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SOME INEQUALITIES FOR UNITARILY INVARIANT NORMS

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Abstract. This paper aims to present some inequalities for unitarily invariant norms. In section 2, we give a refinement of the Cauchy-Schwarz inequality for matrices. In section 3, we obtain an improvement for the result of Bhatia and Kittaneh [Linear Algebra Appl. 308 (2000) 203-211]. In section 4, we establish an improved Heinz inequality for the Hilbert-Schmidt norm. Finally, we present an inequality involving positive definite matrix and Hilbert-Schmidt norm. Then we use it to discuss the conjecture on the Hilbert-Schmidt norm of matrices proposed by Sloane and Harwit and the conjecture is proved for some special matrices.

1. Introduction

Let $M_{m,n}$ be the space of $m\times n$ complex matrices and $M_n = M_{n,n}$. Let D_n be the collections of all $n\times n$ matrices with entries in the interval $[0,1]$. The conjugate transpose of $A\in M_{m,n}$ is the matrix $A^*\in M_{n,m}$. Let $\|\cdot\|$ denote any unitarily invariant norm on M_n. So, $\|UAV\| = \|A\|$ for all $A\in M_n$ and for all unitary matrices $U,V\in M_n$. Two classes of such norms are special important. The first is the class of *Ky Fan k-norm* $\|\cdot\|_{(k)}$ defined as

$$\|A\|_{(k)} = \sum_{j=1}^{k} s_j(A), k = 1,\cdots,n,$$

where, $s_1(A)\geqslant s_2(A)\geqslant\cdots\geqslant s_{n-1}(A)\geqslant s_n(A)$ are the singular values of A, that is, the eigenvalues of the positive semidefinite matrix $|A| = (AA^*)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity. The second is the class of *Schatten p-norm* $\|\cdot\|_p$ defined as

$$\|A\|_p = \left(\sum_{j=1}^{n} s_j^p(A)\right)^{1/p} = (\operatorname{tr}|A|^p)^{1/p}, \quad 1\leqslant p<\infty.$$

For $A = (a_{ij})\in M_n$, the norm

$$\|A\|_2 = \sqrt{\sum_{j=1}^{n} s_j^2(A)} = \sqrt{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\right)} = \sqrt{\operatorname{tr}|A|^2}$$

is also called the Hilbert-Schmidt norm or Frobenius norm (and sometimes written as $\|A\|_F$ for that reason). It plays a basic role in matrix analysis.

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2. Refinement of the Cauchy-Schwarz inequality for matrices

For all $A,B \in M_n$, any real number $r > 0$ and every unitarily invariant norm, Horn and Mathias [2, 3] obtained the following matrix Cauchy-Schwarz inequality

$$|||A^*B|^r||^2 \leqslant ||(AA^*)^r|| \cdot ||(BB^*)^r||. \tag{2.1}$$

Bhatia and Davis [4] (see also [5, p. 267, Theorem IX.5.2]) generalized the inequality (2.1) to the following form

$$|||A^*XB|^r||^2 \leqslant |||AA^*X|^r|| \cdot |||XBB^*|^r|| \tag{2.2}$$

for all $A,B,X \in M_n$ and any real number $r > 0$, which is equivalent to

$$|||A^{1/2}XB^{1/2}|^r||^2 \leqslant |||AX|^r|| \cdot |||XB|^r|| \tag{2.3}$$

for positive semidefinite matrices A,B and arbitrary $X \in M_n$.

Let $A,B,X \in M_n$ such that A and B are positive semidefinite. Then, for every unitarily invariant norm and every positive real number r, the function

$$\varphi(t) = |||A^tXB^{1-t}|^r|| \cdot |||A^{1-t}XB^t|^r||$$

is convex on $[0,1]$ and attains its minimum at $t = \frac{1}{2}$. Consequently, it is decreasing on $\left[0,\frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2},1\right]$. See [6, Theorem 1]. Using the convexity of the function $\varphi(t)$, Hiai and Zhan [6] obtained the following inequality

$$|||A^{1/2}XB^{1/2}|^r||^2 \leqslant |||A^tXB^{1-t}|^r|| \cdot |||A^{1-t}XB^t|^r|| \leqslant |||AX|^r|| \cdot |||XB|^r||, \tag{2.4}$$

which is a refinement of the inequality (2.3).

In this section, we utilize the convexity of the function $\varphi(t)$ to obtain an inequality for unitarily invariant norms that leads to a refinement of the second inequality in (2.4). To do this, we need the following lemma on convex function (see [1, Lemma 2.2]).

LEMMA 2.1. *Let f be a real valued convex function on an interval $[a,b]$ which contains (x_1,x_2). Then for $x_1 \leqslant x \leqslant x_2$, we have*

$$f(x) \leqslant \frac{f(x_2)-f(x_1)}{x_2-x_1}x - \frac{x_1f(x_2)-x_2f(x_1)}{x_2-x_1}.$$

THEOREM 2.1. *Let $A,B,X \in M_n$ such that A and B are positive semidefinite. For every unitarily invariant norm, every positive real number r and every t satisfying $0 \leqslant t \leqslant 1$, we have*

$$\varphi(t) \leqslant (1-2t_0)\,|||AX|^r|| \cdot |||XB|^r|| + 2t_0|||A^{1/2}XB^{1/2}|^r||^2, \tag{2.5}$$

where $t_0 = \min\{t, 1-t\}$.

Proof. If $0 \leqslant t \leqslant \frac{1}{2}$, then by the convexity of the function $\varphi(t)$ and Lemma 2.1, we have

$$\varphi(t) \leqslant \frac{\varphi\left(\frac{1}{2}\right)-\varphi(0)}{\frac{1}{2}-0}t - \frac{0\cdot\varphi\left(\frac{1}{2}\right)-\frac{1}{2}\varphi(0)}{\frac{1}{2}-0}.$$

That is,

$$\varphi(t) \leqslant (1-2t)\,\varphi(0) + 2t\varphi\left(\frac{1}{2}\right),$$

and (2.5) holds.

If $\frac{1}{2} \leqslant t \leqslant 1$, then by the convexity of the function $\varphi(t)$ and Lemma 2.1, we have

$$\varphi(t) \leqslant \frac{\varphi(1) - \varphi\left(\frac{1}{2}\right)}{1-\frac{1}{2}} t - \frac{\frac{1}{2}\varphi(1) - \varphi\left(\frac{1}{2}\right)}{1-\frac{1}{2}}.$$

That is,

$$\varphi(t) \leqslant (2t-1)\,\varphi(1) + 2(1-t)\,\varphi\left(\frac{1}{2}\right),$$

and again (2.5) holds. This completes the proof. $\square$

Now, we give a simple comparison between the upper bound in (2.4) and (2.5).

$$|||AX|^r|| \cdot |||XB|^r|| - (1-2t_0)\,|||AX|^r|| \cdot |||XB|^r|| - 2t_0|||A^{1/2}XB^{1/2}|^r||^2$$

$$= 2t_0\left(|||AX|^r|| \cdot |||XB|^r|| - |||A^{1/2}XB^{1/2}|^r||^2\right) \geqslant 0.$$

So, Theorem 2.1 is a refinement of the second inequality in (2.4).

3. An improvement for the result of Bhatia and Kittaneh

Bhatia and Kittaneh [7] proved that if $A, B \in M_n$ are positive semidefinite, then

$$\left\|A^{3/2}B^{1/2} + A^{1/2}B^{3/2}\right\| \leqslant \frac{1}{2}\left\|(A+B)^2\right\|. \tag{3.1}$$

Meanwhile, the following inequality

$$\|AB\| \leqslant \frac{1}{4}\left\|(A+B)^2\right\| \tag{3.2}$$

was also proved by Bhatia and Kittaneh [7] for positive semidefinite matrices A, B.

In this section, we first obtain an inequality involving unitarily invariant norms. After that, we present an improvement of the inequality (3.2) for the Hilbert-Schmidt norm. To do this, we need the following lemma (see [8, Theorem 2]).

LEMMA 3.1. *Let $A, B, X \in M_n$ such that A and B are positive semidefinite. If $0 \leqslant v \leqslant 1$, then*

$$\left\|A^v X B^{1-v}\right\| \leqslant \|AX\|^v \|XB\|^{1-v}.$$

THEOREM 3.1. *Let $A,B,X \in M_n$ such that A and B are positive semidefinite. If $0 \leqslant v \leqslant 1$, then*

$$4\left\|A^vXB^{1-v}\right\|^2+\left(\|AX\|^{2v}-\|XB\|^{2(1-v)}\right)^2 \leqslant \|AX\|^{4v}+\|XB\|^{4(1-v)}+2\left\|A^vXB^{1-v}\right\|^2.$$

Proof. By Lemma 3.1, we have

$$\begin{aligned}\|AX\|^{4v}+\|XB\|^{4(1-v)}-2\left\|A^vXB^{1-v}\right\|^2 &\geqslant \|AX\|^{4v}+\|XB\|^{4(1-v)}-2\|AX\|^{2v}\|XB\|^{2(1-v)}\\ &=\left(\|AX\|^{2v}-\|XB\|^{2(1-v)}\right)^2 \geqslant 0.\end{aligned}$$

This completes the proof. □

Let $A,B,X \in M_n$ such that A and B are positive semidefinite. Note that

$$\|AX+XB\|_2^2=\|AX\|_2^2+\|XB\|_2^2+2\left\|A^{1/2}XB^{1/2}\right\|_2^2.$$

So, taking $v=1$ and $\|\cdot\|=\|\cdot\|_2$ in Theorem 3.1, then we have the following result.

COROLLARY 3.1. *Let $A,B,X \in M_n$ such that A and B are positive semidefinite. Then*

$$4\left\|A^{1/2}XB^{1/2}\right\|_2^2+(\|AX\|_2-\|XB\|_2)^2 \leqslant \|AX+XB\|_2^2.$$

Now, we give an improvement of the inequality (3.2) for the Hilbert-Schmidt norm.

THEOREM 3.2. *Let $A,B \in M_n$ be positive semidefinite. Then*

$$\sqrt{\|AB\|_2^2+\frac{1}{4}\left(\|A^{3/2}B^{1/2}\|_2-\|A^{1/2}B^{3/2}\|_2\right)^2} \leqslant \frac{1}{4}\left\|(A+B)^2\right\|_2.$$

Proof. Taking

$$X=A^{1/2}B^{1/2}.$$

Then, by Corollary 3.1, we have

$$4\|AB\|_2^2+\left(\left\|A^{3/2}B^{1/2}\right\|_2-\left\|A^{1/2}B^{3/2}\right\|_2\right)^2 \leqslant \left\|A^{3/2}B^{1/2}+A^{1/2}B^{3/2}\right\|_2^2. \tag{3.3}$$

It follows from (3.1) and (3.3) that

$$4\|AB\|_2^2+\left(\left\|A^{3/2}B^{1/2}\right\|_2-\left\|A^{1/2}B^{3/2}\right\|_2\right)^2 \leqslant \frac{1}{4}\left\|(A+B)^2\right\|_2^2.$$

That is,

$$\sqrt{\|AB\|_2^2+\frac{1}{4}\left(\|A^{3/2}B^{1/2}\|_2-\|A^{1/2}B^{3/2}\|_2\right)^2} \leqslant \frac{1}{4}\left\|(A+B)^2\right\|_2.$$

This completes the proof. □

4. Improved Heinz inequality for matrices

Bhatia and Davis proved in [9] that if $A,B,X \in M_n$ such that A and B are positive semidefinite and if $0 \leqslant v \leqslant 1$, then

$$\left\|A^{1/2}XB^{1/2}\right\| \leqslant \left\|\frac{A^{v}XB^{1-v}+A^{1-v}XB^{v}}{2}\right\| \leqslant \left\|\frac{AX+XB}{2}\right\|.$$

Kittaneh and Manasrah [10] proved that if $A,B,X \in M_n$ such that A and B are positive semidefinite, then

$$2\left\|A^{1/2}XB^{1/2}\right\|_2 + a^2 \leqslant \|AX+XB\|_2, \tag{4.1}$$

where $a = \sqrt{\|AX\|_2} - \sqrt{\|XB\|_2}$. This is a refinement of arithmetic-geometric mean inequality for the Hilbert-Schmidt norm. Inequality (4.1) is equivalent to the following inequality

$$4\left\|A^{1/2}XB^{1/2}\right\|_2^2 + 4a^2\left\|A^{1/2}XB^{1/2}\right\|_2 + a^4 \leqslant \|AX+XB\|_2^2, \tag{4.2}$$

where $a = \sqrt{\|AX\|_2} - \sqrt{\|XB\|_2}$.

By (4.1), Kittaneh and Manasrah [10] obtained an improvement of the Heinz inequality for the Hilbert-Schmidt norm which can be stated as follows:

$$\left\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\right\|_2 + 2v_0a^2 \leqslant \|AX+XB\|_2, \tag{4.3}$$

where $v_0 = \min\{v, 1-v\}$ and $a = \sqrt{\|AX\|_2} - \sqrt{\|XB\|_2}$.

By (4.2), we will give another improvement of the Heinz inequality for the Hilbert-Schmidt norm. To do this, we need the following lemma (see [5, p. 265]).

LEMMA 4.1. *Let $A,B,X \in M_n$ such that A and B are positive semidefinite. Then, for each unitarily invariant norm, the function*

$$g(v) = \left\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\right\|$$

is a continuous convex function on $[0,1]$ and attains its minimum at $v = \frac{1}{2}$. Moreover, $g(v)$ is twice differentiable on $(0,1)$.

THEOREM 4.1. *Let $A,B,X \in M_n$ such that A and B are positive semidefinite. If $0 \leqslant v \leqslant 1$, then*

$$\left\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\right\|_2^2 + 8v_0a^2\left\|A^{1/2}XB^{1/2}\right\|_2 + 2v_0a^4 \leqslant \|AX+XB\|_2^2, \tag{4.4}$$

where $v_0 = \min\{v, 1-v\}$ and $a = \sqrt{\|AX\|_2} - \sqrt{\|XB\|_2}$.

Proof. Let

$$f(v) = \|AX+XB\|_2^2 - \left\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\right\|_2^2 = \|AX+XB\|_2^2 - g^2(v).$$

Define

$$\psi(v) = \frac{f(v)}{v_0}, \quad 0 < v < 1.$$

That is,

$$\psi(v) = \begin{cases} \dfrac{f(v)}{v}, & 0 < v \leqslant \dfrac{1}{2}, \\[2ex] \dfrac{f(v)}{1-v}, & \dfrac{1}{2} \leqslant v < 1. \end{cases}$$

So, we have

$$\psi'(v) = \begin{cases} \dfrac{-2vg(v)g'(v) - f(v)}{v^2}, & 0 < v < \dfrac{1}{2}, \\[2ex] \dfrac{-2(1-v)g(v)g'(v) + f(v)}{(1-v)^2}, & \dfrac{1}{2} < v < 1. \end{cases}$$

and

$$\psi'_-\left(\frac{1}{2}\right) = -4f\left(\frac{1}{2}\right) \leqslant 0, \quad \psi'_+\left(\frac{1}{2}\right) = 4f\left(\frac{1}{2}\right) \geqslant 0.$$

Consider the following two functions

$$\omega_1(v) = -2vg(v)g'(v) - f(v), \quad 0 \leqslant v \leqslant \frac{1}{2}$$

and

$$\omega_2(v) = -2(1-v)g(v)g'(v) + f(v), \quad \frac{1}{2} \leqslant v \leqslant 1.$$

Then, we have

$$\omega_1(0) = \omega_2(1) = 0, \omega_1\left(\frac{1}{2}\right) = -\omega_2\left(\frac{1}{2}\right) = -f\left(\frac{1}{2}\right) \leqslant 0.$$

Meanwhile, we obtain

$$\omega_1'(v) = -2v\left(\left(g'(v)\right)^2 + g(v)g''(v)\right) \leqslant 0, \quad 0 \leqslant v \leqslant \frac{1}{2},$$

and

$$\omega_2'(v) = -2(1-v)\left(\left(g'(v)\right)^2 + g(v)g''(v)\right) \leqslant 0, \quad \frac{1}{2} \leqslant v \leqslant 1.$$

Thus,

$$\begin{cases} \psi'(v) \leqslant 0, & 0 < v < \frac{1}{2}, \\ \psi'(v) \geqslant 0, & \frac{1}{2} < v < 1. \end{cases}$$

It follows from the continuity of $\psi(v)$ and the symmetry of $\psi(v)$ about $v = \frac{1}{2}$ that $\psi(v)$ attains its minimum at $v = \frac{1}{2}$. So, by (4.2), we have

$$\psi(v) \geqslant 2\left(\|AX + XB\|_2^2 - 4\left\|A^{1/2}XB^{1/2}\right\|_2^2\right) \geqslant 2\left(4a^2\left\|A^{1/2}XB^{1/2}\right\|_2 + a^4\right).$$

Thus,

$$\left\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\right\|_2^2+8v_0a^2\left\|A^{1/2}XB^{1/2}\right\|_2+2v_0a^4\leqslant\|AX+XB\|_2^2.$$

This completes the proof. □

It should be noticed that neither (4.3) nor (4.4) is in general better than the other.

5. On a conjecture concerning the Hilbert-Schmidt norm of matrices

In this section, we shall mainly adopt the notation and terminology in [14]. For convenience, recall that. The spectral radius of a square matrix A is the nonnegative real number

$$\rho(A)\triangleq\max\{|\lambda|:\lambda \text{ is an eigenvalue of } A\}.$$

A Hadamard matrix is a square matrix with entries equal to ± 1 whose rows and hence columns are mutually orthogonal. In other words, a Hadamard matrix of order n is a $\{1,-1\}$ -matrix A satisfying $AA^T=nI$, where I is the identity matrix [11, p. 126].

In 1976, Sloane and Harwit [12] made the following conjecture. See also [11, p. 130].

CONJECTURE. *If A is a nonsingular matrix of order n all of whose entries are in the interval $[0,1]$, then*

$$\frac{2n}{n+1}\leqslant\left\|A^{-1}\right\|_2.$$

Equality holds if and only if A is an S-matrix.

An S-matrix of order n is a $\{0,1\}$-matrix formed by taking a Hadamard matrix of order $n+1$ in which the entries in the first row and column are 1, changing 1's to 0's and -1's to 1's, and deleting the first row and column [11, p. 130].

This problem arose from weighing designs in optics and statistics. Recently, the conjecture was proved for some special matrices and the following results were obtained.

THEOREM 5.1. [13] *Let $A\in D_n$ be a positive definite matrix and suppose that $\rho(A)\leqslant\frac{\sqrt{2}}{4}n$. Then*

$$\frac{2n}{n+1}\leqslant\left\|A^{-1}\right\|_2.$$

THEOREM 5.2. [14] *Let A be a nonsingular matrix of order n and suppose that the modulus of entries of A is from $[0,1]$. If*

$$\|A\|_2\leqslant\frac{\sqrt{n^3+n^2-n-1}}{n},$$

then

$$\frac{2n}{n+1} \leqslant \left\|A^{-1}\right\|_2.$$

From the Cauchy-Schwarz inequality, we have the following inequality

$$\frac{n}{\|A\|_2} \leqslant \left\|A^{-1}\right\|_2.$$

So, if

$$\|A\|_2 \leqslant \frac{n+1}{2},$$

then, we have

$$\frac{2n}{n+1} \leqslant \left\|A^{-1}\right\|_2.$$

In this section, if A is a positive definite matrix of order n, we first obtain a lower bound for $\left\|A^{-1}\right\|_2$. Then we use it to discuss the conjecture on the Hilbert-Schmidt norm of matrices proposed by Sloane and Harwit and the conjecture is proved for some special matrices. To do this, we need the following lemma.

LEMMA 5.1. *If A is a positive definite matrix of order n, then*

$$n^2 \leqslant \mathrm{tr}A \cdot \mathrm{tr}A^{-1}.$$

Proof. By the harmonic-arithmetic inequality. □

THEOREM 5.3. *If A is a positive definite matrix of order n, then*

$$\frac{n\sqrt{n}}{\mathrm{tr}A} \leqslant \left\|A^{-1}\right\|_2.$$

Proof. Using the Cauchy–Schwarz inequality, we have

$$(\mathrm{tr}A)^2 \leqslant n\,\|A\|_2^2.$$

That is,

$$\mathrm{tr}A \leqslant \sqrt{n}\,\|A\|_2.$$

So,

$$\mathrm{tr}A^{-1} \leqslant \sqrt{n}\left\|A^{-1}\right\|_2. \tag{5.1}$$

By Lemma 5.1 and (5.1), we have

$$\frac{n\sqrt{n}}{\mathrm{tr}A} \leqslant \left\|A^{-1}\right\|_2.$$

This completes the proof. □

COROLLARY 5.1. *If $A \in D_n$ is a positive definite matrix, then*

$$\frac{2n}{n+1} \leqslant \left\|A^{-1}\right\|_2.$$

Proof. Since $A \in D_n$, we have

$$\operatorname{tr}A \leqslant n.$$

So, by Theorem 5.3, we obtain

$$\frac{2n}{n+1} \leqslant \sqrt{n} \leqslant \frac{n\sqrt{n}}{\operatorname{tr}A} \leqslant \left\|A^{-1}\right\|_2.$$

This completes the proof. □

Obviously Corollary 5.1 is a refinement of Theorem 5.1.

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