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NORMS OF MATRIX OPERATORS ON bv_p

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Abstract. In this paper we consider the problem of finding norms of certain matrix operators on sequence space bv_p. In fact, we consider these problems for weighted mean, and generalized Cesaro matrices on sequence space bv_p.

1. Introduction

By w, we denote the space of all real or complex valued sequences. Any vector subspace of w is called a sequence space. We write l_p for the space of all p-absolutely convergent series, with

$$\|x\|_p := \Big(\sum_{k=1}^{\infty} |x_k|^p\Big)^{\frac{1}{p}}.$$

For $0 < p < \infty$, the sequence space bv_p is defined by:

$$bv_p = \{x = (x_k) \in w : \sum_{k=1}^{\infty} |x_k - x_{k-1}|^p < \infty\},$$

where $x_0 = 0$, with

$$\|x\|_{bv_p} := \Big(\sum_{k=1}^{\infty} |x_k - x_{k-1}|^p\Big)^{\frac{1}{p}}.$$

A matrix $A = (a_{n,k})$ defines an operator by $Ax = y$, where $y_n = \sum_{k=1}^{\infty} a_{n,k}x_k$. Provided that this defines an operator from a sequence space X into itself, its norm is defined in the usual way,

$$\|A\|_X := \sup_{\|x\|_X=1} \|Ax\|_X.$$

We will estimate $\|A\|_{bv_p}$ when A is a weighed mean matrix $M_a = (m_{n,k})$ or a generalized Cesaro matrix $C_N = (b^N_{n,k})$. This matrices are defined as follows:

$$m_{nk} := \begin{cases} \frac{a_k}{A_n}, & 1 \leqslant k \leqslant n, \\ 0, & \text{otherwise}, \end{cases} \tag{1.1}$$

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and

$$b^N_{n,k} = \begin{cases} \frac{1}{n+N}, & 1 \leqslant k \leqslant n, \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

Hear $A_n = \sum_{k=1}^{n} a_k$ and $a = (a_n)_{n=1}^{\infty}$ is a non negative sequence with $a_1 > 0$.

We summarize the knowledge in the existing literature concerning the upper and lower bounds of the matrix operators on some sequence spaces. Lyons [10] examined the lower bound for the Cesaro operator on l_2. Bennett [2] studied the lower bounds for matrix operators on ℓ_p for $p \geqslant 1$. Jameson [5] computed the lower bounds and norms of operators on Lorentz sequence space $d(w,1)$. Jameson and Lashkaripour [6,7] studied norms and lower bounds of certain matrix operators on weighted ℓ_p spaces and Lorentz sequence space. In this paper, we determine the norms of some matrix operators such as weighted mean, and generalized Cesaro on sequence space bv_p.

2. Main result

THEOREM 2.1. *Suppose that $a_n \leqslant m a_r$ for all $r \leqslant n$. Then $||M_a||_{bv_p} \leqslant m^{\frac{p-1}{p}}$. In particular, the norms equals 1 when $p = 1$ (for any m)and when (a_n) is decreasing (for any p).*

THEOREM 2.2. *$||C_N||_{bv_p} \leqslant 1$ for all N and all $p \geqslant 1$.*

To prove Theorems 2.1 and 2.2 we need the following theorem.

SCHUR'S THEOREM. ([4], Theorem 275) *suppose that*

$$\sum_{k=1}^{\infty} |c_{n,k}| \leqslant R \quad \textit{for all } n, \qquad \sum_{n=1}^{\infty} |c_{n,k}| \leqslant K \quad \textit{for all } k,$$

(bounds for row and column sums respectively). Let $||C||_p$ denote the norm of C as an operator on ℓ_p. Then $||C||_p \leqslant R^{\frac{p-1}{p}} K^{\frac{1}{p}}$ (and $||C||_1 \leqslant K$).

A lower triangular operator on bv_p can be equated to an operator on ℓ_p, as follows. First, note that if $a_1 + a_2 + \ldots + a_k = A_k$, then

$$\sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{n} (A_n - A_{k-1})(x_k - x_{k-1}), \tag{2.1}$$

with $x_0 = A_0 = 0$ (reversed Abel summation). Now let $y_n = \sum_{k=1}^{n} a_{n,k} x_k$ and write $A_{n,k} = \sum_{j=1}^{k} a_{n,j}$. Also, write

$$u = (x_1, x_2 - x_1, x_3 - x_2, \ldots), \qquad v = (y_1, y_2 - y_1, y_3 - y_2, \ldots),$$

so that $||x||_{bv_p} = ||u||_p$ and $||y||_{bv_p} = ||v||_p$. By (2.1),

$$y_n = \sum_{k=1}^{n} (A_{n,n} - A_{n,k-1}) u_k. \tag{2.2}$$

If also $A_{n,n}=1$ for all n, then $v_n=y_n-y_{n-1}=\sum\limits_{k=1}^{n} c_{n,k}u_k$, where $c_{1,1}=1$,

$$c_{n,k}=A_{n-1,k-1}-A_{n,k-1}. \tag{2.3}$$

for $k\leqslant n$ with $n\geqslant 2$, also $c_{n,1}=0$ for $n\geqslant 2$ and $c_{n,k}=0$ for $k>n$.

Proof of Theorem 2.1. We have $A_{n,k}=\frac{A_k}{A_n}$. Note that (A_n) increases with n, and hence $c_{n,k}\geqslant 0$. Now consider column sums. For $k\geqslant 2$,

$$\sum_{n=k}^{K} c_{n,k}=\sum_{n=k}^{K} A_{k-1}\left(\frac{1}{A_{n-1}}-\frac{1}{A_n}\right)=A_{k-1}\left(\frac{1}{A_{k-1}}-\frac{1}{A_K}\right)\leqslant 1,$$

hence $\sum\limits_{n=k}^{\infty} c_{n,k}\leqslant 1$. For $n\geqslant 2$ the row sum is

$$\sum_{k=1}^{n} c_{n,k}=\sum_{k=1}^{n} A_{k-1}\left(\frac{1}{A_{n-1}}-\frac{1}{A_n}\right)\leqslant (n-1)A_{n-1}\left(\frac{1}{A_{n-1}}-\frac{1}{A_n}\right)=(n-1)\frac{a_n}{A_n}.$$

Since $a_n\leqslant ma_r$ for all $r\leqslant n$, we have $mA_n\geqslant na_n$, hence the row sum is not greater than m. By *Schur's theorem*, $||M_a||_{bv_p}\leqslant m^{\frac{p-1}{p}}$. In particular, if $p=1$ or (a_n) is decreasing, then $||M_a||_{bv_p}\leqslant 1$. Since $M_ax=x$,where $x=(1,1,1,\ldots)$, $||M_a||_{bv_p}=1$. □

Proof of Theorem 2.2. Let $a_{n,k}=\frac{1}{n+N}$ for $1\leqslant k\leqslant n$. By (2.2),

$$y_n=\sum_{k=1}^{n}\frac{n-k+1}{n+N}u_k,$$

hence $v_n=\sum\limits_{k=1}^{n} c_{n,k}u_k$ with $c_{1,1}=\frac{1}{1+N}$ and for $n\geqslant 2$,

$$c_{n,k}=\frac{n-k+1}{n+N}-\frac{n-k}{n+N-1}=(N+k-1)\left(\frac{1}{n+N-1}-\frac{1}{n+N}\right).$$

From this it is easily seen that $\sum\limits_{n=k}^{\infty} c_{n,k}=1$ for all k. Also,

$$\sum_{k=1}^{n} c_{n,k}=\frac{n(n+2N-1)}{2(n+N-1)(n+N)}.$$

This is not greater than 1, and in fact not greater than $\frac{1}{2}$ when $N=0$ or $N\geqslant 1$.By applying *Schur's theorem*, we get the result. □

COROLLARY 2.3. *C_0 is a bounded matrix operator from bv_p into itself with norm $\|C_0\|_{bv_p}=1$.*

Proof. Again, $C_0x=x$ where $x=(1,1,\ldots)$, hence $\|C_0\|_{bv_p}=1$. □

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