SOME INEQUALITIES FOR UNITARILY INVARIANT NORMS

XINGKAI HU

(Communicated by J. J. Koliha)

Abstract. This paper aims to present some inequalities for unitarily invariant norms. In section 2, we give a refinement of the Cauchy-Schwarz inequality for matrices. In section 3, we obtain an improvement for the result of Bhatia and Kittaneh [Linear Algebra Appl. 308 (2000) 203-211]. In section 4, we establish an improved Heinz inequality for the Hilbert-Schmidt norm. Finally, we present an inequality involving positive definite matrix and Hilbert-Schmidt norm. Then we use it to discuss the conjecture on the Hilbert-Schmidt norm of matrices proposed by Sloane and Harwit and the conjecture is proved for some special matrices.

1. Introduction

Let $M_{m,n}$ be the space of $m \times n$ complex matrices and $M_n = M_{n,n}$. Let $D_n$ be the collections of all $n \times n$ matrices with entries in the interval $[0,1]$. The conjugate transpose of $A \in M_{m,n}$ is the matrix $A^* \in M_{n,m}$. Let $\| \cdot \|$ denote any unitarily invariant norm on $M_n$. So, $\|UAV\| = \|A\|$ for all $A \in M_n$ and for all unitary matrices $U,V \in M_n$. Two classes of such norms are special important. The first is the class of Ky Fan $k$-norm $\| \cdot \|_{(k)}$ defined as

$$\|A\|_{(k)} = \sum_{j=1}^{k} s_j(A), k = 1, \ldots, n,$$

where, $s_1(A) \geq s_2(A) \geq \cdots \geq s_{n-1}(A) \geq s_n(A)$ are the singular values of $A$, that is, the eigenvalues of the positive semidefinite matrix $|A| = (AA^*)^{1/2}$, arranged in decreasing order and repeated according to multiplicity. The second is the class of Schatten $p$-norm $\| \cdot \|_p$ defined as

$$\|A\|_p = \left( \sum_{j=1}^{n} s_j^p(A) \right)^{1/p} = (\text{tr}|A|^p)^{1/p}, \quad 1 \leq p < \infty.$$

For $A = (a_{ij}) \in M_n$, the norm

$$\|A\|_2 = \sqrt{\sum_{j=1}^{n} s_j^2(A)} = \sqrt{\left( \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)} = \sqrt{\text{tr}|A|^2}$$

is also called the Hilbert-Schmidt norm or Frobenius norm (and sometimes written as $\|A\|_F$ for that reason). It plays a basic role in matrix analysis.


Keywords and phrases: Unitarily invariant norms, positive semidefinite matrices, Cauchy-Schwarz inequality, convex function, Heinz inequality.
2. Refinement of the Cauchy-Schwarz inequality for matrices

For all $A, B \in M_n$, any real number $r > 0$ and every unitarily invariant norm, Horn and Mathias [2, 3] obtained the following matrix Cauchy-Schwarz inequality

$$ |||A^t B^r|||^2 \leq ||(AA^*)^r|| \cdot ||(BB^*)^r||. \quad (2.1) $$

Bhatia and Davis [4] (see also [5, p. 267, Theorem IX.5.2]) generalized the inequality (2.1) to the following form

$$ |||A^t XB^r|||^2 \leq |||AA^* X^r|| \cdot |||XBB^*^r||| $$

(2.2)

for all $A, B, X \in M_n$ and any real number $r > 0$, which is equivalent to

$$ |||A^{1/2} XB^{1/2}|||^2 \leq |||A^r X^r|| \cdot |||X^r B^r||| $$

(2.3)

for positive semidefinite matrices $A, B$ and arbitrary $X \in M_n$.

Let $A, B, X \in M_n$ such that $A$ and $B$ are positive semidefinite. Then, for every unitarily invariant norm and every positive real number $r$, the function

$$ \phi(t) = |||A^t XB^{1-t}|| \cdot ||A^{1-t} X B^t|| $$

is convex on $[0, 1]$ and attains its minimum at $t = \frac{1}{2}$. Consequently, it is decreasing on $[0, \frac{1}{2}]$ and increasing on $[\frac{1}{2}, 1]$ . See [6, Theorem 1]. Using the convexity of the function $\phi(t)$, Hiai and Zhan [6] obtained the following inequality

$$ |||A^{1/2} XB^{1/2}|||^2 \leq |||A^r X^r|| | ||A^{1-r} X B^r|| \leq |||AX^r|| | ||XB^r||, $$

(2.4)

which is a refinement of the inequality (2.3).

In this section, we utilize the convexity of the function $\phi(t)$ to obtain an inequality for unitarily invariant norms that leads to a refinement of the second inequality in (2.4). To do this, we need the following lemma on convex function (see [1, Lemma 2.2]).

**Lemma 2.1.** Let $f$ be a real valued convex function on an interval $[a, b]$ which contains $(x_1, x_2)$. Then for $x_1 \leq x \leq x_2$, we have

$$ f(x) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} x - \frac{x_1 f(x_2) - x_2 f(x_1)}{x_2 - x_1}. $$

**Theorem 2.1.** Let $A, B, X \in M_n$ such that $A$ and $B$ are positive semidefinite. For every unitarily invariant norm, every positive real number $r$ and every $t$ satisfying $0 \leq t \leq 1$, we have

$$ \phi(t) \leq (1 - 2t_0) |||AX^r|| | ||XB^r|| + 2t_0 |||A^{1/2} XB^{1/2}|||^2, $$

(2.5)

where $t_0 = \min \{t, 1 - t\}$.

**Proof.** If $0 \leq t \leq \frac{1}{2}$, then by the convexity of the function $\phi(t)$ and Lemma 2.1, we have

$$ \phi(t) \leq \frac{\phi\left(\frac{1}{2}\right) - \phi(0)}{\frac{1}{2} - 0} t - \frac{0 \cdot \phi\left(\frac{1}{2}\right) - \frac{1}{2} \phi(0)}{\frac{1}{2} - 0}. $$
That is,
\[ \phi(t) \leq (1 - 2t) \phi(0) + 2t \phi \left( \frac{1}{2} \right), \]
and (2.5) holds.

If \( \frac{1}{2} \leq t \leq 1 \), then by the convexity of the function \( \phi(t) \) and Lemma 2.1, we have
\[ \phi(t) \leq \frac{\phi(1) - \phi \left( \frac{1}{2} \right)}{1 - \frac{1}{2}} t - \frac{1}{2} \phi(1) - \phi \left( \frac{1}{2} \right). \]
That is,
\[ \phi(t) \leq (2t - 1) \phi(1) + 2 (1 - t) \phi \left( \frac{1}{2} \right), \]
and again (2.5) holds. This completes the proof. \( \square \)

Now, we give a simple comparison between the upper bound in (2.4) and (2.5).
\[
\| |AX|^r\| \cdot \| |XB|^r\| - (1 - 2t_0) |||AX|^r|| \cdot \| |XB|^r\| - 2t_0 |||A^{1/2}XB^{1/2}|^r||^2
\]
\[
= 2t_0 \left( |||AX|^r|| \cdot |||XB|^r|| - |||A^{1/2}XB^{1/2}|^r||^2 \right) \geq 0.
\]
So, Theorem 2.1 is a refinement of the second inequality in (2.4).

3. An improvement for the result of Bhatia and Kittaneh

Bhatia and Kittaneh [7] proved that if \( A, B \in M_n \) are positive semidefinite, then
\[
\left\| A^{3/2}B^{1/2} + A^{1/2}B^{3/2} \right\| \leq \frac{1}{2} \left\| (A + B)^2 \right\|.
\]
(3.1)

Meanwhile, the following inequality
\[
\| AB \| \leq \frac{1}{4} \left\| (A + B)^2 \right\|
\]
(3.2)
was also proved by Bhatia and Kittaneh [7] for positive semidefinite matrices \( A, B \).

In this section, we first obtain an inequality involving unitarily invariant norms. After that, we present an improvement of the inequality (3.2) for the Hilbert-Schmidt norm. To do this, we need the following lemma (see [8, Theorem 2]).

**Lemma 3.1.** Let \( A, B, X \in M_n \) such that \( A \) and \( B \) are positive semidefinite. If \( 0 \leq \nu \leq 1 \), then
\[
\| A^\nu X B^{1-\nu} \| \leq \| AX \|^{\nu} \| XB \|^{1-\nu}.
\]
THEOREM 3.1. Let $A, B, X \in M_n$ such that $A$ and $B$ are positive semidefinite. If $0 \leq v \leq 1$, then

$$4 \left| A^{v}XB^{1-v} \right|^2 + \left( \|AX\|^2 - \|XB\|^2(1-v) \right)^2 \leq \|AX\|^{4v} + \|XB\|^{4(1-v)} + 2 \left| A^{v}XB^{1-v} \right|^2.$$ 

Proof. By Lemma 3.1, we have

$$\|AX\|^{4v} + \|XB\|^{4(1-v)} - 2 \left| A^{v}XB^{1-v} \right|^2 \geq \|AX\|^{4v} + \|XB\|^{4(1-v)} - 2 \|AX\|^{2v}\|XB\|^{2(1-v)}$$

$$= \left( \|AX\|^{2v} - \|XB\|^{2(1-v)} \right)^2 \geq 0.$$ 

This completes the proof. \[\square\]

Let $A, B, X \in M_n$ such that $A$ and $B$ are positive semidefinite. Note that

$$\|AX + XB\|^2_2 = \|AX\|^2 + \|XB\|^2 + 2 \left| A^{1/2}XB^{1/2} \right|^2_2.$$ 

So, taking $v = 1$ and $\|\cdot\| = \|\cdot\|_2$ in Theorem 3.1, then we have the following result.

COROLLARY 3.1. Let $A, B, X \in M_n$ such that $A$ and $B$ are positive semidefinite. Then

$$4 \left| A^{1/2}XB^{1/2} \right|^2_2 + \left( \|AX\|_2 - \|XB\|_2 \right)^2 \leq \|AX + XB\|^2_2.$$ 

Now, we give an improvement of the inequality (3.2) for the Hilbert-Schmidt norm.

THEOREM 3.2. Let $A, B \in M_n$ be positive semidefinite. Then

$$\sqrt{\|AB\|^2_2 + \frac{1}{4} \left( \left| A^{3/2}B^{1/2} \right|^2_2 - \left| A^{1/2}B^{3/2} \right|^2_2 \right)} \leq \frac{1}{4} \left( \|A + B\|^2 \right)_2.$$ 

Proof. Taking $X = A^{1/2}B^{1/2}$. Then, by Corollary 3.1, we have

$$4 \|AB\|^2_2 + \left( \left| A^{3/2}B^{1/2} \right|^2_2 - \left| A^{1/2}B^{3/2} \right|^2_2 \right)^2 \leq \left| A^{3/2}B^{1/2} + A^{1/2}B^{3/2} \right|^2_2.$$ 

It follows from (3.1) and (3.3) that

$$4 \|AB\|^2_2 + \left( \left| A^{3/2}B^{1/2} \right|^2_2 - \left| A^{1/2}B^{3/2} \right|^2_2 \right)^2 \leq \frac{1}{4} \left( \|A + B\|^2 \right)_2.$$ 

That is,

$$\sqrt{\|AB\|^2_2 + \frac{1}{4} \left( \left| A^{3/2}B^{1/2} \right|^2_2 - \left| A^{1/2}B^{3/2} \right|^2_2 \right)} \leq \frac{1}{4} \left( \|A + B\|^2 \right)_2.$$ 

This completes the proof. \[\square\]
4. Improved Heinz inequality for matrices

Bhatia and Davis proved in [9] that if $A, B, X \in \mathbb{M}_n$ such that $A$ and $B$ are positive semidefinite and if $0 \leq v \leq 1$, then

$$\left\| A^{1/2}XB^{1/2} \right\| \leq \left\| A^{v}XB^{1-v} + A^{1-v}XB^{v} \right\| \leq \frac{AX + XB}{2}.$$  

Kittaneh and Manasrah [10] proved that if $A, B, X \in \mathbb{M}_n$ such that $A$ and $B$ are positive semidefinite, then

$$2 \left\| A^{1/2}XB^{1/2} \right\|_2 + a^2 \leq \|AX + XB\|_2,$$  \hspace{1cm} (4.1)

where $a = \sqrt{\|AX\|_2} - \sqrt{\|XB\|_2}$. This is a refinement of arithmetic-geometric mean inequality for the Hilbert-Schmidt norm. Inequality (4.1) is equivalent to the following inequality

$$4 \left\| A^{1/2}XB^{1/2} \right\|^2_2 + 4a^2 \left\| A^{1/2}XB^{1/2} \right\|_2 + a^4 \leq \|AX + XB\|^2_2,$$  \hspace{1cm} (4.2)

where $a = \sqrt{\|AX\|_2} - \sqrt{\|XB\|_2}$.

By (4.1), Kittaneh and Manasrah [10] obtained an improvement of the Heinz inequality for the Hilbert-Schmidt norm which can be stated as follows:

$$\left\| A^{v}XB^{1-v} + A^{1-v}XB^{v} \right\|_2 + 2\nu_0a^2 \leq \|AX + XB\|_2,$$  \hspace{1cm} (4.3)

where $\nu_0 = \min \{\nu, 1 - \nu\}$ and $a = \sqrt{\|AX\|_2} - \sqrt{\|XB\|_2}$.

By (4.2), we will give another improvement of the Heinz inequality for the Hilbert-Schmidt norm. To do this, we need the following lemma (see [5, p. 265]).

**Lemma 4.1.** Let $A, B, X \in \mathbb{M}_n$ such that $A$ and $B$ are positive semidefinite. Then, for each unitarily invariant norm, the function

$$g(\nu) = \left\| A^{v}XB^{1-v} + A^{1-v}XB^{v} \right\|$$

is a continuous convex function on $[0, 1]$ and attains its minimum at $\nu = \frac{1}{2}$. Moreover, $g(\nu)$ is twice differentiable on $(0, 1)$.

**Theorem 4.1.** Let $A, B, X \in \mathbb{M}_n$ such that $A$ and $B$ are positive semidefinite. If $0 \leq \nu \leq 1$, then

$$\left\| A^{v}XB^{1-v} + A^{1-v}XB^{v} \right\|^2_2 + 8\nu_0a^2 \left\| A^{1/2}XB^{1/2} \right\|_2 + 2\nu_0a^4 \leq \|AX + XB\|^2_2,$$  \hspace{1cm} (4.4)

where $\nu_0 = \min \{\nu, 1 - \nu\}$ and $a = \sqrt{\|AX\|_2} - \sqrt{\|XB\|_2}$.

**Proof.** Let

$$f(\nu) = \|AX + XB\|^2_2 - \left\| A^{v}XB^{1-v} + A^{1-v}XB^{v} \right\|^2_2 = \|AX + XB\|^2_2 - g^2(\nu).$$
Define
\[ \psi(v) = \frac{f(v)}{v_0}, \quad 0 < v < 1. \]
That is,
\[ \psi(v) = \begin{cases} \frac{f(v)}{v}, & 0 < v \leq \frac{1}{2}, \\ \frac{f(v)}{1-v}, & \frac{1}{2} \leq v < 1. \end{cases} \]
So, we have
\[ \psi'(v) = \begin{cases} -2vg(v)g'(v) - f(v) & 0 < v < \frac{1}{2}, \\ -2(1-v)g(v)g'(v) + f(v) & \frac{1}{2} < v < 1. \end{cases} \]
and
\[ \psi_+ \left( \frac{1}{2} \right) = -4f \left( \frac{1}{2} \right) \leq 0, \quad \psi_- \left( \frac{1}{2} \right) = 4f \left( \frac{1}{2} \right) \geq 0. \]
Consider the following two functions
\[ \omega_1(v) = -2vg(v)g'(v) - f(v), \quad 0 \leq v \leq \frac{1}{2} \]
and
\[ \omega_2(v) = -2(1-v)g(v)g'(v) + f(v), \quad \frac{1}{2} \leq v \leq 1. \]
Then, we have
\[ \omega_1(0) = \omega_2(1) = 0, \quad \omega_1 \left( \frac{1}{2} \right) = -\omega_2 \left( \frac{1}{2} \right) = -f \left( \frac{1}{2} \right) \leq 0. \]
Meanwhile, we obtain
\[ \omega_1'(v) = -2v \left( (g'(v))^2 + g(v)g''(v) \right) \leq 0, \quad 0 \leq v \leq \frac{1}{2}, \]
and
\[ \omega_2'(v) = -2(1-v) \left( (g'(v))^2 + g(v)g''(v) \right) \leq 0, \quad \frac{1}{2} \leq v \leq 1. \]
Thus,
\[ \begin{cases} \psi'(v) \leq 0, & 0 < v < \frac{1}{2}, \\ \psi'(v) \geq 0, & \frac{1}{2} < v < 1. \end{cases} \]
It follows from the continuity of \( \psi(v) \) and the symmetry of \( \psi(v) \) about \( v = \frac{1}{2} \) that \( \psi(v) \) attains its minimum at \( v = \frac{1}{2} \). So, by (4.2), we have
\[ \psi(v) \geq 2 \left( \|AX + XB\|_2^2 - 4 \|A^{1/2}XB^{1/2}\|_2^2 \right) \geq 2 \left( 4a^2 \|A^{1/2}XB^{1/2}\|_2 + a^4 \right). \]
Thus,
\[
\left\| A'XB^{1-v} + A^{1-v}XB' \right\|_2^2 + 8v_0a^2\left\| A^{1/2}XB^{1/2} \right\|_2^2 + 2v_0a^4 \leq \| AX + XB \|_2^2.
\]
This completes the proof. □

It should be noticed that neither (4.3) nor (4.4) is in general better than the other.

5. On a conjecture concerning the Hilbert-Schmidt norm of matrices

In this section, we shall mainly adopt the notation and terminology in [14]. For convenience, recall that. The spectral radius of a square matrix \( A \) is the nonnegative real number
\[
\rho (A) \overset{\Delta}{=} \max \{|\lambda|: \lambda \text{ is an eigenvalue of } A\}.
\]

A Hadamard matrix is a square matrix with entries equal to \( \pm 1 \) whose rows and hence columns are mutually orthogonal. In other words, a Hadamard matrix of order \( n \) is a \( \{1, -1\} \)-matrix \( A \) satisfying \( AA^T = nI \), where \( I \) is the identity matrix [11, p. 126].

In 1976, Sloane and Harwit [12] made the following conjecture. See also [11, p. 130].

**Conjecture.** If \( A \) is a nonsingular matrix of order \( n \) all of whose entries are in the interval \( [0, 1] \), then
\[
\frac{2n}{n+1} \leq \|A^{-1}\|_2.
\]
Equality holds if and only if \( A \) is an \( S \)-matrix.

An \( S \)-matrix of order \( n \) is a \( \{0, 1\} \)-matrix formed by taking a Hadamard matrix of order \( n+1 \) in which the entries in the first row and column are 1, changing 1’s to 0’s and -1’s to 1’s, and deleting the first row and column [11, p. 130].

This problem arose from weighing designs in optics and statistics. Recently, the conjecture was proved for some special matrices and the following results were obtained.

**Theorem 5.1.** [13] Let \( A \in D_n \) be a positive definite matrix and suppose that \( \rho (A) \leq \sqrt{2} n \). Then
\[
\frac{2n}{n+1} \leq \|A^{-1}\|_2.
\]

**Theorem 5.2.** [14] Let \( A \) be a nonsingular matrix of order \( n \) and suppose that the modulus of entries of \( A \) is from \( [0, 1] \). If
\[
\|A\|_2 \leq \frac{\sqrt{n^2 + n^2} - n - 1}{n},
\]
then
\[ \frac{2n}{n+1} \leq \|A^{-1}\|_2. \]

From the Cauchy-Schwarz inequality, we have the following inequality
\[ \frac{n}{\|A\|_2} \leq \|A^{-1}\|_2. \]

So, if \[ \|A\|_2 \leq \frac{n+1}{2}, \]
then, we have
\[ \frac{2n}{n+1} \leq \|A^{-1}\|_2. \]

In this section, if \( A \) is a positive definite matrix of order \( n \), we first obtain a lower bound for \( \|A^{-1}\|_2 \). Then we use it to discuss the conjecture on the Hilbert-Schmidt norm of matrices proposed by Sloane and Harwit and the conjecture is proved for some special matrices. To do this, we need the following lemma.

**Lemma 5.1.** If \( A \) is a positive definite matrix of order \( n \), then
\[ n^2 \leq \text{tr} A \cdot \text{tr} A^{-1}. \]

**Proof.** By the harmonic-arithmetic inequality. \( \Box \)

**Theorem 5.3.** If \( A \) is a positive definite matrix of order \( n \), then
\[ \frac{n\sqrt{n}}{\text{tr} A} \leq \|A^{-1}\|_2. \]

**Proof.** Using the Cauchy–Schwarz inequality, we have
\[ (\text{tr} A)^2 \leq n \|A\|_2^2. \]

That is,
\[ \text{tr} A \leq \sqrt{n} \|A\|_2. \]

So,
\[ \text{tr} A^{-1} \leq \sqrt{n} \|A^{-1}\|_2. \quad (5.1) \]

By Lemma 5.1 and (5.1), we have
\[ \frac{n\sqrt{n}}{\text{tr} A} \leq \|A^{-1}\|_2. \]

This completes the proof. \( \Box \)
COROLLARY 5.1. If $A \in D_n$ is a positive definite matrix, then

$$\frac{2n}{n+1} \leq \|A^{-1}\|_2.$$

Proof. Since $A \in D_n$, we have

$$\text{tr}A \leq n.$$

So, by Theorem 5.3, we obtain

$$\frac{2n}{n+1} \leq \sqrt{n} \leq \frac{n\sqrt{n}}{\text{tr}A} \leq \|A^{-1}\|_2.$$

This completes the proof. □

Obviously Corollary 5.1 is a refinement of Theorem 5.1.

REFERENCES


(Received January 12, 2012)