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SOME SHARP INEQUALITIES INVOLVING RECIPROCALS OF THE SEIFFERT AND OTHER MEANS

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Abstract. In the paper, by establishing the monotonicity of some functions involving the sine and cosine functions, we find some new sharp inequalities involving the reciprocals of the Seiffert, contra-harmonic, arithmetic, geometric and root-square means of two positive real numbers a and b with $a \neq b$.

1. Introduction

Let $C = \frac{a^2+b^2}{a+b}$, $A = \frac{a+b}{2}$, $G = \sqrt{ab}$, $S = \sqrt{\frac{a^2+b^2}{2}}$ be the contra-harmonic, arithmetic, geometric and root-square means of two positive real numbers a and b with $a \neq b$.

For $a,b > 0$ with $a \neq b$, the first Seiffert mean P and the second Seiffert mean T (see [11], [12, eq. (2.4)] and [13], respectively) are defined as follows

$$P = A\frac{t}{\arcsin t}, \tag{1.1}$$

$$T = A\frac{t}{\arctan t}. \tag{1.2}$$

where

$$t = \frac{a-b}{a+b}. \tag{1.3}$$

Recently, the Seiffert's mean has been the subject of intensive research.

In [6], the authors proved that inequality

$$\alpha S + (1-\alpha)A < T < \beta S + (1-\beta)A \tag{1.4}$$

holds for all $a,b > 0$ with $a \neq b$ if and only if $\alpha \leqslant \frac{4-\pi}{(\sqrt{2}-1)\pi}$ and $\beta \geqslant \frac{2}{3}$.

In [20], the double inequality

$$\frac{1}{2}(A+G) < P < \frac{2}{3}A + \frac{1}{3}G$$

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for all $a,b>0$ with $a\neq b$ was given.

In [14], the following inequality

$$P>\frac{3AG}{A+2G},$$

which is equivalent to

$$\frac{1}{P}<\frac{1}{3}\frac{1}{G}+\frac{2}{3}\frac{1}{A}, \tag{1.5}$$

for all $a,b>0$ with $a\neq b$ was given.

For more information on this topic, please refer to [4-10, 15-19].

In the paper, by establishing the monotonicity of some functions involving the sine and cosine functions, we find some new sharp inequalities involving reciprocals of the Seiffert, contra-harmonic, arithmetic, geometric, and root-square means of two positive real numbers a and b with $a\neq b$.

2. Lemmas

For establishing the monotonicity of some functions involving the sine and cosine functions, we need some lemmas below.

LEMMA 2.1. *The Bernoulli numbers B_{2n} for $n\in\mathbb{N}$ have the property*

$$(-1)^{n-1}B_{2n}=|B_{2n}|, \tag{2.1}$$

where the Bernoulli numbers B_i for $i\geqslant 0$ are defined by

$$\frac{x}{e^x-1}=\sum_{i=0}^{\infty}\frac{B_i}{i!}x^i=1-\frac{x}{2}+\sum_{i=1}^{\infty}B_{2i}\frac{x^{2i}}{(2i)!},\quad |x|<2\pi. \tag{2.2}$$

Proof. In [3, p. 16 and p. 56], it is listed that for $q\geqslant 1$

$$\zeta(2q)=(-1)^{q-1}\frac{(2\pi)^{2q}}{(2q)!}\frac{B_{2q}}{2}, \tag{2.3}$$

where ζ is the Riemann zeta function defined by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}. \tag{2.4}$$

From (2.3), the formula (2.1) follows. $\square$

LEMMA 2.2. ([1, p. 75, 4.3.70]) *For $0<|x|<\pi$,*

$$\cot x=\frac{1}{x}-\sum_{n=1}^{\infty}\frac{2^{2n}|B_{2n}|}{(2n)!}x^{2n-1}. \tag{2.5}$$

LEMMA 2.3. *For* $0<|x|<\pi$,

$$\frac{1}{\sin^2 x}=\frac{1}{x^2}+\sum_{n=1}^{\infty}\frac{2^{2n}(2n-1)|B_{2n}|}{(2n)!}x^{2(n-1)}. \tag{2.6}$$

Proof. Since

$$\frac{1}{\sin^2 x}=\csc^2 x=-\frac{\mathrm{d}}{\mathrm{d}x}(\cot x),$$

the formula (2.6) follows from differentiating (2.5). □

LEMMA 2.4. *For* $0<|x|<\pi$, *we have*

$$\frac{\cos x}{\sin^3 x}=\frac{1}{x^3}-\sum_{n=2}^{\infty}\frac{(2n-1)(n-1)2^{2n}|B_{2n}|}{(2n)!}x^{2n-3}. \tag{2.7}$$

Proof. The equality (2.7) follows from combination of

$$\frac{\cos x}{\sin^3 x}=-\frac{1}{2}\left(\frac{1}{\sin^2 x}\right)'$$

with Lemma 2.3. □

LEMMA 2.5. [2, p. 292, Lemma 1] *Let* f *and* g *be continuous on* $[a,b]$ *and differentiable in* (a,b) *such that* $g'(x)\neq 0$ *in* (a,b). *If* $\frac{f'(x)}{g'(x)}$ *is increasing* (*or decreasing*) *in* (a,b), *then the functions* $\frac{f(x)-f(b)}{g(x)-g(b)}$ *and* $\frac{f(x)-f(a)}{g(x)-g(a)}$ *are also increasing* (*or decreasing*) *in* (a,b).

3. Some trigonometric inequalities

For finding some new sharp inequalities involving the Seiffert, contra-harmonic, arithmetic, geometric, and root-square means of two positive real numbers a and b with $a\neq b$, we need the following monotonicity of some functions involving the sine and cosine functions.

THEOREM 1. *For* $x\in(0,\pi/2)$, *the function*

$$h_1(x)=\frac{\cos x(x-\sin x\cos x)}{\sin^3 x} \tag{3.1}$$

is strictly decreasing and satisfies

$$\lim_{x\to 0^+}h_1(x)=\frac{2}{3}\quad \text{and}\quad \lim_{x\to\pi/2^-}h_1(x)=0. \tag{3.2}$$

Proof. The function $h_1(x)$ may be rewritten as

$$h_1(x)=\frac{x\cos x}{\sin^3 x}-\frac{1}{\sin^2 x}+1$$

for $x\in(0,\pi/2)$. By using (2.6) and (2.7), we have

$$\begin{aligned}h_1(x)&=\frac{1}{x^2}-\sum_{n=2}^{\infty}\frac{2^{2n}(2n-1)(n-1)}{(2n)!}|B_{2n}|x^{2n-2}-\frac{1}{x^2}-\sum_{n=1}^{\infty}\frac{2^{2n}(2n-1)}{(2n)!}|B_{2n}|x^{2n-2}+1\\&=-\sum_{n=1}^{\infty}\frac{n(2n-1)2^{2n}}{(2n)!}|B_{2n}|x^{2n-2}+1.\end{aligned}$$

So the function $h_1(x)$ is strictly decreasing on $(0,\pi/2)$.

The limits in (3.2) may be concluded from L'Hôspital rule and standard argument. The proof of Theorem 1 is complete. □

THEOREM 2. *For $x\in(0,\pi/2)$, the function*

$$h_2(x)=\frac{\cos x(x-\sin x)}{\sin x(1-\cos x)} \tag{3.3}$$

is strictly decreasing, with

$$\lim_{x\to 0^+}h_2(x)=\frac{1}{3}\quad \textit{and}\quad \lim_{x\to\pi/2^-}h_2(x)=0. \tag{3.4}$$

Proof. It is obvious that

$$h_2(x)=1+\frac{f_1(x)}{f_2(x)},$$

where

$$f_1(x)=x\cot x-1\quad\text{and}\quad f_2(x)=1-\cos x.$$

Easy computations give

$$\frac{f_1'(x)}{f_2'(x)}=\frac{\sin x\cos x-x}{\sin^3 x}\triangleq\frac{f_3(x)}{f_4(x)}$$

and

$$\frac{f_3'(x)}{f_4'(x)}=-\frac{2}{3\cos x}.$$

Since $\frac{1}{\cos x}$ is increasing on $\left(0,\frac{\pi}{2}\right)$, the function $\frac{f_3'(x)}{f_4'(x)}$ is strictly decreasing on $\left(0,\frac{\pi}{2}\right)$. Hence, By Lemma 2.5 and the continuity of $h_2(x)$ at $x=\frac{\pi}{2}$, we see that $h_2(x)$ is strictly decreasing on $(0,\pi/2)$.

The limits in (3.4) can be deduced from L'Hôspital rule and standard argument. The proof of Theorem 2 is complete. □

4. New inequalities involving Seiffert and other means

In this section we will find some new sharp inequalities involving reciprocals of the Seiffert, contra-harmonic, arithmetic, geometric, and root-square means of two positive real numbers a and b with $a \neq b$.

THEOREM 3. *The double inequality*

$$\frac{\alpha}{G} + \frac{(1-\alpha)}{A} < \frac{1}{P} < \frac{\beta}{G} + \frac{(1-\beta)}{A} \tag{4.1}$$

holds for all $a, b > 0$ *with* $a \neq b$ *if and only if* $\alpha = 0$ *and* $\beta \geqslant \frac{1}{3}$.

Proof. The double inequality (4.1) is the same as

$$\alpha < \frac{\frac{A}{P} - 1}{\frac{A}{G} - 1} < \beta.$$

Without loss of generality, we assume that $a > b > 0$. Put $t = \frac{a-b}{a+b}$. Then $t \in (0,1)$ and

$$\frac{\frac{A}{P} - 1}{\frac{A}{G} - 1} = \frac{\frac{\arcsin t}{t} - 1}{\sqrt{\frac{1}{1-t^2}} - 1}.$$

Let $t = \sin\theta$ for $\theta \in \left(0, \frac{\pi}{2}\right)$. Then

$$\frac{\frac{A}{P} - 1}{\frac{A}{G} - 1} = \frac{\frac{\theta}{\sin\theta} - 1}{\frac{1}{\cos\theta} - 1} = \frac{\cos\theta(\theta - \sin\theta)}{\sin\theta(1-\cos\theta)}.$$

By Theorem 2, we obtain Theorem 3. $\square$

THEOREM 4. *The double inequality*

$$\frac{\alpha}{A} + \frac{(1-\alpha)}{C} < \frac{1}{T} < \frac{\beta}{A} + \frac{(1-\beta)}{C} \tag{4.2}$$

holds for all $a, b > 0$ *with* $a \neq b$ *if and only if* $\alpha \leqslant \frac{\pi}{2} - 1$ *and* $\beta \geqslant \frac{2}{3}$.

Proof. It is sufficient to show

$$\alpha < \frac{\frac{A}{T} - \frac{A}{C}}{1 - \frac{A}{C}} < \beta.$$

Without loss of generality, we assume that $a > b > 0$. Let $t = \frac{a-b}{a+b}$. Then $t \in (0,1)$ and

$$\frac{\frac{A}{T} - \frac{A}{C}}{1 - \frac{A}{C}} = \frac{\frac{\arctan t}{t} - \frac{1}{1+t^2}}{1 - \frac{1}{1+t^2}}.$$

Let $t=\tan\theta$ for $\theta\in\left(0,\frac{\pi}{4}\right)$. Then

$$\frac{\frac{A}{T}-\frac{A}{C}}{1-\frac{A}{C}}=\frac{\frac{\theta}{\tan\theta}-\cos^2\theta}{\sin^2\theta}=\frac{\cos\theta(\theta-\sin\theta\cos\theta)}{\sin^3\theta}.$$

By Theorem 1 and $h_1\left(\frac{\pi}{4}\right)=\frac{\pi}{2}-1$, we obtain Theorem 4. $\square$

THEOREM 5. *The double inequality*

$$\frac{\alpha}{A}+\frac{(1-\alpha)}{S}<\frac{1}{T}<\frac{\beta}{A}+\frac{(1-\beta)}{S} \tag{4.3}$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha\leqslant\frac{\pi-2\sqrt{2}}{4-2\sqrt{2}}$ and $\beta\geqslant\frac{1}{3}$.

Proof. The inequality (4.3) is equivalent to

$$\alpha<\frac{\frac{A}{T}-\frac{A}{S}}{1-\frac{A}{S}}<\beta.$$

Without loss of generality, we assume that $a>b>0$. Let $t=\frac{a-b}{a+b}$. Then $t\in(0,1)$ and

$$\frac{\frac{A}{T}-\frac{A}{S}}{1-\frac{A}{S}}=\frac{\frac{\arctan t}{t}-\sqrt{\frac{1}{1+t^2}}}{1-\sqrt{\frac{1}{1+t^2}}}.$$

Let $t=\tan\theta$ for $\theta\in\left(0,\frac{\pi}{4}\right)$. Then

$$\frac{\frac{A}{T}-\frac{A}{S}}{1-\frac{A}{S}}=\frac{\frac{\theta}{\tan\theta}-\cos\theta}{1-\cos\theta}=\frac{\cos\theta(\theta-\sin\theta)}{\sin\theta(1-\cos\theta)}. \tag{4.4}$$

By Theorem 2 and $h_2\left(\frac{\pi}{4}\right)=\frac{\pi-2\sqrt{2}}{4-2\sqrt{2}}$, we obtain Theorem 5. $\square$

REMARK 4.1. E. Neuman pointed out that (1.5), a special case of Theorem 3 for $\beta=\frac{1}{3}$ follows from the inequality

$$\left(A^2G\right)^{\frac{1}{3}}<P, \tag{4.5}$$

(see [12]) by taking reciprocals and next using the inequality of arithmetic and geometric means. Similarly, using

$$\left(S^2A\right)^{1/3}<T, \tag{4.6}$$

(see [12]) one obtains

$$\frac{1}{T}<\frac{2}{3}\frac{1}{S}+\frac{1}{3}\frac{1}{A}. \tag{4.7}$$

The well known fact that $G<P<A$ and $A<T<S$ is utilized to claim that $\beta\geqslant\frac{1}{3}$ is an optimal value in Theorems 3 and 5.

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