Journal of Mathematical Inequalities
Volume 6, Number 4 (2012), 645–654 doi:10.7153/jmi-06-63

A SIMPLE PROOF OF OPPENHEIM'S DOUBLE INEQUALITY RELATING TO THE COSINE AND SINE FUNCTIONS

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(*Communicated by N. Elezović*)

Abstract. In the paper, the authors provide a simple proof of Oppenheim's double inequality relating to the cosine and sine functions. In passing, the authors survey this topic.

1. Introduction and main results

In [14], the following problem was posed: For each $p > 0$ there is a greatest q and a least r such that

$$\frac{q\sin x}{1+p\cos x} \leqslant x \leqslant \frac{r\sin x}{1+p\cos x} \tag{1}$$

for $0 \leqslant x \leqslant \frac{\pi}{2}$. Determine q and r as functions of p.

In [3], it was explicitly obtained that

1. the least value of r required by the problem is

$$r = \frac{\pi}{2} \quad \text{when } p \leqslant \frac{\pi}{2} - 1,$$
$$r = p+1 \quad \text{when } p \geqslant \frac{\pi}{2} - 1;$$

2. the required greatest value of q is

$$q = p+1 \quad \text{when } p \leqslant \frac{1}{2},$$
$$q = \frac{\pi}{2} \quad \text{when } p \geqslant \frac{\pi}{2}.$$

Mathematics subject classification (2010): Primary 33B10; secondary 26D05.

Keywords and phrases: Simple proof, Oppenheim's double inequality, cosine function, sine function, monotonicity.

The first author was partially supported by the China Scholarship Council. The present investigation was supported in part by the Natural Science Foundation Project of Chongqing, China under Grant CSTC2011JJA00024, the Research Project of Science and Technology of Chongqing Education Commision, China under Grant KJ120625, and the Fund of Chongqing Normal University, China under Grant 10XLR017 and 2011XLZ07..

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In [11, p. 238, 3.4.15], it was listed that

$$\frac{(p+1)\sin x}{1+p\cos x} \leqslant x \leqslant \frac{(\pi/2)\sin x}{1+p\cos x} \tag{2}$$

for $0 < p \leqslant \frac{1}{2}$ and $0 \leqslant x \leqslant \frac{\pi}{2}$.

In [17, p. 521, (26)], by Čebyšev's integral inequality, it was constructed that

$$\frac{\sin x}{x} \geqslant \frac{1+\cos x}{2} \tag{3}$$

and

$$\frac{\sin x}{x} \geqslant \frac{1+2\cos x}{3} + \frac{x\sin x}{6} \tag{4}$$

for $0 < x \leqslant \frac{\pi}{2}$. The inequality (3) can be rewritten as

$$\frac{2\sin x}{1+\cos x} \geqslant x, \quad 0 \leqslant x \leqslant \frac{\pi}{2}. \tag{5}$$

In [24], it was pointed out that the inequality

$$\frac{3\sin x}{2+\cos x} < x \tag{6}$$

was discovered by Nicolaus de Cusa (1401–1464) using certain geometrical constructions. In [23], the inequality (6) was generalized as follows: For $a,b,c > 0$ such that $2b \leqslant c \leqslant a+b$,

$$\frac{c\sin x}{a+b\cos x} < x, \quad 0 < x < \frac{\pi}{2}. \tag{7}$$

This is equivalent to the left-hand side inequality in (1) for $2p \leqslant q \leqslant 1+p$.

In [27, Theorem 7], a complete answer to the above problem was obtained as follows: Let $0 \leqslant x \leqslant \frac{\pi}{2}$ and $p > 0$, then the inequality (1) holds in cases:

1. When $0 < p < \frac{1}{2}$, we have $q = p+1$, $r = \frac{\pi}{2}$;
2. When $\frac{1}{2} \leqslant p < \frac{\pi}{2} - 1$, we have $q = 4p(1-p^2)$, $r = \frac{\pi}{2}$;
3. When $\frac{\pi}{2} - 1 \leqslant p < \frac{2}{\pi}$, we have $q = 4p(1-p^2)$, $r = p+1$;
4. When $\frac{2}{\pi} \leqslant p < \infty$, we have $q = \frac{\pi}{2}$, $r = p+1$.

The aim of this paper is to provide a simple proof of the inequality (1).

Our main results may be recited as the following theorems.

THEOREM 1. *For $p > 0$ and $x \in \left(0, \frac{\pi}{2}\right]$, let*

$$f_p(x) = \frac{\sin x}{x(1+p\cos x)}. \tag{8}$$

1. *When* $p \geqslant \frac{2}{\pi}$, *the function* $f_p(x)$ *is strictly increasing;*

2. *When* $0 < p \leqslant \frac{1}{2}$, *the function* $f_p(x)$ *is strictly decreasing;*

3. *When* $\frac{1}{2} < p < \frac{2}{\pi}$, *the function* $f_p(x)$ *has a unique maximum;*

4. *When* $p \leqslant 0$, *the reciprocal of* $f_p(x)$ *is strictly increasing.*

As straightforward consequences of Theorem 1, the following inequalities may be derived immediately.

THEOREM 2. *If* $p \geqslant \frac{2}{\pi}$, *then*

$$\frac{(\pi/2)\sin x}{1+p\cos x} \leqslant x \leqslant \frac{(1+p)\sin x}{1+p\cos x}, \quad 0 \leqslant x \leqslant \frac{\pi}{2}; \tag{9}$$

If $p \leqslant \frac{1}{2}$, *the double inequality* (9) *reverses; If* $\frac{1}{2} < p < \frac{2}{\pi}$, *then*

$$\frac{4p(1-p^2)\sin x}{1+p\cos x} \leqslant x \leqslant \frac{\max\{\pi/2, 1+p\}\sin x}{1+p\cos x}. \tag{10}$$

The constants $\frac{\pi}{2}$ *and* $1+p$ *in* (9) *and* (10) *are the best possible.*

2. Simple proofs of Theorems 1 and 2

Now we start to demonstrate our simple proofs of Theorems 1 and 2.

Proof of Theorem 1. A direct differentiation yields

$$\begin{aligned} f_p'(x) &= \frac{(x-\sin x\cos x)[p-(\sin x - x\cos x)/(x-\sin x\cos x)]}{x^2(p\cos x+1)^2} \\ &\triangleq \frac{(x-\sin x\cos x)[p-h(x)]}{(px\cos x+x)^2}, \\ h'(x) &= \frac{2\left[2x^2+x\sin(2x)+2\cos(2x)-2\right]\sin x}{[2x-\sin(2x)]^2} \\ &\triangleq \frac{2g(x)\sin x}{[2x-\sin(2x)]^2}, \\ g'(x) &= 2\cos(2x)x+4x-3\sin(2x), \\ g''(x) &= 8(\tan x - x)\sin x\cos x \\ &> 0 \end{aligned} \tag{11}$$

on $\left(0, \frac{\pi}{2}\right)$. So the function $g'(x)$ is strictly increasing on $\left(0, \frac{\pi}{2}\right)$. Further, from $g'(0) = 0$, it follows that $g'(x) > 0$ and that $g(x)$ is strictly increasing on $\left(0, \frac{\pi}{2}\right)$. Owing to $g(0) = 0$, the functions $g(x)$ and $h'(x)$ are positive on $\left(0, \frac{\pi}{2}\right)$. As a result, the function $h(x)$ is strictly increasing on $\left(0, \frac{\pi}{2}\right)$. Due to $\lim_{x\to 0^+} h(x) = \frac{1}{2}$ and $h\left(\frac{\pi}{2}\right) = \frac{2}{\pi}$, it is concluded that

1. when $p \geqslant \frac{2}{\pi}$, the derivative $f_p'(x)$ is positive on $\left(0, \frac{\pi}{2}\right)$, and so the function $f_p(x)$ is strictly increasing on $\left(0, \frac{\pi}{2}\right]$;
2. when $p \leqslant \frac{1}{2}$, the derivative $f_p'(x)$ is negative on $\left(0, \frac{\pi}{2}\right)$, and so the function $f_p(x)$ is strictly decreasing on $\left(0, \frac{\pi}{2}\right]$;
3. when $\frac{1}{2} < p < \frac{2}{\pi}$, the derivative $f_p'(x)$ has a unique zero on $\left(0, \frac{\pi}{2}\right)$, and so the function $f_p(x)$ has a unique maximum on $\left(0, \frac{\pi}{2}\right]$.

On the other hand, when $p \leqslant 0$, a direct differentiation gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{f_p(x)}\right] &= (1 - x\cot x)(p\cot x + \csc x) - px \\ &> (1 - x\cot x)\left(\frac{p}{x} + \frac{1}{x}\right) - px \\ &= \frac{1 - x\cot x + p(1 - x^2 - x\cot x)}{x} \\ &> 0,\end{aligned}$$

where $1 - x\cot x > 0$ and $1 - x^2 - x\cot x < 0$ on $\left(0, \frac{\pi}{2}\right)$. This means that the reciprocal of $f_p(x)$ is strictly increasing on $\left(0, \frac{\pi}{2}\right)$ for $p \leqslant 0$. The proof of Theorem 1 is complete. □

Proof of Theorem 2. It is easy to see that

$$\lim_{x\to 0^+} f_p(x) = \frac{1}{1+p} \quad \text{and} \quad f_p\left(\frac{\pi}{2}\right) = \frac{2}{\pi}.$$

By Theorem 1, it follows that

1. when $p \geqslant \frac{2}{\pi}$, we have
$$\frac{1}{1+p} < \frac{\sin x}{x(1 + p\cos x)} \leqslant \frac{2}{\pi} \tag{12}$$
on $\left(0, \frac{\pi}{2}\right]$, which may be rewritten as the inequality (9);
2. when $p \leqslant \frac{1}{2}$, the inequality (12) reverses;
3. when $\frac{1}{2} < p < \frac{2}{\pi}$, we have
$$\frac{\sin x}{x(1 + p\cos x)} > \min\left\{\frac{1}{1+p}, \frac{2}{\pi}\right\}$$
on $\left(0, \frac{\pi}{2}\right)$, which may be rearranged as the right-hand side inequality in (10).

The left-hand side inequality in (10) can be deduced by the same argument as in [27, p. 60]. The proof of Theorem 2 is complete. □

3. Remarks

After proving our theorems, we give several remarks on them.

REMARK 1. For $p \leqslant \frac{1}{2}$, the reversed version of the inequality (9) may be rewritten as

$$\frac{2(1+p\cos x)}{\pi} < \frac{\sin x}{x} \leqslant \frac{(1+p\cos x)}{1+p}, \quad 0 < x \leqslant \frac{\pi}{2}. \tag{13}$$

Integrating on both sides of (13) gives

$$1+\frac{2}{\pi}p < \int_0^{\pi/2} \frac{\sin x}{x}\,\mathrm{d}x < \frac{\pi+2p}{1+p}, \quad p \leqslant \frac{1}{2}.$$

Hence, taking $p = \frac{1}{2}$ in the above inequality leads to

$$1.31\cdots = 1+\frac{1}{\pi} < \int_0^{\pi/2} \frac{\sin x}{x}\,\mathrm{d}x < \frac{2(\pi+1)}{3} = 2.76\cdots. \tag{14}$$

Similarly, if integrating and letting $p = \frac{2}{\pi}$ in (9), then

$$1.34\cdots = \frac{4+\pi^2}{2(2+\pi)} < \int_0^{\pi/2} \frac{\sin x}{x}\,\mathrm{d}x < 1+\left(\frac{2}{\pi}\right)^2 = 1.40\cdots. \tag{15}$$

REMARK 2. For $\frac{1}{2} < p < \frac{2}{\pi}$, the inequality (10) may be rearranged as

$$\min\left\{\frac{2}{\pi}, \frac{1}{1+p}\right\}(1+p\cos x) \leqslant \frac{\sin x}{x} \leqslant \frac{1+p\cos x}{4p(1-p^2)}, \quad 0 < x \leqslant \frac{\pi}{2}. \tag{16}$$

As done in Remaek 1, integrating gives

$$\min\left\{\frac{2}{\pi}, \frac{1}{1+p}\right\}\left(p+\frac{\pi}{2}\right) < \int_0^{\pi/2} \frac{\sin x}{x}\,\mathrm{d}x < \frac{2p+\pi}{8p\left(1-p^2\right)}, \quad \frac{1}{2} < p < \frac{2}{\pi}.$$

Maximizing the lower bound and minimizing the upper bound in the above double inequality reduce to

$$1.36\cdots = 2\left(1-\frac{1}{\pi}\right) < \int_0^{\pi/2} \frac{\sin x}{x}\,\mathrm{d}x < \frac{2p_0+\pi}{8p_0\left(1-p_0^2\right)} = 1.37\cdots, \tag{17}$$

where

$$
\begin{aligned}
p_0 &= \frac{\pi}{4}\left\{\cos\left[\frac{1}{3}\arctan\left(\frac{4\sqrt{\pi^2-4}}{\pi^2-8}\right)\right]+\sqrt{3}\sin\left[\frac{1}{3}\arctan\left(\frac{4\sqrt{\pi^2-4}}{\pi^2-8}\right)\right]-1\right\}\\
&=0.52\cdots.
\end{aligned}
$$

Comparing inequalities (14), (15) and (17) shows that the inequality (10) or (16) is more accurate in whole.

The inequality (17) improves inequalities

$$
1.33\cdots=\frac{4}{3}<\int_0^{\pi/2}\frac{\sin x}{x}\,\mathrm{d}x<\frac{\pi+1}{3}=1.38\cdots \tag{18}
$$

and

$$
\int_0^{\pi/2}\frac{\sin x}{x}\,\mathrm{d}x>\frac{\pi+5}{6}=1.35\cdots \tag{19}
$$

obtained in [17, p. 521, (32)] and [20].

REMARK 3. In [11, p. 247, 3.4.31], it was listed that the inequality

$$
\arcsin x>\frac{6\left(\sqrt{1+x}-\sqrt{1-x}\right)}{4+\sqrt{1+x}+\sqrt{1-x}}>\frac{3x}{2+\sqrt{1-x^2}} \tag{20}
$$

holds for $0<x<1$. It was also pointed out in [11, p. 247, 3.4.31] that these inequalities are due to R. E. Shafer, but no a related reference is provided. By now we do not know the very original source of inequalities in (20).

In the first part of the short paper [6], the inequality between the very ends of (20) was recovered and an upper bound for the arc sine function was also established as

$$
\frac{3x}{2+\sqrt{1-x^2}}\leqslant\arcsin x\leqslant\frac{\pi x}{2+\sqrt{1-x^2}},\quad 0\leqslant x\leqslant 1. \tag{21}
$$

Therefore, we call (21) Shafer-Fink's double inequality for the arc sine function.

In [10], the right-hand side inequality in (21) was improved to

$$
\arcsin x\leqslant\frac{\pi x/(\pi-2)}{2/(\pi-2)+\sqrt{1-x^2}},\quad 0\leqslant x\leqslant 1. \tag{22}
$$

As done in [27], by taking $t=\sin x$ in Theorem 2, inequalities in (21), (22), and the following Shafer-Fink type inequalities may be derived readily:

$$
\frac{\pi(4-\pi)x}{2/(\pi-2)+\sqrt{1-x^2}}\leqslant\arcsin x,\quad 0\leqslant x\leqslant 1; \tag{23}
$$

$$
\frac{\pi x/2}{1+\sqrt{1-x^2}}\leqslant\arcsin x,\quad 0\leqslant x\leqslant 1. \tag{24}
$$

All corresponding bounds in (21), (22), (23), and (24) are not included each other.

Above-mentioned fact strongly shows us that Oppenheim type inequalities and Shafer-Fink type inequalities can be converted to each other.

REMARK 4. For $\frac{1}{2} < p < \frac{2}{\pi}$, let

$$P_x(p) = \frac{p(1-p^2)}{1+p\cos x}, \quad 0 < x < \frac{\pi}{2}.$$

Then a straightforward calculation gives

$$1-3p^2-2p^3 < (1+p\cos x)^2 P_x'(p) = 1-3p^2-2p^3\cos x < 1-3p^2.$$

This implies that

1. when $\frac{2}{\pi} > p \geqslant \frac{\sqrt{3}}{3}$ the function $p \mapsto P_x(p)$ is decreasing;
2. when $\frac{1}{2} < p < \frac{\sqrt{3}}{3}$ the function $p \mapsto P_x(p)$ attains its maximum $\frac{1}{4}\sec^3\left(\frac{x}{3}\right)$ at the point $\left[\cos\left(\frac{2}{3}x\right) - \frac{1}{2}\right]\sec x$ for $0 < x < \frac{\pi}{2}$.

Combining this with the fact that the function $p \mapsto \frac{1+p}{1+p\cos x}$ is increasing, we derive from the inequality (10) the following double inequalities for $0 < x < \frac{\pi}{2}$:

$$\frac{8\sin x}{3\left(\sqrt{3}+\cos x\right)} < x < \frac{\pi\sin x}{2+(\pi-2)\cos x}, \tag{25}$$

$$\sin x\sec^3\left(\frac{x}{3}\right) < x < \frac{\pi\sin x}{2+(\pi-2)\cos x}. \tag{26}$$

Similarly, from (9) and its revision, we deduce the following sharp double inequalities for $0 < x < \frac{\pi}{2}$:

$$\frac{\pi^2\sin x}{2(\pi+2\cos x)} < x < \frac{(\pi+2)\sin x}{\pi+2\cos x}, \tag{27}$$

$$\frac{\pi\sin x}{2+\cos x} > x > \frac{3\sin x}{2+\cos x}. \tag{28}$$

The famous software MATHEMATICA 7.0 shows that the lower bound in the inequality (26) is better than the corresponding ones in (25) and (28), but the left-hand side inequalities in (26) and (27) are not contained each other. Furthermore, the upper bound in (26) is better than the corresponding ones in (27) and (28). Accordingly, the best and sharp double inequality deduced from Theorem 2 may be stated as

$$\max\left\{\frac{\pi^2\sin x}{2(\pi+2\cos x)}, \sin x\sec^3\left(\frac{x}{3}\right)\right\} < x < \frac{\pi\sin x}{2+(\pi-2)\cos x} \tag{29}$$

or

$$\max\left\{\frac{2(\pi+2\cos x)}{\pi^2}, \cos^3\left(\frac{x}{3}\right)\right\} > \frac{\sin x}{x} > \frac{2+(\pi-2)\cos x}{\pi} \tag{30}$$

for $0<x<\frac{\pi}{2}$. Replacing $\sin x$ by t in (29) gives

$$\max\left\{\frac{\pi^2 t}{2\left(\pi+2\sqrt{1-t^2}\right)}, t\sec^3\left(\frac{\arcsin t}{3}\right)\right\} < \arcsin t < \frac{\pi t}{2+(\pi-2)\sqrt{1-t^2}}, \quad 0<t<\frac{\pi}{2}. \tag{31}$$

The right-hand side inequality in (31) is a recovery of the inequality (22) in [10].

REMARK 5. Recently, some new Shafer-Fink type inequalities and generalizations of Oppenheim's inequality are procured in [2, 8, 15, 19, 28].

For more information on this topic, please refer to [7, 22, 25, 26], [21, Sections 1.7, 7.5 and 7.6], and closely related references therein.

REMARK 6. In [1, 16], the following L'Hôpital rule for monotonicity was established: Let $f(x)$ and $g(x)$ be continuous functions on $[a,b]$ and differentiable on (a,b) such that $g'(x)\ne 0$ on (a,b). If $\frac{f'(x)}{g'(x)}$ is increasing (or decreasing respectively) on (a,b), then the functions $\frac{f(x)-f(b)}{g(x)-g(b)}$ and $\frac{f(x)-f(a)}{g(x)-g(a)}$ are also increasing (or decreasing respectively) on (a,b). This conclusion has been employed in a lot of literature such as [9, 12, 13] and closely related references therein. This conclusion can also be utilized to prove the increasing monotonicity of the function $h(x)$ in the proof of Theorem 1 as follows.

Let $h_1(x)=\sin x - x\cos x$ and $h_2(x)=x-\sin x\cos x$ on $\left[0,\frac{\pi}{2}\right]$. Then

$$h_1'(x)=x\sin x, \quad h_2'(x)=2\sin^2 x,$$

and so

$$\frac{h_1'(x)}{h_2'(x)}=\frac{x}{2\sin x}$$

is strictly increasing on $\left(0,\frac{\pi}{2}\right)$. Consequently, the function

$$h(x)=\frac{h_1(x)}{h_2(x)}=\frac{h_1(x)-h_1(0)}{h_2(x)-h_2(0)},$$

defined in (11), is strictly increasing on $\left(0,\frac{\pi}{2}\right)$.

REMARK 7. We note that our approach used in Section 2 is simpler and more elementary than those in [2, 3, 4, 6, 10, 15, 27, 28] and closely related references therein.

REMARK 8. It is noted that there are some applications in [5] of this type of inequalities obtained in [10].

REMARK 9. A trivial remark is that the surname "Oppenheim" was mistaken for "Oppeheim" in [27].

REMARK 10. This paper is a slightly modified version of the preprint [18].

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(Received February 12, 2012)

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