpha(2n)}{\psi(2n)}.$$

Therefore we have the following inequality:

$$A^{-\frac{t_{2n}}{2}}\left\{A^{\frac{t_{2n}}{2}}\left(A^{-\frac{t_{2n-1}}{2}}\cdots\left(A^{\frac{t_2}{2}}\left(A^{-\frac{t_1}{2}}B^{p_1}A^{-\frac{t_1}{2}}\right)^{p_2}A^{\frac{t_2}{2}}\right)^{p_3}\cdots A^{-\frac{t_{2n-1}}{2}}\right)^{p_{2n}}A^{\frac{t_{2n}}{2}}\right\}^{\frac{\alpha(2n)}{\psi(2n)}}A^{-\frac{t_{2n}}{2}}$$

$$\leqslant A^{-\frac{t_{2n-1}}{2}}\left\{A^{\frac{t_{2n-2}}{2}}\left(A^{-\frac{t_{2n-3}}{2}}\cdots\left(A^{\frac{t_2}{2}}\left(A^{-\frac{t_1}{2}}B^{p_1}A^{-\frac{t_1}{2}}\right)^{p_2}A^{\frac{t_2}{2}}\right)^{p_3}\cdots A^{-\frac{t_{2n-3}}{2}}\right)^{p_{2n-2}}\right.$$

$$\left.\times A^{\frac{t_{2n-2}}{2}}\right\}^{\frac{\alpha(2n-2)}{\psi(2n-2)}}A^{-\frac{t_{2n-1}}{2}}. \qquad \square$$

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