Journal of Mathematical Inequalities
Volume 6, Number 4 (2012), 625–629 doi:10.7153/jmi-06-60

A LOWER BOUND FOR THE SMALLEST SINGULAR VALUE

LIMIN ZOU

(*Communicated by J. Liu*)

Abstract. In this paper, we obtain a lower bound for the smallest singular value of nonsingular matrices which is better than the bound presented by Yu and Gu [Linear Algebra Appl. 252(1997)25-38]. Meanwhile, we give some numerical examples which will show the effectiveness of our result.

1. Introduction

Let M_n be the space of $n \times n$ complex matrices. Let σ_i $(i = 1, \cdots, n)$ be the singular values of $A \in M_n$ and suppose that $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_{n-1} \geqslant \sigma_n \geqslant 0$. For $A = [a_{ij}] \in M_n$, the Frobenius norm of A is defined by

$$\|A\|_F = \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{1/2}.$$

The relationship between the Frobenius norm and singular values is

$$\|A\|_F^2 = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2.$$

It is well known that lower bounds for the smallest singular value σ_n of a nonsingular matrix $A \in M_n$ have many potential theoretical and practical applications [1–2].

Let $A \in M_n$ be nonsingular. Yu and Gu [3] obtained a lower bound for σ_n as follows:

$$\sigma_n \geqslant l = |\det A| \cdot \left(\frac{n-1}{\|A\|_F^2} \right)^{(n-1)/2} > 0. \tag{1.1}$$

The inequality (1.1) is also shown in [11].

In this paper, we obtain a lower bound for the smallest singular value of nonsingular matrices. It is better than (1.1). Meanwhile, we give some numerical examples which will show the effectiveness of our result.

Mathematics subject classification (2010): 15A18, 15A42, 15A60.
Keywords and phrases: Singular values, Frobenius norm, determinant.

Paper JMI-06-60

2. Main result

THEOREM 2.1. *Let $A \in M_n$ be nonsingular. Then*

$$\sigma_n \geqslant |\det A| \cdot \left(\frac{n-1}{\|A\|_F^2 - l^2} \right)^{(n-1)/2}. \tag{2.1}$$

Proof. Let

$$K = \frac{\sigma_1^2}{p_1} \frac{\sigma_1^2}{q_1} \cdots \frac{\sigma_k^2}{p_k} \frac{\sigma_k^2}{q_k} \sigma_{k+1}^2 \cdots \sigma_{n-1}^2, \quad 1 \leqslant k \leqslant n-1. \tag{2.2}$$

where

$$\frac{1}{p_i} + \frac{1}{q_i} = 1,\ p_i > 0,\ q_i > 0,\ i = 1, \cdots, k.$$

By the arithmetic-geometric mean inequality and (2.2), we have

$$K \leqslant \left(\frac{\|A\|_F^2 - \sigma_n^2}{n+k-1} \right)^{n+k-1}. \tag{2.3}$$

Note that (2.2) can be rewritten as follows:

$$K = \frac{\sigma_1^2}{\sigma_n^2} \prod_{i=2}^k \sigma_i^2 \prod_{i=1}^k \frac{1}{p_i q_i} |\det(A)|^2, \tag{2.4}$$

where for $k = 1$, $\prod_{i=2}^k \sigma_i^2 = 1$. It follows from (2.3) and (2.4) that

$$\frac{\sigma_1^2}{\sigma_n^2} \leqslant \frac{\prod_{i=1}^k p_i q_i}{\prod_{i=2}^k \sigma_i^2} \cdot \frac{1}{|\det(A)|^2} \cdot \left(\frac{\|A\|_F^2 - \sigma_n^2}{n+k-1} \right)^{n+k-1}. \tag{2.5}$$

By (1.1) and (2.5), we have

$$\frac{\sigma_1^2}{\sigma_n^2} \leqslant \frac{\prod_{i=1}^k p_i q_i}{\prod_{i=2}^k \sigma_i^2} \cdot \frac{1}{|\det(A)|^2} \cdot \left(\frac{\|A\|_F^2 - l^2}{n+k-1} \right)^{n+k-1}. \tag{2.6}$$

For each k $(1 \leqslant k \leqslant n-1)$, we define

$$f(p_1, \cdots, p_k, q_1, \cdots, q_k) = \prod_{i=1}^k p_i q_i,$$

where

$$\frac{1}{p_i} + \frac{1}{q_i} = 1,\ p_i > 0,\ q_i > 0.$$

The relationship above suggests that we study the following optimization problem:

$$\begin{cases} \min f(p_1,\cdots,p_k,q_1,\cdots,q_k) \\ s.t. \begin{cases} \frac{1}{p_i}+\frac{1}{q_i}=1 \\ p_i>0,\ q_i>0,\ 1\leqslant i\leqslant k. \end{cases} \end{cases}$$

Let

$$L(p_1,\cdots,p_k,q_1,\cdots,q_k,\lambda_1,\cdots,\lambda_k)=\prod_{i=1}^{k} p_i q_i+\sum_{i=1}^{k}\lambda_i\left(\frac{1}{p_i}+\frac{1}{q_i}-1\right).$$

We search for the stationary points of L. We have

$$\frac{\partial L}{\partial p_j}=\prod_{i=1,i\neq j}^{k} p_i\prod_{i=1}^{k} q_i-\frac{\lambda_j}{p_j^2}=0,\quad 1\leqslant j\leqslant k,$$

$$\frac{\partial L}{\partial q_j}=\prod_{i=1,i\neq j}^{k} q_i\prod_{i=1}^{k} p_i-\frac{\lambda_j}{q_j^2}=0,\quad 1\leqslant j\leqslant k.$$

Thus, we obtain $p_j=q_j=2,\ 1\leqslant j\leqslant k$. Therefore

$$\min f(p_1,\cdots,p_k,q_1,\cdots,q_k)=4^k. \tag{2.7}$$

It follows from (2.6) and (2.7) that

$$\frac{\sigma_1}{\sigma_n}\leqslant\frac{2^k}{\prod_{i=2}^{k}\sigma_i}\cdot\frac{1}{|\det A|}\cdot\left(\frac{\|A\|_F^2-l^2}{n+k-1}\right)^{(n+k-1)/2}. \tag{2.8}$$

Putting $k=n-1$ in (2.8), we have

$$\frac{\sigma_1}{\sigma_n}\leqslant\frac{2^{n-1}}{\prod_{i=2}^{n-1}\sigma_i}\cdot\frac{1}{|\det A|}\cdot\left(\frac{\|A\|_F^2-l^2}{2(n-1)}\right)^{n-1}.$$

That is

$$\frac{1}{\sigma_n^2}\leqslant\frac{1}{|\det A|^2}\cdot\left(\frac{\|A\|_F^2-l^2}{n-1}\right)^{n-1}.$$

Hence

$$\sigma_n\geqslant|\det A|\cdot\left(\frac{n-1}{\|A\|_F^2-l^2}\right)^{(n-1)/2}.$$

This completes the proof. □

3. Numerical examples

There are many lower bounds for the smallest singular value in the literature [4–10]. They are different from (1.1) and (2.1). These bounds are incomparable.

In this section, we give some numerical examples to show that (2.1) is better than Theorem 2 of [4], Theorem 3.1 of [5], and Theorem 4.1 of [6] in some cases.

EXAMPLE 3.1. [4] Let

$$A = \begin{bmatrix} 10 & 2 \\ -2 & 2 \end{bmatrix}.$$

We calculate that the true value of the smallest singular value of A is $\sigma_2(A) = 2.3246$. By Theorem 2 of [4], we have

$$\sigma_2 \geqslant 2.0000.$$

By Theorem 3.1 of [5], we have

$$\sigma_2 \geqslant 2.3217.$$

By (2.1), we have

$$\sigma_2 \geqslant 2.3217.$$

EXAMPLE 3.2. [4] Let

$$A = \begin{bmatrix} 10 & 1 & 2 \\ 2 & 20 & 3 \\ 20 & 1 & 10 \end{bmatrix}.$$

It is not difficult to calculate that the determinant of A is 1214. We calculate that the true value of the smallest singular value of A is $\sigma_3 = 2.4909$. By Theorem 2 of [4], we have

$$\sigma_3 \geqslant 0.6227.$$

By Theorem 3.1 of [5], we have

$$\sigma_3 \geqslant 2.0694.$$

By (2.1), we have

$$\sigma_3 \geqslant 2.3961.$$

EXAMPLE 3.3. [4] Let

$$A = \begin{bmatrix} 0.75 & 0.5 & 0.4 \\ 0.5 & 1 & 0.6 \\ 0 & 0.5 & 1 \end{bmatrix}.$$

Obviously, A is an upper Hessenberg matrix. It is not difficult to calculate that the determinant of A is 0.3750. We calculate that the true value of the smallest singular value of A is $\sigma_3 = 0.2977$. By Theorem 2 of [4], we have

$$\sigma_3 \geqslant 0.0560.$$

By (2.3), Theorem 4.1 and (2.6) of [6], we have

$$\sigma_3 \geqslant 0.1500.$$

By Theorem 3.1 of [5], we have

$$\sigma_3 \geqslant 0.1547.$$

By (2.1), we have

$$\sigma_3 \geqslant 0.1977.$$

Acknowledgments

The author wishes to express his heartfelt thanks to the referees for their detailed and helpful suggestions for revising the manuscript. At the same time, we are grateful for the suggestions of Jiewu Huang. This research was supported by Natural Science Foundation Project of Chongqing Municipal Education Commission (KJ121103).

REFERENCES

[1] R. A. HORN, C. R. JOHNSON, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985.
[2] R. A. HORN, C. R. JOHNSON, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.
[3] Y. YU, D. GU, *A note on a lower bound for the smallest singular value*, Linear Algebra Appl. **252** (1997), 25–38.
[4] T. HUANG, *Estimation of* $\|A^{-1}\|_\infty$ *and the smallest singular value*, Comput. Math. Appl. **55** (2008), 1075–1080.
[5] L. ZOU, Y. JIANG, *Estimation of the eigenvalues and the smallest singular value of matrices*, Linear Algebra Appl. **433** (2010), 1203–1211.
[6] J. M. PENA, T. SZULC, *Loewner matrix ordering in estimation of the smallest singular value*, Electron J Linear Al. **22** (2011), 466–479.
[7] J. M. VARAH, *A lower bound for the smallest singular value*, Linear Algebra Appl. **11** (1975), 3–5.
[8] C. R. JOHNSON, *A Gersgorin-type lower bound for the smallest singular value*, Linear Algebra Appl. **112** (1989), 1–7.
[9] C. R. JOHNSON, T. SZULC, *Further lower bounds for the smallest singular value*, Linear Algebra Appl. **272** (1998), 169–179.
[10] L. Y. KOLOTILINA, *Bounds for the singular values of a matrix involving its sparsity pattern*, J. Math. Sci. **137**, 3 (2006), 4794–4800.
[11] G. PIAZZA, T. POLITI, *An upper bound for the condition number of a matrix in spectral norm*, J. Comput. Appl. Math. **143** (2002), 141–144.

(Received January 19, 2012)

Limin Zou
School of Mathematics and Statistics
Chongqing Three Gorges University
Chongqing, 404100
P. R. China
e-mail: limin-zou@163.com