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A NOTE ON A CERTAIN BIVARIATE MEAN

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Abstract. Weighted arithmetic or geometric means of two bivariate means are used to obtain lower and upper bounds for a bivariate mean introduced by Neuman and Sándor. Bounds involving weighted arithmetic means are sharp.

1. Introduction

In recent years a significant progress has been made in theory of bivariate means with special emphasis on inequalities involving those mens. In particular, means being the special cases of the Schwab - Borchardt mean, have attracted attention of several researchers. These iterative means, defined, e.g. in [2], include in particular the logarithmic mean and two Seiffert means defined in [18] and in [19]. In [12] the authors have introduced another bivariate mean, which is also a special case of the Schwab-Borchardt mean and denoted the latter by M. In what follows we will denote the mean under discussion by NS rather then by M, as suggested by an anonymous referee of this paper. Recall that for $x > 0$ and $y > 0$

$$NS = A\frac{u}{\sinh^{-1}u}, \tag{1.1}$$

where

$$A = \frac{x+y}{2}$$

is the arithmetic mean of x and y and

$$u = \frac{x-y}{x+y} \tag{1.2}$$

(see [12, (2.6)]). Clearly $|u| < 1$.

This mean has been studied extensively in [12], [13] and in [10]. Recently, B.-Y. Long and Y.-M. Chu (see [8]) have obtained lower and upper bounds for the mean NS in terms of the generalized logarithmic mean. Inequalities for quotients involving mean NS are obtained in [15]. It has been shown in [11] that the mean NS is bounded from above by members of a one-parameter family of bivariate means. The latter are defined

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in terms of inverse Jacobian elliptic functions. We omit further details. Let us mention also that in [8] the authors called the mean NS the Neuman - Sándor mean.

This paper deals with inequalities which connect mean NS with weighted arithmetic or geometric means of two other bivariate means. Preliminaries are given in Section 2. The main results are established in Section 3. For inequalities similar to those established in this section and involving different means, the interested reader is referred to [3], [4], [5], [6], [7], and the references therein.

2. Preliminaries

In what follows the letters Q and C will stand for the root-mean-square and for the contra-harmonic mean, respectively, of $x > 0$ and $y > 0$. Recall that

$$Q = \sqrt{\frac{x^2+y^2}{2}}, \quad C = \frac{x^2+y^2}{x+y}. \tag{2.1}$$

It is known that

$$A < NS < Q < C \tag{2.2}$$

provided $x \neq y$. The first two inequalities in (2.2) are established in [12, (2.10)] while the third one follows from the Comparison Theorem for Gini means (see [16]).

It has been demonstrated in [12] that the mean NS is the common limit of two recursive sequences whose initial terms are equal to Q and A. We omit further details.

All bivariate means which appear in this paper are strict, homogeneous of degree one and they are monotonic, i.e., they increase (decrease) with an increase (decrease) of each of their variables.

In the next section we will utilize the following lemmas. The first one (see, e.g. [17]) reads as follows.

LEMMA 2.1. *Suppose that the power series* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $g(x) = \sum_{n=0}^{\infty} b_n x^n$ ($b_n > 0$ *for all* $n \geqslant 0$) *both converge for* $|x| < \infty$. *Then the function* $f(x)/g(x)$ *is (strictly) increasing (decreasing) for* $x > 0$ *if the sequence* $\{a_n/b_n\}_{n=0}^{\infty}$ *is (strictly) increasing (decreasing).*

The second lemma, often called L'Hospital's - type rule for monotonicity, can be found, e.g. in [1].

LEMMA 2.2. *Let the functions* f *and* g *be continuous on* $[a,b]$, *differentiable on* (a,b) *and such that* $g'(t) \neq 0$ *on* (a,b). *If* $\frac{f'(t)}{/g'(t)}$ *is (strictly) increasing (decreasing) on* (a,b), *then the functions* $\frac{f(t)-f(b)}{g(t)-g(b)}$ *and* $\frac{f(t)-f(a)}{g(t)-g(a)}$ *are also (strictly) increasing (decreasing) on* (a,b).

3. Bounds for the mean NS

In this section we will deal with problems of finding sharp bounds for NS, where the bounding terms are either additive or multiplicative convex combinations of A and Q or A and C.

Theorems 3.1 and 3.3, established below, provide a generalization of the two-sided inequality

$$Q^{\frac{1}{3}}A^{\frac{2}{3}} < NS < \frac{1}{3}Q + \frac{2}{3}A$$

which follows from [12, (2.8), (3.10)].

Throughout the sequel we will always assume that variables x and y of utilized bivariate means are not equal. This in turn implies that the quantity u defined in (1.2) is not equal to zero. For the later use let us note that with $u = \sinh t$ the inequality $|u| < 1$ implies $|t| < t^*$, where $t^* = \ln(\sqrt{2}+1) = 0.8814\ldots$. One can easily verify that $\sinh t^* = 1$.

Our first result reads as follows.

THEOREM 3.1. *The inequality*

$$\alpha Q + (1-\alpha)A < NS < \beta Q + (1-\beta)A \tag{3.1}$$

holds true if and only if $0 \leqslant \alpha \leqslant \dfrac{1-t^*}{(\sqrt{2}-1)t^*} = 0.3249\ldots$ *and* $1 \geqslant \beta \geqslant \dfrac{1}{3}$.

Proof. Let us rewrite (3.1) as

$$\alpha < \frac{NS - A}{Q - A} < \beta. \tag{3.2}$$

It follows from (1.2) and (2.1) that

$$Q = A\sqrt{1+u^2}. \tag{3.3}$$

Substituting (1.1) and (3.3) into (3.2) we obtain

$$\alpha < \frac{u - \sinh^{-1} u}{(\sinh^{-1} u)\sqrt{1+u^2} - \sinh^{-1} u} < \beta.$$

With $u = \sinh t$ the last two-sided inequality can be written as

$$\alpha < \varphi(t) < \beta, \tag{3.4}$$

where

$$\varphi(t) = \frac{\sinh t - t}{t\cosh t - t} \tag{3.5}$$

$(|t| < t^*)$. Since the function $\varphi(t)$ is an even function, it suffices to investigate its behavior on the interval $(0, t^*)$. Using power series $\sinh t = \sum_{n=0}^{\infty} t^{2n+1}/(2n+1)!$ and $\cosh t = \sum_{n=0}^{\infty} t^{2n}/(2n)!$ we can express (3.5) as follows

$$\varphi(t) = \frac{\sum_{n=1}^{\infty} t^{2n+1}/(2n+1)!}{\sum_{n=1}^{\infty} t^{2n+1}/(2n)!}.$$

With $a_n = 1/(2n+1)!$ and $b_n = 1/(2n)!$ we have $a_n/b_n = (2n)!/(2n+1)! = 1/(2n+1)$. Thus the sequence $\{a_n/b_n\}_{n=1}^{\infty}$ is strictly decreasing and so is the function $\varphi(t)$, where the last statement follows from Lemma 2.1. This in turn implies that

$$\lim_{t\to(t^*)^-} \varphi(t) = \frac{1-t^*}{(\sqrt{2}-1)t^*}$$

and

$$\lim_{t\to 0^+} \varphi(t) = \frac{1}{3}.$$

This in conjunction with (3.4) gives the asserted result. □

In the next theorem we give a counterpart of Theorem 3.1 for the following means $\{$A, NS, C$\}$.

THEOREM 3.2. *The two-sided inequality*

$$\alpha C + (1-\alpha)A < NS < \beta C + (1-\beta)A \tag{3.6}$$

is satisfied if and only if $0 \leqslant \alpha \leqslant \frac{1-t^*}{t^*} = 0.1345...$ *and* $1 \geqslant \beta \geqslant \frac{1}{6}$.

Proof. We will follow, to some extend, lines introduced in the proof of Theorem 3.1. First we divide each member of (3.6) by A and next rearrange terms to obtain

$$\alpha < \frac{\frac{NS}{A}-1}{\frac{C}{A}-1} < \beta.$$

Use of (1.1) together with $C/A = 1+u^2$ followed by a substitution $u = \sinh t$ $(|t| < t^*)$ gives

$$\alpha < \varphi(t) < \beta, \tag{3.7}$$

where

$$\varphi(t) = \frac{\sinh t - t}{t(\sinh t)^2}. \tag{3.8}$$

Differentiation yields

$$\frac{(\sinh t)^3}{\cosh t}\varphi'(t) = 2 - \frac{\sinh t}{t} - \frac{\sinh t}{t}\frac{\tanh t}{t} =: g(t). \tag{3.9}$$

We will show now that $g(t) < 0$ for $t \in (0,t^*)$. To this aim we will utilize the left inequality in

$$(\cosh t)^{\frac{1}{3}} < \frac{\sinh t}{t} < \frac{2+\cosh t}{3} \tag{3.10}$$

(see, e.g., [9], [14]), which is due to Lazarević. It is easy to see that the latter inequality is equivalent to the following one

$$1 < \left(\frac{\sinh t}{t}\right)\left(\frac{\sinh t}{t}\frac{\tanh t}{t}\right).$$

Extracting square roots and next applying inequality of the arithmetic and geometric means we obtain

$$1 < \sqrt{\left(\frac{\sinh t}{t}\right)\left(\frac{\sinh t}{t}\frac{\tanh t}{t}\right)} < \frac{1}{2}\left(\frac{\sinh t}{t} + \frac{\sinh t}{t}\frac{\tanh t}{t}\right).$$

Thus $g(t) < 0$. This in conjunction with (3.9) gives $\varphi'(t) < 0$ and in consequence that $\varphi(t)$ is strictly decreasing on $(0, t^*)$. This in turn implies that

$$\lim_{t \to 0^+} \varphi(t) = \frac{1 - t^*}{t^*}$$

and

$$\lim_{t \to (t^*)^-} \varphi(t) = \frac{1}{6}.$$

Making use of (3.7) we conclude that in order for the inequalities (3.6) to be valid it is necessary and sufficient that $0 \leqslant \alpha \leqslant \frac{1 - t^*}{t^*} = 0.1345...$ and $1 \geqslant \beta \geqslant \frac{1}{6}$. □

We shall now prove two inequalities which can be regarded as the complementary results to those contained in Theorems 3.1 and 3.2. To be more specific, the weighted arithmetic means bounding NS will be now replaced by the weighted geometric means of two other means. We have the following.

THEOREM 3.3. *The following simultaneous inequality*

$$Q^{\alpha} A^{1-\alpha} < NS < Q^{\beta} A^{1-\beta} \tag{3.11}$$

holds true if $0 \leqslant \alpha \leqslant \frac{1}{3}$ *and* $1 \geqslant \beta \geqslant \frac{\ln((2+\sqrt{2})/3)}{\ln\sqrt{2}} = 0.3732...$.

Proof. First we write (3.11) in the form

$$\left(\frac{Q}{A}\right)^{\alpha} < \frac{NS}{A} < \left(\frac{Q}{A}\right)^{\beta}. \tag{3.12}$$

Making use of (1.1) and (3.2) we have $Q/A = \sqrt{1+u^2} = \cosh t$ and $NS/A = u/\sinh^{-1} u = \sinh t/t$, where $u = \sinh t$ $(|t| < t^*)$. Taking logarithms of each part of (3.12) we can rewrite the latter as

$$\alpha < \varphi(t) < \beta, \tag{3.13}$$

where

$$\varphi(t)=\frac{\ln\left(\frac{NS}{A}\right)}{\ln\left(\frac{Q}{A}\right)}=\frac{\ln\left(\frac{\sinh t}{t}\right)}{\ln(\cosh t)}. \tag{3.14}$$

Making use of (3.10) and (3.14) we obtain

$$\frac{1}{3}<\varphi(t)<\psi(t), \tag{3.15}$$

where

$$\psi(t)=\ln\left(\frac{2+\cosh t}{3}\right)/\ln(\cosh t)=:f(t)/g(t).$$

Hence

$$\frac{f'(t)}{g'(t)}=\frac{\cosh t}{2+\cosh t}=1-\frac{2}{2+\cosh t}.$$

This shows that the function $f'(t)/g'(t)$ is strictly increasing on the interval $(0,t^*)$. Making use of Lemma 2.2 we conclude that the function $\psi(t)$ is also strictly increasing on the same interval. Thus $\psi(t)\leqslant\psi(t^*)=\dfrac{\ln((2+\sqrt{2})/3)}{\ln\sqrt{2}}=0.3732\ldots$. This in conjunction with (3.15) yields the asserted result. □

We close this section with the following.

THEOREM 3.4. *The following inequality*

$$C^{\alpha}A^{1-\alpha}<NS<C^{\beta}A^{1-\beta} \tag{3.16}$$

is valid if $0\leqslant\alpha\leqslant\dfrac{1}{6}$ *and* $1\geqslant\beta\geqslant\dfrac{\ln((2+\sqrt{2})/3)}{\ln 2}=0.1865\ldots$.

Proof. It follows from (3.16) that

$$\alpha<\frac{\ln\left(\frac{NS}{A}\right)}{\ln\left(\frac{C}{A}\right)}<\beta. \tag{3.17}$$

One can easily verify that the identity $\dfrac{C}{A}=\left(\dfrac{Q}{A}\right)^2$ holds true. Making use of appropriate formulas used in the proof of Theorem 3.3 we obtain $\dfrac{NS}{A}=\dfrac{\sinh t}{t}$ and $\dfrac{C}{A}=\cosh^2 t$. This in conjunction with (3.17) yields

$$\alpha<\lambda(t)<\beta,$$

where $\lambda(t)=\dfrac{\varphi(t)}{2}$ and $\varphi(t)$ is the same as in the proof of Theorem 3.3. Thus the upper bound for α and the lower bound for β are equal the half of the corresponding bounds in Theorem 3.3. The assertion now follows. □

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