REMARKS ON THE PAPER “JENSEN’S INEQUALITY AND NEW ENTROPY BOUNDS” OF S. SIMIĆ

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Abstract. The purpose of this paper is twofold. The first is to give a brief account of the results preceding the main results from [14] and [15]. The second is to give generalizations and improvements of these results.

1. Introduction

Let $I \subset \mathbb{R}$ be an interval, $(x_i)_{i=1}^n$ a sequence such that $x_i \in I$, $i = 1, \ldots, n$, and $(p_i)_{i=1}^n$ a sequence of positive weights with $\sum_{i=1}^n p_i = 1$. For a convex function $f : I \rightarrow \mathbb{R}$, the Jensen inequality states

$$0 \leq \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right).$$

The following theorems are proved in [15].

**Theorem 1.** If $f$ is convex on $I$, then

$$\max_{1 \leq \mu < \nu \leq n} \left[ p_\mu f(x_\mu) + p_\nu f(x_\nu) - (p_\mu + p_\nu) f\left(\frac{p_\mu x_\mu + p_\nu x_\nu}{p_\mu + p_\nu}\right)\right]$$

$$\leq \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right)$$

and this bound is sharp.

**Theorem 2.** If $(x_i)_{i=1}^n \in [a, b]^n$, then

$$\sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right) \leq f(a) + f(b) - 2 f\left(\frac{a + b}{2}\right) := S_f(a, b)$$


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However, Theorem 1 is contained in the following theorem and corollary, which are in fact Theorem 3.14 and Corollary 3.15 published in [9, p. 87]. The statement on sharpness in Theorem 1 is obvious (equality is attained for $n = 2$).

**THEOREM 3.** Let $f : U \to \mathbb{R}$ be a convex function, where $U$ is a convex set in a real linear space $M$. Let $I$ and $J$ be finite subsets in $\mathbb{N}$, such that $I \cap J = \emptyset$. Let $(x_i)_{i \in I \cup J}$ be a sequence such that $x_i \in U$, $i \in I \cup J$ and $(p_i)_{i \in I \cup J}$ a real sequence such that $P_I > 0$, $P_J > 0$ and $P_{I \cup J} > 0$, where $P_K = \sum_{k \in K} p_k$ for $K \subseteq I \cup J$. If $\frac{1}{P_I} \sum_{i \in I} p_i x_i \in U$, $\frac{1}{P_J} \sum_{i \in J} p_i x_i \in U$, $\frac{1}{P_{I \cup J}} \sum_{i \in I \cup J} p_i x_i \in U$. Then

$$F(I \cup J) \leq F(I) + F(J),$$

where $F(K) = P_K f \left( \frac{1}{P_K} \sum_{k \in K} p_k x_k \right) - \sum_{k \in K} p_k f(x_k)$.

Notice that the function $F$ from Theorem 3 describes the opposite Jensen’s difference than the differences in Theorems 1, 2.

**COROLLARY 1.** Let $f$ be a convex function on $U$, where $U$ is a convex set in an arbitrary real linear space $M$. Let $(x_i)_{i=1}^n$ be a sequence such that $x_i \in U$, $i = 1, \ldots, n$, and $(p_i)_{i=1}^n$ a real sequence. If $p_i \geq 0$, $i = 1, \ldots, n$ and $I_k = \{1, \ldots, k\}$, then

$$F(I_n) \leq F(I_{n-1}) \leq \cdots \leq F(I_2) \leq 0$$

and

$$F(I_n) \leq \min_{1 \leq i < j \leq n} \left\{ (p_i + p_j) f \left( \frac{p_i x_i + p_j x_j}{p_i + p_j} \right) - p_i f(x_i) - p_j f(x_j) \right\}.$$

**REMARK 4.** It should be noted that results related to Theorem 3 and Corollary 1 and implying Theorem 1 were published previously in [12], [13], [6], [2], [8] and [1].

Theorem 2 was proved by the same author in paper published one year before [15]. It was the main result in [14]. This fact wasn’t noted in [15]. It was noted in [4] that this main result from [14], and therefore Theorem 2 can be derived from Corollaries 3 and 4 from [5]. Moreover, these Corollaries give the following improvements of Theorem 2.

**THEOREM 5.** Let $[a, b] \subset \mathbb{R}$, $(x_i)_{i=1}^n$ be a sequence such that $x_i \in [a, b]$, $i = 1, \ldots, n$ and $(p_i)_{i=1}^n$ a sequence of positive weights with $\sum_{i=1}^n p_i = 1$. If $f : [a, b] \to \mathbb{R}$ is a convex function, then

$$\sum_{i=1}^n p_i f(x_i) - f \left( \sum_{i=1}^n p_i x_i \right) \leq f \left( a + b - \sum_{i=1}^n p_i x_i \right) - 2f \left( \frac{a + b}{2} \right) + \sum_{i=1}^n p_i f(x_i) \leq f(a) + f(b) - 2f \left( \frac{a + b}{2} \right) = S_f(a, b)$$
To state further improvements of Theorem 2 and also for the rest of the paper, we need the following notions.

Let $E$ be a non-empty set and $L$ be a linear class of real-valued functions $f: E \to \mathbb{R}$ having the properties:

(L1) $(\forall a, b \in \mathbb{R}) (\forall f, g \in L) af + bg \in L$

(L2) $1 \in L$ (that is if $f(t) = 1$ for all $t \in E$, then $f \in L$)

We consider positive linear functionals $A: L \to \mathbb{R}$, or in other words we assume:

(A1) $(\forall f, g \in L) (\forall a, b \in \mathbb{R}) A(af + bg) = aA(f) + bA(g)$ (linearity)

(A2) $(\forall f \in L) (f \geq 0 \implies A(f) \geq 0)$ (positivity)

If additionally the condition $A(1) = 1$ is satisfied, we say that $A$ is positive normalized linear functional.

The following generalization and improvement of Theorem 2 was proved in [4].

**Theorem 6.** Let $L$ satisfy (L1) and (L2) and let $\Phi$ be a convex function on $I = [a, b]$. Then for any positive normalized linear functional $A$ on $L$ and for any $g \in L$ such that $\Phi(g) \in L$ we have

$$A(\Phi(g)) - \Phi(A(g)) \leq \left\{ \frac{1}{2} + \frac{1}{b - a} \left| \frac{a + b}{2} - A(g) \right| \right\} S_\Phi(a, b).$$

The main purpose of the paper is to give generalizations and improvements of Theorems 1, 2 and an improvement of Theorem 6.

2. Improvements

In the following theorem we give two generalizations of Theorem 1. For similar results obtained by different methods see [3].

**Theorem 7.** Let $f$ be a convex function on $U$, where $U$ is a convex set in an arbitrary real linear space $M$. Let $(x_i)_{i=1}^n$ be a sequence such that $x_i \in U$, $i \in I_n = \{1, \ldots, n\}$, and $(p_i)_{i=1}^n$ a positive sequence.

(i) If $\mathcal{S}$ is a family of subsets of $I_n$, then

$$F(I_n) \leq \min_{S \in \mathcal{S}} (F(S) + F(I_n \setminus S)) \leq \min_{S \in \mathcal{S}} F(S) + \max_{S \in \mathcal{S}} F(I_n \setminus S) \leq \min_{S \in \mathcal{S}} F(S).$$

(ii) If $\mathcal{S}$ is a family of disjoint subsets of $I_n$, then

$$F(I_n) \leq \sum_{S \in \mathcal{S}} F(S),$$
where $F$ is the function defined in Theorem 3.

Proof. (i) Simple consequence of Theorem 3.

(ii) Obviously $I_n = \bigcup_{S \in \mathcal{F}} S \cup \bigcup_{i \notin \mathcal{F}} \{i\}$, so (3) follows using Theorem 3 and $F(\{i\}) = 0$. \qed

Improvement of Theorem 2 and Theorem 6 will be obtained using the following lemma.

**Lemma 1.** Let $\phi$ be a convex function on an interval $I$, $x, y \in I$ and $p, q \in [0, 1]$ such that $p + q = 1$. Then

$$\min \{p, q\} S_{\phi}(x, y) \leq p \phi(x) + q \phi(y) - \phi(px + qy) \leq \max \{p, q\} S_{\phi}(x, y). \quad (4)$$

Proof. This is a special case of Theorem 1 from [7, p.717] for $n = 2$. \qed

We will also need to equip our linear class $L$ from above with an additional property denoted by $(L_3)$:

$$(L_3) \quad (\forall f, g \in L)(\min \{f, g\} \in L \wedge \max \{f, g\} \in L) \text{ (lattice property)}$$

Obviously, $(\mathbb{R}^E, \leq)$ (with standard ordering) is a lattice. It can also be easily verified that a subspace $X \subseteq \mathbb{R}^E$ is a lattice if and only if $x \in X$ implies $|x| \in X$. This is a simple consequence of identities

$$\min \{x, y\} = \frac{1}{2}(x + y - |x - y|), \quad \max \{x, y\} = \frac{1}{2}(x + y + |x - y|).$$

Next theorem is our main result.

**Theorem 8.** Let $L$ satisfy $(L_1)$, $(L_2)$ and $(L_3)$ and let $\Phi$ be a convex function on $I = [a, b]$. Then for any positive normalized linear functional $A$ on $L$ and for any $g \in L$ such that $\Phi(g) \in L$ we have

$$A(\Phi(g)) - \Phi(A(g)) \leq \frac{1}{b - a} \left\{ \frac{a + b}{2} - A(g) \right\} + A \left( \frac{a + b}{2} - g \right) \right\} S_{\Phi}(a, b). \quad (5)$$

Proof. First observe that $\Phi(g) \in L$ also means that the composition $\Phi(g)$ is well defined, hence $g(E) \subseteq [a, b]$. It follows $A(g) \in [a, b]$.

Let the functions $p, q: [a, b] \to \mathbb{R}$ be defined by

$$p(x) = \frac{b - x}{b - a}, \quad q(x) = \frac{x - a}{b - a}$$

For any $x \in [a, b]$ we can write

$$\Phi(x) = \Phi \left( \frac{b - x}{b - a}a + \frac{x - a}{b - a}b \right) = \Phi(p(x)a + q(x)b)$$
Since $A(g) \in [a, b]$ we have $\Phi(A(g)) = \Phi(p(A(g))a + q(A(g))b)$ and by Lemma 1

$$\Phi(A(g)) \geq p(A(g))\Phi(a) + q(A(g))\Phi(b) - \max\{p(A(g)), q(A(g))\} S_\Phi(a, b)$$

$$= p(A(g))\Phi(a) + q(A(g))\Phi(b) - \left\{ \frac{1}{2} + \frac{|a+b|}{b-a} \right\} S_\Phi(a, b) \quad (6)$$

Again by Lemma 1 we have

$$\Phi(x) \leq p(x)\Phi(a) + q(x)\Phi(b) - \min\{p(x), q(x)\} S_\Phi(a, b).$$

We have $p(g), q(g) \in L$ and applying $A$ to the above inequality, we obtain

$$A(\Phi(g)) \leq A(p(g))\Phi(a) + A(q(g))\Phi(b) - A \left( \min\{p(g), q(g)\} \right) S_\Phi(a, b)$$

$$= A(p(g))\Phi(a) + A(q(g))\Phi(b) - A \left( \frac{1}{2} - \frac{|g - \frac{a+b}{2}|}{b-a} \right) S_\Phi(a, b)$$

$$= p(A(g))\Phi(a) + q(A(g))\Phi(b) - A \left( \frac{1}{2} - \frac{|g - \frac{a+b}{2}|}{b-a} \right) S_\Phi(a, b) \quad (7)$$

Now, from inequalities (6) and (7) we get desired inequality (5) \quad \Box

Since $A(g) \in [a, b]$, it is obvious that Theorem 8 is an improvement of Theorem 6 and a generalization and an improvement of Theorem 2.

REFERENCES


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