

## REMARKS ON THE PAPER “JENSEN’S INEQUALITY AND NEW ENTROPY BOUNDS” OF S. SIMIĆ

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*Abstract.* The purpose of this paper is twofold. The first is to give a brief account of the results preceding the main results from [14] and [15]. The second is to give generalizations and improvements of these results.

### 1. Introduction

Let  $I \subset \mathbb{R}$  be an interval,  $(x_i)_{i=1}^n$  a sequence such that  $x_i \in I$ ,  $i = 1, \dots, n$ , and  $(p_i)_{i=1}^n$  a sequence of positive weights with  $\sum_1^n p_i = 1$ . For a convex function  $f: I \rightarrow \mathbb{R}$ , the Jensen inequality states

$$0 \leq \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right).$$

The following theorems are proved in [15].

**THEOREM 1.** *If  $f$  is convex on  $I$ , then*

$$\begin{aligned} \max_{1 \leq \mu < \nu \leq n} \left[ p_\mu f(x_\mu) + p_\nu f(x_\nu) - (p_\mu + p_\nu) f\left(\frac{p_\mu x_\mu + p_\nu x_\nu}{p_\mu + p_\nu}\right) \right] \\ \leq \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right) \end{aligned}$$

*and this bound is sharp.*

**THEOREM 2.** *If  $(x_i)_{i=1}^n \in [a, b]^n$ , then*

$$\sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right) \leq f(a) + f(b) - 2f\left(\frac{a+b}{2}\right) := S_f(a, b)$$

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However, Theorem 1 is contained in the following theorem and corollary, which are in fact Theorem 3.14 and Corollary 3.15 published in [9, p. 87]. The statement on sharpness in Theorem 1 is obvious (equality is attained for  $n = 2$ ).

**THEOREM 3.** *Let  $f : U \rightarrow \mathbb{R}$  be a convex function, where  $U$  is a convex set in a real linear space  $M$ . Let  $I$  and  $J$  be finite subsets in  $\mathbb{N}$ , such that  $I \cap J = \emptyset$ . Let  $(x_i)_{i \in I \cup J}$  be a sequence such that  $x_i \in U$ ,  $i \in I \cup J$  and  $(p_i)_{i \in I \cup J}$  a real sequence such that  $P_I > 0$ ,  $P_J > 0$  and  $P_{I \cup J} > 0$ , where  $P_K = \sum_{k \in K} p_k$  for  $K \subseteq I \cup J$ . If  $\frac{1}{P_I} \sum_{i \in I} p_i x_i \in U$ ,  $\frac{1}{P_J} \sum_{i \in J} p_i x_i \in U$ ,  $\frac{1}{P_{I \cup J}} \sum_{i \in I \cup J} p_i x_i \in U$ . Then*

$$F(I \cup J) \leq F(I) + F(J), \quad (1)$$

where  $F(K) = P_K f\left(\frac{1}{P_K} \sum_{k \in K} p_k x_k\right) - \sum_{k \in K} p_k f(x_k)$ .

Notice that the function  $F$  from Theorem 3 describes the opposite Jensen's difference than the differences in Theorems 1, 2.

**COROLLARY 1.** *Let  $f$  be a convex function on  $U$ , where  $U$  is a convex set in an arbitrary real linear space  $M$ . Let  $(x_i)_{i=1}^n$  be a sequence such that  $x_i \in U$ ,  $i = 1, \dots, n$ , and  $(p_i)_{i=1}^n$  a real sequence. If  $p_i \geq 0$ ,  $i = 1, \dots, n$  and  $I_k = \{1, \dots, k\}$ , then*

$$F(I_n) \leq F(I_{n-1}) \leq \dots \leq F(I_2) \leq 0$$

and

$$F(I_n) \leq \min_{1 \leq i < j \leq n} \left\{ (p_i + p_j) f\left(\frac{p_i x_i + p_j x_j}{p_i + p_j}\right) - p_i f(x_i) - p_j f(x_j) \right\}.$$

**REMARK 4.** It should be noted that results related to Theorem 3 and Corollary 1 and implying Theorem 1 were published previously in [12], [13], [6], [2], [8] and [1].

Theorem 2 was proved by the same author in paper published one year before [15]. It was the main result in [14]. This fact wasn't noted in [15]. It was noted in [4] that this main result from [14], and therefore Theorem 2 can be derived from Corollaries 3 and 4 from [5]. Moreover, these Corollaries give the following improvements of Theorem 2.

**THEOREM 5.** *Let  $[a, b] \subset \mathbb{R}$ ,  $(x_i)_{i=1}^n$  be a sequence such that  $x_i \in [a, b]$ ,  $i = 1, \dots, n$  and  $(p_i)_{i=1}^n$  a sequence of positive weights with  $\sum_{i=1}^n p_i = 1$ . If  $f : [a, b] \rightarrow \mathbb{R}$  is a convex function, then*

$$\begin{aligned} & \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right) \\ & \leq f\left(a + b - \sum_{i=1}^n p_i x_i\right) - 2f\left(\frac{a+b}{2}\right) + \sum_{i=1}^n p_i f(x_i) \\ & \leq f(a) + f(b) - 2f\left(\frac{a+b}{2}\right) = S_f(a, b) \end{aligned}$$

To state further improvements of Theorem 2 and also for the rest of the paper, we need the following notions.

Let  $E$  be a non-empty set and  $L$  be a linear class of real-valued functions  $f: E \rightarrow \mathbb{R}$  having the properties:

$$(L1) \quad (\forall a, b \in \mathbb{R})(\forall f, g \in L) \quad af + bg \in L$$

$$(L2) \quad \mathbf{1} \in L \text{ (that is if } f(t) = 1 \text{ for all } t \in E, \text{ then } f \in L)$$

We consider positive linear functionals  $A: L \rightarrow \mathbb{R}$ , or in other words we assume:

$$(A1) \quad (\forall f, g \in L)(\forall a, b \in \mathbb{R}) \quad A(af + bg) = aA(f) + bA(g) \text{ (linearity)}$$

$$(A2) \quad (\forall f \in L)(f \geq 0 \implies A(f) \geq 0) \text{ (positivity)}$$

If additionally the condition  $A(\mathbf{1}) = 1$  is satisfied, we say that  $A$  is positive normalized linear functional.

The following generalization and improvement of Theorem 2 was proved in [4].

**THEOREM 6.** *Let  $L$  satisfy (L1) and (L2) and let  $\Phi$  be a convex function on  $I = [a, b]$ . Then for any positive normalized linear functional  $A$  on  $L$  and for any  $g \in L$  such that  $\Phi(g) \in L$  we have*

$$A(\Phi(g)) - \Phi(A(g)) \leq \left\{ \frac{1}{2} + \frac{1}{b-a} \left| \frac{a+b}{2} - A(g) \right| \right\} S_{\Phi}(a, b).$$

The main purpose of the paper is to give generalizations and improvements of Theorems 1, 2 and an improvement of Theorem 6.

## 2. Improvements

In the following theorem we give two generalizations of Theorem 1. For similar results obtained by different methods see [3].

**THEOREM 7.** *Let  $f$  be a convex function on  $U$ , where  $U$  is a convex set in an arbitrary real linear space  $M$ . Let  $(x_i)_{i=1}^n$  be a sequence such that  $x_i \in U, i \in I_n = \{1, \dots, n\}$ , and  $(p_i)_{i=1}^n$  a positive sequence.*

(i) *If  $\mathcal{S}$  is a family of subsets of  $I_n$ , then*

$$F(I_n) \leq \min_{S \in \mathcal{S}} (F(S) + F(I_n \setminus S)) \leq \min_{S \in \mathcal{S}} F(S) + \max_{S \in \mathcal{S}} F(I_n \setminus S) \leq \min_{S \in \mathcal{S}} F(S). \quad (2)$$

(ii) *If  $\mathcal{S}$  is a family of disjoint subsets of  $I_n$ , then*

$$F(I_n) \leq \sum_{S \in \mathcal{S}} F(S), \quad (3)$$

where  $F$  is the function defined in Theorem 3.

*Proof.* (i) Simple consequence of Theorem 3.

(ii) Obviously  $I_n = \cup_{S \in \mathcal{S}} S \cup_{i \notin \cup_{S \in \mathcal{S}} S} \{i\}$ , so (3) follows using Theorem 3 and  $F(\{i\}) = 0$ .  $\square$

Improvement of Theorem 2 and Theorem 6 will be obtained using the following lemma.

LEMMA 1. *Let  $\phi$  be a convex function on an interval  $I$ ,  $x, y \in I$  and  $p, q \in [0, 1]$  such that  $p + q = 1$ . Then*

$$\min\{p, q\}S_\phi(x, y) \leq p\phi(x) + q\phi(y) - \phi(px + qy) \leq \max\{p, q\}S_\phi(x, y). \quad (4)$$

*Proof.* This is a special case of Theorem 1 from [7, p.717] for  $n = 2$ .  $\square$

We will also need to equip our linear class  $L$  from above with an additional property denoted by (L3):

$$(L3) \quad (\forall f, g \in L)(\min\{f, g\} \in L \wedge \max\{f, g\} \in L) \text{ (lattice property)}$$

Obviously,  $(\mathbb{R}^E, \leq)$  (with standard ordering) is a lattice. It can also be easily verified that a subspace  $X \subseteq \mathbb{R}^E$  is a lattice if and only if  $x \in X$  implies  $|x| \in X$ . This is a simple consequence of identities

$$\min\{x, y\} = \frac{1}{2}(x + y - |x - y|), \quad \max\{x, y\} = \frac{1}{2}(x + y + |x - y|).$$

Next theorem is our main result.

THEOREM 8. *Let  $L$  satisfy (L1), (L2) and (L3) and let  $\Phi$  be a convex function on  $I = [a, b]$ . Then for any positive normalized linear functional  $A$  on  $L$  and for any  $g \in L$  such that  $\Phi(g) \in L$  we have*

$$\begin{aligned} & A(\Phi(g)) - \Phi(A(g)) \leq \\ & \leq \frac{1}{b-a} \left\{ \left| \frac{a+b}{2} - A(g) \right| + A \left( \left| \frac{a+b}{2} - g \right| \right) \right\} S_\Phi(a, b). \end{aligned} \quad (5)$$

*Proof.* First observe that  $\Phi(g) \in L$  also means that the composition  $\Phi(g)$  is well defined, hence  $g(E) \subseteq [a, b]$ . It follows  $A(g) \in [a, b]$ .

Let the functions  $p, q: [a, b] \rightarrow \mathbb{R}$  be defined by

$$p(x) = \frac{b-x}{b-a}, \quad q(x) = \frac{x-a}{b-a}$$

For any  $x \in [a, b]$  we can write

$$\Phi(x) = \Phi \left( \frac{b-x}{b-a}a + \frac{x-a}{b-a}b \right) = \Phi(p(x)a + q(x)b)$$

Since  $A(g) \in [a, b]$  we have  $\Phi(A(g)) = \Phi(p(A(g))a + q(A(g))b)$  and by Lemma 1

$$\begin{aligned} \Phi(A(g)) &\geq p(A(g))\Phi(a) + q(A(g))\Phi(b) - \max\{p(A(g)), q(A(g))\}S_{\Phi}(a, b) \\ &= p(A(g))\Phi(a) + q(A(g))\Phi(b) - \left\{ \frac{1}{2} + \frac{\left| \frac{a+b}{2} - A(g) \right|}{b-a} \right\} S_{\Phi}(a, b) \end{aligned} \tag{6}$$

Again by Lemma 1 we have

$$\Phi(x) \leq p(x)\Phi(a) + q(x)\Phi(b) - \min\{p(x), q(x)\}S_{\Phi}(a, b).$$

We have  $p(g), q(g) \in L$  and applying  $A$  to the above inequality, we obtain

$$\begin{aligned} A(\Phi(g)) &\leq A(p(g))\Phi(a) + A(q(g))\Phi(b) - A(\min\{p(g), q(g)\})S_{\Phi}(a, b) \\ &= A(p(g))\Phi(a) + A(q(g))\Phi(b) - A\left(\frac{1}{2} - \frac{\left| g - \frac{a+b}{2} \right|}{b-a}\right) S_{\Phi}(a, b) \\ &= p(A(g))\Phi(a) + q(A(g))\Phi(b) - A\left(\frac{1}{2} - \frac{\left| g - \frac{a+b}{2} \right|}{b-a}\right) S_{\Phi}(a, b) \end{aligned} \tag{7}$$

Now, from inequalities (6) and (7) we get desired inequality (5)  $\square$

Since  $A(g) \in [a, b]$ , it is obvious that Theorem 8 is an improvement of Theorem 6 and a generalization and an improvement of Theorem 2.

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