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BOUNDEDNESS FOR THE MULTI–COMMUTATORS OF CALDERÓN–ZYGMUND OPERATORS

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Abstract. In this paper, the authors study the multi-commutators $\mathscr{T}^{A_1,\cdots,A_k}$ generalized by the Calderón-Zygmund operator T and the (m_i+1)-th remainders of Taylor series of the functions A_i whose m_i-th derivatives belong to BMO spaces for $m_i \geqslant 0$ and $i=1,2,\cdots,k$. The boundedness from the weighted central Morrey space $B_p(w)$ to the weighted central BMO space $CMO(w)$ for these multi-commutators was derived. As corollary, the L^p-boundedness for the multi-commutators has been obtained.

1. Introduction

Let T be a non-convolution type singular integral operator with a standard Calderón-Zygmund kernel $K(x,y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{x=y\}$, this means that, for any $f \in L^2(\mathbb{R}^n)$ with compact support and for $x \notin \operatorname{supp}(f)$, $Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy$, with

$$|K(x,y)| \leqslant \frac{C}{|x-y|^n} \tag{1.1}$$

and, for all $2|x-z| \leqslant |y-z|$,

$$|K(x,y)-K(z,y)|+|K(y,x)-K(y,z)| \leqslant C\frac{|x-z|^{\varepsilon}}{|y-z|^{n+\varepsilon}}, \tag{1.2}$$

with some positive constants C and $0<\varepsilon \leqslant 1$. We note that, if the singular integral operator T is bounded on $L^2(\mathbb{R}^n)$, then T is called a (non-convolution type) Calderón-Zygmund operator. It's well-known that a Calderón-Zygmund operator is bounded on $L^p(\mathbb{R}^n)$, $1<p<\infty$, and is of weak $(1,1)$ boundedness. The commutator of a Calderón-Zygmund operator T and a BMO function b, $[b,T](f) = bT(f) - T(bf)$, was first studied by Coifman, Rochberg and Weiss [2] who proved that $\|[b,T](f)\|_{L^p} \leqslant C\|b\|_{BMO}\|f\|_{L^p}$ for all $1<p<\infty$.

We define the following multilinear commutator generalized by the Calderón-Zygmund operator T and the function A,

$$T^A(f)(x) := \int_{\mathbb{R}^n} \frac{R_{m+1}(A;x,y)}{|x-y|^m} K(x,y)f(y)dy, \tag{1.3}$$

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where $R_{m+1}(A;x,y)$ denotes the $(m+1)$-th remainder of Taylor series of A at x about y, more precisely,

$$R_{m+1}(A;x,y) = A(x) - \sum_{|\gamma| \leqslant m} \frac{1}{\gamma!} D^{\gamma} A(y)(x-y)^{\gamma},$$

and $D^{\gamma}(A) \in BMO(\mathbb{R}^n)$ for all multi-indices $|\gamma| = m \geqslant 0$. It's clear that, in case $m = 0$, $T^A(f) = [A,T](f)$ is the classical commutator mentioned above.

Since L^{∞} is properly contained in BMO, we see that in general the kernel of the operator T^A fails to satisfy the standard kernel estimates (1.1) and (1.2), and one can not get the L^p-boundedness for T^A from the standard Calderón-Zygmund theory. In 2002, Lu and Yan proved in [5] that if T^A is bounded on $L^2(\mathbb{R}^n)$ with the bound $C\sum_{|\gamma|=m} \|D^{\gamma}A\|_{BMO}$, then T^A is bounded from $L^{\infty}(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$ with the same bound, which in turn implies that T^A is bounded on $L^p(\mathbb{R}^n)$ for $2 \leqslant p < \infty$.

One aim of the paper is to derive the weighted CMO estimates for the operator T^A, which imply the endpoint estimates for the operator T^A and so generalize the Lu-Yan's results above. In fact, we consider the following more general multi-commutator $\mathscr{T}^{\vec{A}}$ generalized by the Calderón-Zygmund operator T and the functions $A_1, \cdots, A_k$,

$$\mathscr{T}^{\vec{A}}(f)(x) := \int_{\mathbb{R}^n} K(x,y) \prod_{j=1}^{k} \frac{R_{m_j+1}(A_j;x,y)}{|x-y|^{m_j}} f(y)dy \tag{1.4}$$

where $R_{m_j+1}(A_j;x,y)$ denotes the (m_j+1)-th remainder of Taylor series of A_j at x about y, and $D^{\gamma}(A_j) \in BMO(\mathbb{R}^n)$ for all multi-indices $|\gamma| = m_j \geqslant 0$, and $j = 1, 2, \cdots, k$.

In particular if $m_1 = m_2 = \cdots = m_k = 0$, we write $\mathscr{T}^{\vec{A}}$ as $\mathscr{T}_0^{\vec{A}}$, i.e.

$$\mathscr{T}_0^{\vec{A}}(f)(x) := \int_{\mathbb{R}^n} K(x,y) \left[\prod_{j=1}^{k} [A_j(x) - A_j(y)] \right] f(y)dy \tag{1.5}$$

which was introduced by Pérez and Trujillo-González [6] in 2002 who proved that the operator $\mathscr{T}_0^{\vec{A}}$ is bounded on $L^p(w)$ for $w \in A_p$ and $1 < p < \infty$ whenever all $A_j \in BMO(\mathbb{R}^n)$. Recently in [7] the authors proved that $\mathscr{T}_0^{\vec{A}}$ is bounded on central Morrey spaces if each A_j belong to CMO spaces, $j = 1, 2, \cdots, k$.

For the multi-indices $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n)$, we will always use notations $|\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_n$, $\gamma! = \gamma_1!\gamma_2!\cdots\gamma_n!$, and $x^{\gamma} = x_1^{\gamma_1} x_2^{\gamma_2} \cdots x_n^{\gamma_n}$, and $D^{\gamma} = \frac{\partial^{|\gamma|}}{\partial^{\gamma_1} x_1 \partial^{\gamma_2} x_2 \cdots \partial^{\gamma_n} x_n}$. For a cube Q and a locally integrable function f, let $f_Q = \frac{1}{|Q|}\int_Q f(x)dx$ and the sharp maximal function

$$M^{\#}(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy.$$

A function f is said to belong to $BMO(\mathbb{R}^n)$ if $M^{\#}(f) \in L^{\infty}(\mathbb{R}^n)$, and the norm is defined by $\|f\|_{BMO} = \|M^{\#}(f)\|_{L^{\infty}}$.

Denote by $Q(x,r)$ the cube in $\mathbb{R}^n$ with side length r and center point x and with sides parallel to the axes. For a non-negative weight functions w, we denote the weighted central BMO space by $CMO(w)$, which is the space of those functions $f \in L_{loc}(\mathbb{R}^n)$ such that

$$\|f\|_{CMO(w)} := \sup_{r\geqslant 1} \frac{1}{w(Q(0,r))} \int_{Q(0,r)} |f(x) - f_{Q(0,r)}| w(x)dx < \infty.$$

It is easy to see that

$$\|f\|_{CMO(w)} \approx \sup_{r\geqslant 1} \inf_{c\in\mathbb{R}} \frac{1}{w(Q(0,r))} \int_{Q} |f(x) - c| w(x)dx.$$

We remark that the CMO space is the dual space of the atomic space associated with the Beurling algebra, which is in some sense a local version of BMO at origin. But, they have quite different properties, for example, there is no analogy of the famous John-Nirenberg inequality of BMO for the CMO space, see [1] and [7] for the details.

Let $1 < p < \infty$ and w be a non-negative weight function, we define the weighted central Morrey space $B_p(w)$ by

$$\|f\|_{B_p(w)} := \sup_{r\geqslant 1} \left(\frac{1}{w(Q(0,r))} \int_{Q(0,r)} |f(y)|^p w(y)dy \right)^{\frac{1}{p}} < \infty.$$

As is easily seen, the spaces $CMO(w)$ and $B_p(w)$ reflect local regularity of the function more precisely than the Lebesgue space. We also denote by $C\dot{M}O(w)$ and $\dot{B}_p(w)$ the homogeneous versions of the weighted central bounded mean oscillation space $CMO(w)$ and the weighted central Morrey space $B_p(w)$, which can be defined by taking the supremum over $r > 0$ in the definitions above instead of $r \geqslant 1$.

Now we state our main theorems as follows:

THEOREM 1.1. *Let $m_j \geqslant 1$ and $D^\gamma(A_j) \in BMO(\mathbb{R}^n)$ for multi-indices $|\gamma| = m_j$ and $j = 1,2,\cdots,k$, and let $w \in A_1$, a Muchenhoupt weight. Suppose that $\mathscr{T}^{\vec{A}}$ is bounded on $L^{p_0}(\mathbb{R}^n)$ with the bound $C\prod_{j=1}^k \sum_{|\gamma|=m_j} \|D^\gamma A_j\|_{BMO}$ for some $1 < p_0 < \infty$. Then the operator $\mathscr{T}^{\vec{A}}$ is bounded from $B_p(w)$ to $CMO(w)$, and bounded from $\dot{B}_p(w)$ to $C\dot{M}O(w)$ for any p with $p_0 < p < \infty$. Moreover,*

$$\left\| \mathscr{T}^{\vec{A}}(f) \right\|_{CMO(w)} \leqslant C \prod_{j=1}^{k} \sum_{|\gamma|=m_j} \|D^\gamma A_j\|_{BMO} \|f\|_{B_p(w)} \tag{1.6}$$

and

$$\left\| \mathscr{T}^{\vec{A}}(f) \right\|_{C\dot{M}O(w)} \leqslant C \prod_{j=1}^{k} \sum_{|\gamma|=m_j} \|D^\gamma A_j\|_{BMO} \|f\|_{\dot{B}_p(w)} \tag{1.7}$$

with the absolute positive constant C.

COROLLARY 1.2. *With the same conditions of Theorem 1.1 for $w = 1$, then the multi-commutator $\mathscr{T}^{\vec{A}}$ is bounded from $L^\infty(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$, and*

$$\left\| \mathscr{T}^{\vec{A}}(f) \right\|_{BMO(\mathbb{R}^n)} \leqslant C \prod_{j=1}^{k} \sum_{|\gamma|=m_j} \|D^\gamma A_j\|_{BMO} \|f\|_{L^\infty(\mathbb{R}^n)} \tag{1.8}$$

with the absolute positive constant C. Moreover, the multi-commutator $\mathscr{T}^{\vec{A}}$ is bounded on $L^p(\mathbb{R}^n)$ for any $p_0 \leqslant p < \infty$, and

$$\left\| \mathscr{T}^{\vec{A}}(f) \right\|_{L^p(\mathbb{R}^n)} \leqslant C \prod_{j=1}^{k} \sum_{|\gamma|=m_j} \|D^\gamma A_j\|_{BMO} \|f\|_{L^p(\mathbb{R}^n)} \tag{1.9}$$

with the absolute positive constant C.

We remark that Cohen and Gosselin [3] have proved the $L^{p_0}(\mathbb{R}^n)$-boundedness for the multi-commutator $\mathscr{T}^{\vec{A}}$ in the special case $k(x,y) = \Omega(x-y)/|x-y|^n$ with the three conditions: (1) $\Omega \in Lip(\mathbb{S}^{n-1})$, (2) Ω is homogeneous of degree zero, (3) $\int_{\mathbb{S}^{n-1}} \Omega(x)x^\alpha dx = 0$ for $|\alpha| = m$. By means of the "$T1$" theorem, S. Hofmann [4] extended Cohen-Gosselin's result and obtained the weighted $L^p(w)$-boundedness for the multi-commutator $\mathscr{T}^{\vec{A}}$ with the same conditions as in [3] and $w \in A_p$. But for general kernel $k(x,y)$, the $L^{p_0}(\mathbb{R}^n)$-boundedness for the multi-commutator $\mathscr{T}^{\vec{A}}$ remains unknown. Here we point out that the results of Corollary 1.2 above have extended the related works of Lu and Yan in [5] in which only the case $k = 1$ had been considered.

In Theorem 1.1, we assume that all $m_j \geqslant 1$ for the multi-commutator $\mathscr{T}^{\vec{A}}(f)$ of form (1.4). For the cases that some $m_j = 0$, we may equivalently study the following multi-commutator

$$\mathbb{T}^{\vec{A},\vec{B}}(f)(x) := \int_{\mathbb{R}^n} K(x,y) \left[\prod_{j=1}^{k} \frac{R_{m_j+1}(A_j;x,y)}{|x-y|^{m_j}} \right] \left[\prod_{i=1}^{l} [B_i(x) - B_i(y)] \right] f(y)dy \tag{1.10}$$

We will deduce the following weighted CMO estimates and L^p-boundedness for the operator $\mathbb{T}^{\vec{A},\vec{B}}$. For convenience, we denote by C_i^l the family of all subsets $\sigma = \{\sigma_1, \sigma_2, \cdots, \sigma_i\}$ of i different elements of $\{1,2,\cdots,l\}$, and let $\sigma' = \{1,2,\cdots,l\} \backslash \sigma$ and $\vec{B_\sigma} = \{B_{\sigma_1}, B_{\sigma_2}, \cdots, B_{\sigma_i}\}$.

THEOREM 1.3. *Let $D^\gamma(A_j) \in BMO(\mathbb{R}^n)$ for multi-indices $|\gamma| = m_j$ and $m_j \geqslant 1$, $j = 1,2,\cdots,k$. Suppose that $\mathscr{T}^{\vec{A}}$ is bounded on $L^{p_0}(\mathbb{R}^n)$ for some $1 < p_0 < \infty$ with the bound $C\prod_{j=1}^{k} \sum_{|\gamma|=m_j} \|D^\gamma A_j\|_{BMO}$. If $w \in A_1$ and $B_i \in BMO(\mathbb{R}^n)$, $i = 1,2,\cdots,l$, then the operators $\mathbb{T}^{\vec{A},\vec{B}}$ of form (1.10) have the property that*

$$\begin{aligned}\left\|\mathbb{T}^{\overrightarrow{A},\overrightarrow{B}}(f)\right\|_{CMO(w)} \leqslant & C\prod_{j=1}^{k}\sum_{|\gamma|=m_j}\|D^{\gamma}A_j\|_{BMO}\prod_{i=1}^{l}\|B_i\|_{BMO}\|f\|_{B_p(w)} \\ & + C\prod_{i=1}^{l}\|B_i\|_{BMO}\left\|\mathscr{T}^{\overrightarrow{A}}(f)\right\|_{B_p(w)} \\ & + C\sum_{i=1}^{l-1}\sum_{\sigma\in C_i^l}\prod_{j\in\sigma'}\|B_j\|_{BMO}\left\|\mathbb{T}^{\overrightarrow{A},\overrightarrow{B_\sigma}}(f)\right\|_{B_p(w)}\end{aligned} \tag{1.11}$$

with the absolute positive constant C.

COROLLARY 1.4. *With the same conditions as in Theorem 1.3, then the operators $\mathbb{T}^{\overrightarrow{A},\overrightarrow{B}}$ of form (1.10) are bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for any p with $p_0 \leqslant p < \infty$, moreover,*

$$\left\|\mathbb{T}^{\overrightarrow{A},\overrightarrow{B}}(f)\right\|_{L^p(\mathbb{R}^n)} \leqslant C\prod_{j=1}^{k}\sum_{|\gamma|=m_j}\|D^{\gamma}A_j\|_{BMO}\prod_{i=1}^{l}\|B_i\|_{BMO}\|f\|_{L^p(\mathbb{R}^n)} \tag{1.12}$$

with the absolute positive constant C.

From Corollary 1.4 one can obtain the L^p boundedness $(1 < p < \infty)$ for the multilinear commutator $\mathscr{T}_0^{\overrightarrow{B}}$ for any Calderón-Zygmund operator T and the BMO functions $\overrightarrow{B}$, which was showed by Pérez and Trujillo-González [6].

The paper is organized as follows: in the next section we will give the proofs of Theorem 1.1 and Corollary 1.2; and in Section 3, we will prove Theorem 1.3 Corollary 1.4. In this paper we always use the letter C to denote a positive constant which is independent of the main parameters, but it may vary from line to line.

2. The proofs of Theorem 1.1 and Corollary 1.2

Before giving the proof of the theorem, we need first recall some useful notations and lemmas. The non-negative locally integrable function w belongs to the Muckenhoupt weight class, denoted by $w \in A_p$, if for any cube $Q \subset \mathbb{R}^n$,

$$\frac{1}{|Q|}\int_Q w(x)dx\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}dx\right)^{p-1} < \infty, \quad 1 < p < \infty,$$

and

$$\frac{1}{|Q|}\int_Q w(x)dx \leqslant C\inf_{x\in Q} w(x), \quad p = 1.$$

One knows that $A_p \subset A_q$ if $1 \leqslant p < q < \infty$, and that $w \in A_p$ for some $1 < p < q$ if $w \in A_q$ with $q > 1$.

LEMMA 2.1. [3] *Let $m \geqslant 1$ be integer, and A be a function on $\mathbb{R}^n$ with m-th order derivatives in $L^q(\mathbb{R}^n)$ for some $q > n$. Then*

$$|R_m(A;x,y)| \leqslant C_{m,n}|x-y|^m \sum_{|\gamma|=m} \left(\frac{1}{|\widetilde{Q}(x,y)|} \int_{\widetilde{Q}(x,y)} |D^\gamma A(z)|^q dz \right)^{\frac{1}{q}} \tag{2.1}$$

where $\widetilde{Q}(x,y)$ is the cube centered at x with edges parallel to the axes and having diameter $5\sqrt{n}|x-y|$.

LEMMA 2.2. *Let w be an A_p weight with $1 < p < \infty$, and assume $f \in B_p(w)$ and $g \in BMO(\mathbb{R}^n)$. Then there exists $1 < s < \infty$, for any cubes Q, such that*

$$\frac{1}{|Q|} \int_Q |f(x)g(x)|dx \leqslant C\|f\|_{B_p(w)} \left(\frac{1}{|Q|} \int_Q |g(x)|^s dx \right)^{1/s} \tag{2.2}$$

with the constant $C > 0$ independent of f, g and Q.

Proof. Recall that $w \in A_p$ implies $w \in A_{p-\varepsilon}$ for some small ε with $1 < p-\varepsilon < p$. We let $s = \frac{p}{\varepsilon}$ and $q = \frac{p}{p-1-\varepsilon}$, and so $\frac{1}{p}+\frac{1}{q}+\frac{1}{s} = 1$. By the Hölder inequality and the $A_{p-\varepsilon}$ weight property of w, we can deduce that

$$\begin{aligned}
\frac{1}{|Q|} \int_Q |f(x)g(x)|dx &= \frac{1}{|Q|} \int_Q |f(x)|w(x)^{\frac{1}{p}} |g(x)|w(x)^{-\frac{1}{p}} dx \\
&\leqslant \left(\frac{1}{|Q|} \int_Q |f(x)|^p w(x)dx \right)^{\frac{1}{p}} \left(\frac{1}{|Q|} \int_Q |g(x)|^s dx \right)^{\frac{1}{s}} \left(\frac{1}{|Q|} \int_Q w(x)^{-\frac{q}{p}} dx \right)^{\frac{1}{q}} \\
&\leqslant C \left(\frac{1}{w(Q)} \int_Q |f(x)|^p w(x)dx \right)^{\frac{1}{p}} \left(\frac{1}{|Q|} \int_Q |g(x)|^s dx \right)^{\frac{1}{s}} \\
&\leqslant C\|f\|_{B_p(w)} \left(\frac{1}{|Q|} \int_Q |g(x)|^s dx \right)^{\frac{1}{s}}
\end{aligned}$$

which yields the lemma.

Now we give *the proof of Theorem* 1.1. Our proof depends on a little technical. The spirit of the estimates for the operator $\mathscr{T}^{\overrightarrow{A}}$ is treated on the local parts and the nonlocal parts respectively. Without loss generality, we let $k = 2$, i.e., $\overrightarrow{A} = (A,B)$, and just consider the following multi-commutator

$$T^{A,B}(f)(x) := \int_{\mathbb{R}^n} \frac{R_{m_1+1}(A;x,y)R_{m_2+1}(B;x,y)}{|x-y|^{m_1+m_2}} K(x,y)f(y)dy$$

for the case $m_1, m_2 = 1, 2, 3, \cdots$.

It is sufficient to prove that there exists an absolute constant $C>0$ independent of f and Q, such that

$$\begin{aligned}&\frac{1}{w(Q)}\int_Q\left|T^{A,B}(f)(x)-C_0\right|w(x)dx\\&\qquad\leqslant C\left(\sum_{\substack{|\gamma|=m_1\\|\beta|=m_2}}\|D^\gamma A\|_{BMO}\|D^\beta B\|_{BMO}\right)\|f\|_{B_p(w)}\end{aligned}$$

holds for any cube $Q=Q(0,r_0)$ with $r_0\geqslant 1$, where C_0 is a real number which will be determined later.

Let $A^Q(x)=A(x)-\sum_{|\gamma|=m_1}\frac{1}{\gamma!}(D^\gamma A)_Q x^\gamma$ and $B^Q(x)=B(x)-\sum_{|\beta|=m_2}\frac{1}{\beta!}(D^\beta B)_Q x^\beta$, then it's easy to deduce that

$$R_{m_1+1}(A;x,y)=R_{m_1+1}(A^Q;x,y),\quad R_{m_2+1}(B;x,y)=R_{m_2+1}(B^Q;x,y) \tag{2.3}$$

and

$$D^\gamma A^Q=D^\gamma A-(D^\gamma A)_Q,\ \ D^\beta B^Q=D^\beta B-(D^\beta B)_Q \tag{2.4}$$

for all $|\gamma|=m_1$ and $|\beta|=m_2$, respectively. One also has

$$\begin{aligned}R_{m_1+1}(A^Q;x,y)&=R_{m_1}(A^Q;x,y)-\sum_{|\gamma|=m_1}\frac{1}{\gamma!}D^\gamma A^Q(y)(x-y)^\gamma,\\R_{m_2+1}(B^Q;x,y)&=R_{m_2}(B^Q;x,y)-\sum_{|\beta|=m_2}\frac{1}{\beta!}D^\beta B^Q(y)(x-y)^\beta.\end{aligned} \tag{2.5}$$

Fix the cube $Q=Q(0,r_0)$, let $\widetilde{Q}=10\sqrt{n}Q$ and write $f=f_1+f_2$ with $f_1=f\chi_{\widetilde{Q}}$ and $f_2=f\chi_{\mathbb{R}^n\setminus\widetilde{Q}}$, where χ_E denotes the characteristic function of set E, then we can write that $T^{A,B}(f)(x)=T^{A,B}(f_1)(x)+T^{A,B}(f_2)(x)$. Take $x_0\in\partial(2Q)$, a boundary point of $2Q$, we have

$$\begin{aligned}&\frac{1}{w(Q)}\int_Q\left|T^{A,B}(f)(x)-T^{A,B}(f_2)(x_0)\right|w(x)dx\\&\qquad\leqslant\frac{1}{w(Q)}\int_Q\left|T^{A,B}(f_1)(x)\right|w(x)dx\\&\qquad\quad+\frac{1}{w(Q)}\int_Q\left|T^{A^QB^Q}(f_2)(x)-T^{A,B}(f_2)(x_0)\right|w(x)dx\\&\qquad=:I_1+I_2\end{aligned} \tag{2.6}$$

For $1 < p_0 < p < \infty$ we let $\frac{1}{p_0} = \frac{1}{p} + \frac{1}{q}$, then by the Hölder inequality, the L^{p_0}-boundedness of $T^{A,B}$ and the properties of A_1 weight it follows that

$$\begin{aligned}
I_1 &\leqslant \frac{1}{w(Q)} \left(\int_{\mathbb{R}^n} |T^{A,B} f_1(x)|^{p_0} dx \right)^{\frac{1}{p_0}} \left(\frac{1}{|Q|} \int_Q w(x)^{p_0'} dx \right)^{\frac{1}{p_0'}} |Q|^{\frac{1}{p_0'}} \\
&\leqslant C \frac{1}{w(Q)} \left(\int_{\mathbb{R}^n} |f_1(x)|^{p_0} dx \right)^{\frac{1}{p_0}} \left(\frac{1}{|Q|} \int_Q w(x) dx \right) |Q|^{\frac{1}{p_0'}} \\
&\leqslant C \left(\frac{1}{|\widetilde{Q}|} \int_{\widetilde{Q}} |f(x)|^{p_0} dx \right)^{\frac{1}{p_0}} \\
&\leqslant C \left(\frac{1}{|\widetilde{Q}|} \int_{\widetilde{Q}} |f(x)|^{p} w(x) dx \right)^{\frac{1}{p}} \left(\frac{1}{|\widetilde{Q}|} \int_{\widetilde{Q}} w(x)^{-\frac{q}{p_0}} w(x) dx \right)^{\frac{1}{q}} \\
&\leqslant C \left(\frac{1}{w(\widetilde{Q})} \int_{\widetilde{Q}} |f(x)|^{p} w(x) dx \right)^{\frac{1}{p}} \\
&\leqslant C \| f \|_{B_{p(w)}}
\end{aligned}$$

with the constant C controlled by $\sum_{\substack{|\gamma|=m_1 \\ |\beta|=m_2}} \|D^\gamma A\|_{BMO} \|D^\beta B\|_{BMO}$.

For I_2, we will give the pointwise estimates of $T^{A,B}(f_2)(x) - T^{A,B}(f_2)(x_0)$ for $x \in Q$ and $x_0 \in \partial(2Q)$. Using the inequalities (2.3) and (2.5), we can write

$$\begin{aligned}
&T^{A,B}(f_2)(x) - T^{A,B}(f_2)(x_0) \\
= &\int_{\mathbb{R}^n} \frac{R_{m_1+1}(A^Q;x,y)}{|x-y|^{m_1}} \frac{R_{m_2+1}(B^Q;x,y)}{|x-y|^{m_2}} K(x,y) f_2(y) dy \\
&- \int_{\mathbb{R}^n} \frac{R_{m_1+1}(A^Q;x_0,y)}{|x_0-y|^{m_1}} \frac{R_{m_2+1}(B^Q;x_0,y)}{|x_0-y|^{m_2}} K(x_0,y) f_2(y) dy \\
= &\int_{\mathbb{R}^n} \left[\frac{K(x,y)}{|x-y|^{m_1+m_2}} - \frac{K(x_0,y)}{|x_0-y|^{m_1+m_2}} \right] R_{m_1}(A^Q;x,y) R_{m_2}(B^Q;x,y) f_2(y) dy \\
&+ \int_{\mathbb{R}^n} \frac{K(x_0,y)}{|x_0-y|^{m_1+m_2}} \left[R_{m_1}(A^Q;x,y) R_{m_2}(B^Q;x,y) \right. \\
&\left. - R_{m_1}(A^Q;x_0,y) R_{m_2}(B^Q;x_0,y) \right] f_2(y) dy \\
&- \sum_{|\gamma|=m_1} \frac{1}{\gamma!} \int_{\mathbb{R}^n} \left[\frac{(x-y)^\gamma K(x,y)}{|x-y|^{m_1+m_2}} - \frac{(x_0-y)^\gamma K(x_0,y)}{|x_0-y|^{m_1+m_2}} \right] \\
&\times R_{m_2}(B^Q;x,y) D^\gamma A^Q(y) f_2(y) dy \\
&+ \sum_{|\gamma|=m_1} \frac{1}{\gamma!} \int_{\mathbb{R}^n} \left[R_{m_2}(B^Q;x,y) - R_{m_2}(B^Q;x_0,y) \right] \\
&\times \frac{(x_0-y)^\gamma K(x_0,y)}{|x_0-y|^{m_1+m_2}} D^\gamma A^Q(y) f_2(y) dy
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{|\beta|=m_2}\frac{1}{\beta!}\int_{\mathbb{R}^n}\left[\frac{(x-y)^{\beta}K(x,y)}{|x-y|^{m_1+m_2}}-\frac{(x_0-y)^{\beta}K(x_0,y)}{|x_0-y|^{m_1+m_2}}\right]\\
&\times R_{m_1}(A^Q;x,y)D^{\beta}B^Q(y)f_2(y)dy\\
&+\sum_{|\beta|=m_2}\frac{1}{\beta!}\int_{\mathbb{R}^n}\left[R_{m_1}(A^Q;x,y)-R_{m_1}(A^Q;x_0,y)\right]\\
&\times\frac{(x_0-y)^{\beta}K(x_0,y)}{|x_0-y|^{m_1+m_2}}D^{\beta}B^Q(y)f_2(y)dy\\
&+\sum_{\substack{|\gamma|=m_1\\|\beta|=m_2}}\frac{1}{\gamma!\beta!}\int_{R^n}\left[\frac{(x-y)^{\gamma+\beta}K(x,y)}{|x-y|^{m_1+m_2}}-\frac{(x_0-y)^{\gamma+\beta}K(x_0,y)}{|x_0-y|^{m_1+m_2}}\right]\\
&\times D^{\gamma}A^Q(y)D^{\beta}B^Q(y)f_2(y)dy\\
=:&\ J_1+J_2+J_3+J_4+J_5+J_6+J_7
\end{aligned}
$$

Before continuing the proof, it's worthy to point out that, by Lemma 2.1, the inequality (2.4) and the following inequality

$$
|b_{Q_1}-b_{Q_2}|\leqslant C\log(|Q_2|/|Q_1|)\|b\|_{BMO}\quad\text{for any cubes } Q_1\subset Q_2,
$$

we have that, if $x\in 2Q$ and $y\in 2^k\widetilde{Q}\backslash 2^{k-1}\widetilde{Q}$, $k=1,2,\cdots$, and $m_1\geqslant 1$, then

$$
\begin{aligned}
R_{m_1}(A^Q;x,y)&\leqslant C|x-y|^{m_1}\sum_{|\gamma|=m_1}(\|D^{\gamma}A\|_{BMO}+|(D^{\gamma}A)_{\widetilde{Q}(x,y)}-(D^{\gamma}A)_Q|)\\
&\leqslant Ck|x-y|^{m_1}\sum_{|\gamma|=m_1}\|D^{\gamma}A\|_{BMO}
\end{aligned}
\tag{2.7}
$$

and similarly,

$$
R_{m_2}(B^Q;x,y)\leqslant Ck|x-y|^{m_2}\sum_{|\beta|=m_2}\|D^{\beta}B\|_{BMO}
\tag{2.8}
$$

for $x\in 2Q$ and $y\in 2^k\widetilde{Q}\backslash 2^{k-1}\widetilde{Q}$, $k=1,2,\cdots$, and $m_2\geqslant 1$.

Note that $|x-y|\approx|x_0-y|$ for $x\in Q$, $x_0\in\partial(2Q)$ and $y\in\mathbb{R}^n\backslash\widetilde{Q}$, we obtain by the conditions on the kernel K that

$$
\left|\frac{K(x,y)}{|x-y|^{m_1+m_2}}-\frac{K(x_0,y)}{|x_0-y|^{m_1+m_2}}\right|\leqslant\frac{C|x-x_0|^{\varepsilon}}{|x_0-y|^{m_1+m_2+n+\varepsilon}}
\tag{2.9}
$$

for some $\varepsilon>0$. Hence, by the inequality (2.9), and the inequalities (2.7) and (2.8), one has

$$
\begin{aligned}
J_1&\leqslant C\int_{\mathbb{R}^n}\frac{|x-x_0|^{\varepsilon}}{|x_0-y|^{m_1+m_2+n+\varepsilon}}|R_{m_1}(A^Q;x,y)||R_{m_2}(B^Q;x,y)||f_2(y)|dy\\
&\leqslant C\sum_{\substack{|\gamma|=m_1\\|\beta|=m_2}}\|D^{\gamma}A\|_{BMO}\|D^{\beta}B\|_{BMO}\sum_{k=1}^{\infty}k^2\int_{2^k\widetilde{Q}\backslash 2^{k-1}\widetilde{Q}}\frac{|x-x_0|^{\varepsilon}}{|x_0-y|^{n+\varepsilon}}|f(y)|dy
\end{aligned}
$$

Noting, by Lemma 2.2 in case $g=1$,

$$\int_{2^k\widetilde{Q}\setminus 2^{k-1}\widetilde{Q}} \frac{|x-x_0|^{\varepsilon}}{|x_0-y|^{n+\varepsilon}}|f(y)|dy \leqslant \frac{Cr_0^{\varepsilon}}{(2^k r_0)^{n+\varepsilon}}\int_{2^k\widetilde{Q}}|f(y)|dy \leqslant \frac{C}{2^{k\varepsilon}}\|f\|_{B_p(w)}$$

and thus

$$\begin{aligned} J_1 &\leqslant C\sum_{\substack{|\gamma|=m_1\\|\beta|=m_2}}\|D^{\gamma}A\|_{BMO}\|D^{\beta}B\|_{BMO}\sum_{k=1}^{\infty}\frac{k^2}{2^{k\varepsilon}}\|f\|_{B_p(w)}\\ &\leqslant C\sum_{\substack{|\gamma|=m_1\\|\beta|=m_2}}\|D^{\gamma}A\|_{BMO}\|D^{\beta}B\|_{BMO}\|f\|_{B_p(w)} \end{aligned}$$

To estimate J_2, we note

$$\begin{aligned} J_2 &\leqslant \int_{\mathbb{R}^n}\frac{K(x_0,y)}{|x_0-y|^{m_1+m_2}}f_2(y)\Bigg[\big(R_{m_1}(A^Q;x,y)-R_{m_1}(A^Q;x_0,y)\big)R_{m_2}(B^Q;x,y)\\ &\quad+(R_{m_2}(B^Q;x,y)-R_{m_2}(B^Q;x_0,y))R_{m_1}(A^Q;x_0,y)\Bigg]dy\\ &=\int_{\mathbb{R}^n}\frac{K(x_0,y)}{|x_0-y|^{m_1+m_2}}f_2(y)\left[R_{m_1}(A^Q;x,y)-R_{m_1}(A^Q;x_0,y)\right]R_{m_2}(B^Q;x,y)dy\\ &\quad+\int_{\mathbb{R}^n}\frac{K(x_0,y)}{|x_0-y|^{m_1+m_2}}f_2(y)\left[R_{m_2}(B^Q;x,y)-R_{m_2}(B^Q;x_0,y)\right]R_{m_1}(A^Q;x_0,y)dy\\ &=: J_{21}+J_{22}. \end{aligned}$$

We will need the following inequality

$$R_m(F;x,y)-R_m(F;x_0,y)=R_m(F;x,x_0)+\sum_{0<|\beta|<m}\frac{1}{\beta!}R_{m-|\beta|}(D^{\beta}F;x_0,y)(x-x_0)^{\beta} \tag{2.10}$$

for any function F and the integer $m\geqslant 1$. In fact, for any integer $m\geqslant 1$,

$$\begin{aligned} R_m(F;x,y) &= F(x)-\sum_{|\gamma|<m}\frac{1}{\gamma!}D^{\gamma}F(y)(x-y)^{\gamma}\\ &=F(x)-\sum_{|\gamma|<m}\frac{1}{\gamma!}D^{\gamma}F(y)\sum_{\alpha+\beta=\gamma}\frac{\gamma!}{\alpha!\beta!}(x_0-y)^{\alpha}(x-x_0)^{\beta}\\ &=F(x)-\sum_{|\beta|<m}\frac{1}{\beta!}(x-x_0)^{\beta}\sum_{|\alpha|<m-|\beta|}\frac{1}{\alpha!}D^{\alpha}(D^{\beta}F)(y)(x_0-y)^{\alpha}\\ &=F(x)-\sum_{|\beta|<m}\frac{1}{\beta!}\left[D^{\beta}F(x_0)-R_{m-|\beta|}(D^{\beta}F;x_0,y)\right](x-x_0)^{\beta}\\ &=R_m(F;x,x_0)+\sum_{|\beta|<m}\frac{1}{\beta!}R_{m-|\beta|}(D^{\beta}F;x_0,y)(x-x_0)^{\beta}, \end{aligned}$$

which yields the equality (2.10). Using equality (2.10), Lemma 2.1 and the inequality (2.7) we have that if $x \in Q$, $x_0 \in \partial(2Q)$ and $y \in 2^k\widetilde{Q}\backslash 2^{k-1}\widetilde{Q}$, $k = 1,2,\cdots$, and the integer $m_1 \geqslant 1$, then

$$\begin{aligned}
&|R_{m_1}(A^Q;x,y) - R_{m_1}(A^Q;x_0,y)| \\
&\leqslant |R_{m_1}(A^Q;x,x_0)| + \sum_{0<|\beta|<m_1} \frac{1}{\beta!}|R_{m_1-|\beta|}(D^\beta A^Q;x_0,y)(x-x_0)^\beta| \\
&\leqslant C\sum_{|\gamma|=m_1} |x-x_0|^{m_1}\|D^\gamma A\|_{BMO} \\
&\quad + Ck\sum_{0<|\beta|<m_1}\sum_{|\gamma|=m_1} |x-x_0|^{|\beta|}|x_0-y|^{m_1-|\beta|}\|D^\gamma A\|_{BMO} \\
&\leqslant Ck|x-x_0|\sum_{|\gamma|=m_1} |x_0-y|^{m_1-1}\|D^\gamma A\|_{BMO}.
\end{aligned}$$

This and the inequality (2.8), and the fact $|x-y| \sim |x_0-y|$ for $x \in Q$, $x_0 \in \partial(2Q)$ and $y \notin \widetilde{Q}$, and Lemma 2.2 imply that

$$\begin{aligned}
J_{21} &\leqslant C\sum_{|\beta|=m_2}\|D^\beta B\|_{BMO}\sum_{k=1}^{\infty} k\int_{2^k\widetilde{Q}\backslash 2^{k-1}\widetilde{Q}} \frac{|K(x_0,y)|}{|x_0-y|^{m_1}}|f_2(y)| \\
&\qquad \times \left|R_{m_1}(A^Q;x,y) - R_{m_1}(A^Q;x_0,y)\right| dy \\
&\leqslant C\sum_{|\beta|=m_2}\|D^\beta B\|_{BMO}\sum_{k=1}^{\infty} k^2\int_{2^k\widetilde{Q}\backslash 2^{k-1}\widetilde{Q}} \sum_{|\gamma|=m_1}\frac{|x-x_0|}{|x_0-y|^{n+1}}\|D^\gamma A\|_{BMO}|f_2(y)|dy \\
&\leqslant C\sum_{|\beta|=m_2}\|D^\beta B\|_{BMO}\sum_{|\gamma|=m_1}\|D^\gamma A\|_{BMO}\sum_{k=1}^{\infty}\frac{k^2}{2^k}\frac{1}{|2^k\widetilde{Q}|}\int_{2^k\widetilde{Q}}|f_2(y)|dy \\
&\leqslant C\sum_{|\gamma|=m_1}\|D^\gamma A\|_{BMO}\sum_{|\beta|=m_2}\|D^\beta B\|_{BMO}\|f\|_{B_p(w)}.
\end{aligned}$$

Similarly one has the same estimates for J_{22} and so

$$J_2 \leqslant |J_{21}| + |J_{22}| \leqslant C\sum_{|\gamma|=m_1}\|D^\gamma A\|_{BMO}\sum_{|\beta|=m_2}\|D^\beta B\|_{BMO}\|f\|_{B_p(w)}.$$

For J_3, by the conditions on the kernel k and the fact that $|x-y| \approx |x_0-y|$ for $x \in Q$, $x_0 \in \partial(2Q)$ and $y \in \mathbb{R}^n\backslash\widetilde{Q}$, we have for $|\gamma| = m_1$,

$$\left|\frac{(x-y)^\gamma K(x,y)}{|x-y|^{m_1+m_2}} - \frac{(x_0-y)^\gamma K(x_0,y)}{|x_0-y|^{m_1+m_2}}\right| \leqslant \frac{C|x-x_0|^\varepsilon}{|x_0-y|^{m_2+n+\varepsilon}}$$

for some $\varepsilon > 0$. This together with the inequality (2.8) and Lemma 2.2 gives that

$$\begin{aligned}
J_3 &\leqslant C\sum_{|\gamma|=m_1}\int_{\mathbb{R}^n}\frac{|x-x_0|^\varepsilon}{|x_0-y|^{m_2+n+\varepsilon}}|R_{m_2}(B^Q;x,y)||D^\gamma A^Q(y)||f_2(y)|dy \\
&\leqslant C\sum_{|\gamma|=m_1}\sum_{|\beta|=m_2}\|D^\beta B\|_{BMO}\sum_{k=1}^{\infty} k\int_{2^k\widetilde{Q}\backslash 2^{k-1}\widetilde{Q}}\frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon}}|D^\gamma A^Q(y)||f(y)|dy
\end{aligned}$$

$$\leqslant C\sum_{|\gamma|=m_1}\sum_{|\beta|=m_2}\|D^\beta B\|_{BMO}\sum_{k=1}^{\infty}\frac{k}{2^{k\varepsilon}}\frac{1}{|2^k\widetilde{Q}|}\int_{2^k\widetilde{Q}}|D^\gamma A^Q(y)||f(y)|dy$$

$$\leqslant C\sum_{|\gamma|=m_1}\sum_{|\beta|=m_2}\|D^\beta B\|_{BMO}\|f\|_{B_p(w)}\sum_{k=1}^{\infty}\frac{k}{2^{k\varepsilon}}\left(\frac{1}{|2^k\widetilde{Q}|}\int_{2^k\widetilde{Q}}|D^\gamma A^Q(y)|^s dy\right)^{\frac{1}{s}}$$

$$\leqslant C\sum_{|\gamma|=m_1}\sum_{|\beta|=m_2}\|D^\beta B\|_{BMO}\|f\|_{B_p(w)}\sum_{k=1}^{\infty}\frac{k}{2^{k\varepsilon}}\left(\|D^\gamma A\|_{BMO}+|(D^\gamma A)_{2^k\widetilde{Q}}-(D^\gamma A)_Q|\right)$$

$$\leqslant C\sum_{|\gamma|=m_1}\sum_{|\beta|=m_2}\|D^\beta B\|_{BMO}\|f\|_{B_p(w)}\sum_{k=1}^{\infty}\frac{k^2}{2^{k\varepsilon}}\|D^\gamma A\|_{BMO}$$

$$\leqslant C\sum_{|\gamma|=m_1}\|D^\gamma A\|_{BMO}\sum_{|\beta|=m_2}\|D^\beta B\|_{BMO}\|f\|_{B_p(w)}.$$

Using the inequality (2.10) and size condition on K with the similar argument as J_{21}, and using Lemma 2.2, we obtain that

$$J_4\leqslant C\sum_{|\beta|=m_2}\|D^\beta B\|_{BMO}\sum_{|\gamma|=m_1}\sum_{k=1}^{\infty}k\int_{2^k\widetilde{Q}\setminus 2^{k-1}\widetilde{Q}}\frac{|x-x_0|}{|x_0-y|^{n+1}}|D^\gamma A^Q(y)||f(y)|dy$$

$$\leqslant C\sum_{|\gamma|=m_1}\sum_{|\beta|=m_2}\|D^\beta B\|_{BMO}\sum_{k=1}^{\infty}\frac{k}{2^k}\frac{1}{|2^k\widetilde{Q}|}\int_{2^k\widetilde{Q}}|D^\gamma A^Q(y)||f(y)|dy$$

$$\leqslant C\sum_{|\gamma|=m_1}\sum_{|\beta|=m_2}\|D^\beta B\|_{BMO}\sum_{k=1}^{\infty}\frac{k}{2^k}\|f\|_{B_p(w)}\left(\frac{1}{|2^k\widetilde{Q}|}\int_{2^k\widetilde{Q}}|D^\gamma A^Q(y)|^s dy\right)^{1/s}$$

$$\leqslant C\sum_{|\gamma|=m_1}\|D^\gamma A\|_{BMO}\sum_{|\beta|=m_2}\|D^\beta B\|_{BMO}\|f\|_{B_p(w)}$$

In a similar way to J_3 and J_4, one has

$$J_5+J_6\leqslant C\sum_{|\gamma|=m_1}\|D^\gamma A\|_{BMO}\sum_{|\beta|=m_2}\|D^\beta B\|_{BMO}\|f\|_{B_p(w)}$$

At last for J_7, we choose $1<s_1,s_2<\infty$ satisfying $\frac{1}{s_1}+\frac{1}{s_2}=\frac{1}{s}$, where s appears in Lemma 2.2, then the conditions on kernel K, Lemma 2.2 and the Hölder inequality follow that

$$J_7\leqslant C\sum_{|\gamma|=m_1}\sum_{|\beta|=m_2}\sum_{k=1}^{\infty}\int_{2^k\widetilde{Q}\setminus 2^{k-1}\widetilde{Q}}\frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon}}|D^\gamma A^Q(y)||D^\beta B^Q(y)||f_2(y)|dy$$

$$\leqslant C\sum_{|\gamma|=m_1}\sum_{|\beta|=m_2}\sum_{k=1}^{\infty}\frac{1}{2^{k\varepsilon}}\frac{1}{|2^k\widetilde{Q}|}\int_{2^k\widetilde{Q}}|D^\gamma A^Q(y)||D^\beta B^Q(y)||f(y)|dy$$

$$\leqslant C\sum_{|\gamma|=m_1}\sum_{|\beta|=m_2}\sum_{k=1}^{\infty}\frac{1}{2^{k\varepsilon}}\left(\frac{1}{|2^k\widetilde{Q}|}\int_{2^k\widetilde{Q}}|D^\gamma A(y)-(D^\gamma A)_{\widetilde{Q}}|^{s_1}dy\right)^{\frac{1}{s_1}}$$

$$\times\left(\frac{1}{|2^k\widetilde{Q}|}\int_{2^k\widetilde{Q}}|D^\beta B(y)-(D^\beta B)_{\widetilde{Q}}|^{s_2}dy\right)^{\frac{1}{s_2}}\|f\|_{B_p(w)}$$
$$\leqslant C\sum_{|\gamma|=m_1}\sum_{|\beta|=m_2}\sum_{k=1}^{\infty}\frac{k^2}{2^{k\varepsilon}}\|D^\gamma A\|_{BMO}\|D^\beta B\|_{BMO}\|f\|_{B_p(w)}$$
$$\leqslant C\sum_{|\gamma|=m_1}\|D^\gamma A\|_{BMO}\sum_{|\beta|=m_2}\|D^\beta B\|_{BMO}\|f\|_{B_p(w)}$$

Combining the estimates of J_i, $i=1,2,\cdots,7$, then we get

$$I_2\leqslant\frac{1}{w(Q)}\int_Q\sum_{i=1}^{7}|J_i|w(x)dx\leqslant C\sum_{|\gamma|=m_1}\|D^\gamma A\|_{BMO}\sum_{|\beta|=m_2}\|D^\beta B\|_{BMO}\|f\|_{B_p(w)}$$

Moreover, we have proved that

$$\frac{1}{w(Q)}\int_Q\left|T^{A,B}(f)(x)-T^{A,B}(f_2)(x_0)\right|w(x)dx\leqslant|I_1|+|I_2|$$
$$\leqslant C\sum_{|\gamma|=m_1}\|D^\gamma A\|_{BMO}\sum_{|\beta|=m_2}\|D^\beta B\|_{BMO}\|f\|_{B_p(w)}$$

with the constant $C>0$ independent of f and Q, which implies the desired inequality (1.6) in Theorem 1.1. The inequality (1.7) can be deduced by the same arguments above. The proof of the theorem is complete. □

The proof Corollary 1.2. Carefully repeating the proof above, we actually obtain that

$$\frac{1}{|Q|}\int_Q|T^{\vec{A}}f(x)-T^{\vec{A}}f(x_0)|dx\leqslant C\prod_{j=1}^{k}\sum_{|\gamma|=m_j}\|D^\gamma A_j\|_{BMO}\|f\|_{L^\infty(\mathbb{R}^n)}$$

for any cube $Q\subset\mathbb{R}^n$ and some point $x_0\in\partial(2Q)$. This implies the desired inequality (1.8) of the corollary.

Moreover, using the inequality (1.8) and the interpolation theorem, we can easily see the L^p-boundedness of the multi-commutators $\mathscr{T}^{\vec{A}}$ and the inequality (1.9) for $p_0\leqslant p<\infty$. □

3. The proofs of Theorem 1.3 and Corollary 1.4

The following lemma will be used in this section.

LEMMA 3.1. *If $w\in A_p$, $1\leqslant p<\infty$, then for $F\in BMO$ and $1<s<\infty$ we have*

$$\left(\frac{1}{w(Q)}\int_Q|F(x)-F_Q|^s w(x)dx\right)^{1/s}\leqslant C\|F\|_{BMO}$$

with the constant C independent of the cube Q.

Proof. We note that the A_p weight w satisfies the reverse Hölder inequality,

$$\left(\frac{1}{|Q|}\int_Q w(x)^t dx\right)^{1/t} \leqslant \frac{C}{|Q|}\int_Q w(x)dx$$

with some $1<t<\infty$ and for all cube Q. From this and the Hölder inequality, we can see that, for any function F, $1<s<\infty$ and any cube Q,

$$\begin{aligned}\left(\frac{1}{w(Q)}\int_Q |F(x)-F_Q|^s w(x)dx\right)^{1/s} &\leqslant C\left(\frac{1}{|Q|}\int_Q |F(x)-F_Q|^{st'}dx\right)^{1/(st')}\\ &\leqslant C\|F\|_{BMO}\end{aligned}$$

with the constant $C>0$ independent of F and Q, where $\frac{1}{t}+\frac{1}{t'}=1$. □

We are ready to give *the proof of Theorem* 1.3. For $l=1$, the multi-commutator $\mathbb{T}^{\vec{A},\vec{B}}$ reduces to the following form

$$\mathbb{T}(f)(x)=\int_{\mathbb{R}^n} K(x,y)(B(x)-B(y))\prod_{j=1}^{k}\frac{R_{m_j+1}(A_j;x,y)}{|x-y|^{m_j}}f(y)dy=[B,\mathscr{T}^{\vec{A}}](f)(x)$$

where $\mathscr{T}^{\vec{A}}(f)$ is the multi-commutator of form (1.4) for all $m_j\geqslant 1$. For any fixed $r\geqslant 1$, denote $Q(0,r)$ simply by Q, and $\widetilde{Q}=10\sqrt{n}Q$. We write

$$f(x)=f(x)\chi_{\widetilde{Q}}+f(x)\left(1-\chi_{\widetilde{Q}}\right)=: f_1(x)+f_2(x).$$

Recall the notation $B^Q(x)=B(x)-B_Q$ and take a point $x_0\in\partial(2Q)$, we have $\mathbb{T}(f)(x)=B^Q(x)\mathscr{T}^{\vec{A}}(f)(x)-\mathscr{T}^{\vec{A}}(B^Q f)(x)$, and by the boundedness for $\mathscr{T}^{\vec{A}}$ we get

$$\begin{aligned}&\frac{1}{w(Q)}\int_Q |\mathbb{T}(f)(x)-\mathscr{T}^{\vec{A}}(B^Q f_2)(x_0)|w(x)dx\\ &\leqslant \frac{1}{w(Q)}\int_Q |B^Q(x)\mathscr{T}^{\vec{A}}(f)(x)|w(x)dx+\frac{1}{w(Q)}\int_Q |\mathscr{T}^{\vec{A}}(B^Q f_1)(x)|w(x)dx\\ &\quad+\frac{1}{w(Q)}\int_Q |\mathscr{T}^{\vec{A}}(B^Q f_2)(x)-\mathscr{T}^{\vec{A}}(B^Q f_2)(x_0)|w(x)dx\\ &=: V_1+V_2+V_3\end{aligned}$$

For $1<p_0<p<\infty$, then applying the Hölder inequality, the L^{p_0}-boundedness of $\mathscr{T}^{\vec{A}}$, and the properties of A_1 weight, we get from Lemma 3.1 that

$$V_1+V_2\leqslant C\|B\|_{BMO}\left(\|\mathscr{T}^{\vec{A}}(f)\|_{B_p(w)}+\prod_{j=1}^{k}\sum_{|\gamma|=m_j}\|D^\gamma A_j\|_{BMO}\|f\|_{B_p(w)}\right).$$

Repeating the proof of Theorem 1.1, and noting by Lemma 2.2 that for any locally integrable function g,

$$\begin{aligned}
&\frac{1}{|2^k\widetilde{Q}|}\int_{2^k\widetilde{Q}}|g(x)B^Q(x)f_2(x)|dx\\
&\leqslant C\left(\frac{1}{|2^k\widetilde{Q}|}\int_{2^k\widetilde{Q}}|g(x)B^Q(x)|^s dx\right)^{\frac{1}{s}}\|f\|_{B_p(w)} \qquad (3.1)\\
&\leqslant Ck\left(\frac{1}{|2^k\widetilde{Q}|}\int_{2^k\widetilde{Q}}|g(x)|^{2s} dx\right)^{\frac{1}{2s}}\|B\|_{BMO}\|f\|_{B_p(w)}
\end{aligned}$$

with some $1<s<\infty$, we thus use the same argument in the estimates of I_2 in last section to obtain that

$$|\mathscr{T}^{\overrightarrow{A}}(B^Qf_2)(x)-\mathscr{T}^{\overrightarrow{A}}(B^Qf_2)(x_0)|\leqslant C\prod_{j=1}^{k}\sum_{|\gamma|=m_j}\|D^\gamma A_j\|_{BMO}\|B\|_{BMO}\|f\|_{B_p(w)}$$

which implies

$$V_3\leqslant C\prod_{j=1}^{k}\sum_{|\gamma|=m_j}\|D^\gamma A_j\|_{BMO}\|B\|_{BMO}\|f\|_{B_p(w)} \qquad (3.2)$$

Hence we have deduced that

$$\begin{aligned}
&\frac{1}{w(Q)}\int_Q|\mathbb{T}(f)(x)-\mathscr{T}^{\overrightarrow{A}}(B^Qf_2)(x_0)|w(x)dx\\
&\leqslant C\|B\|_{BMO}\left(\|\mathscr{T}^{\overrightarrow{A}}(f)\|_{B_p(w)}+\prod_{j=1}^{k}\sum_{|\gamma|=m_j}\|D^\gamma A_j\|_{BMO}\|f\|_{B_p(w)}\right) \qquad (3.3)
\end{aligned}$$

which implies the desired inequality (1.11) when $l=1$.

Next we consider the case that $l>1$ and all $m_j\geqslant 1$. Without loss of generality, by induction principle we may only consider the case $l=2$, i.e.

$$\mathbb{T}^{\overrightarrow{A},\overrightarrow{B}}(f)(x)=\int_{\mathbb{R}^n}K(x,y)\prod_{j=1}^{k}\frac{R_{m_j+1}(A_j;x,y)}{|x-y|^{m_j}}\prod_{i=1}^{2}(B_i(x)-B_i(y))f(y)dy$$

Since $B_i(x)-B_i(y)=B_i^Q(x)-B_i^Q(y)$, we can decompose $\mathbb{T}^{\overrightarrow{A},\overrightarrow{B}}$ into four parts as follows:

$$\begin{aligned}
\mathbb{T}^{\overrightarrow{A},\overrightarrow{B}}(f)(x)&=-B_1^Q(x)B_2^Q(x)\mathscr{T}^{\overrightarrow{A}}(f)(x)+B_2^Q(x)\,[B_1,\mathscr{T}^{\overrightarrow{A}}](f)(x)\\
&\quad+B_1^Q(x)\,[B_2,\mathscr{T}^{\overrightarrow{A}}](f)(x)+\mathscr{T}^{\overrightarrow{A}}\left(B_1^QB_2^Qf\right)(x) \qquad (3.4)\\
&=:\mathbb{T}_1f(x)+\mathbb{T}_2f(x)+\mathbb{T}_3f(x)+\mathbb{T}_4f(x).
\end{aligned}$$

Letting $\frac{1}{p}+\frac{1}{q}+\frac{1}{t}=1$ for some $1<q,t<\infty$ and the assumption $1<p<\infty$, and applying the Hölder inequality and Lemma 3.1, we obtain that

$$\begin{aligned}\frac{1}{w(Q)}\int_Q|\mathbb{T}_1(f)(x)|\,w(x)dx &\leqslant \left(\frac{1}{w(Q)}\int_Q|B_1(x)-B_{1_Q}|^q wdx\right)^{\frac{1}{q}}\\ &\times\left(\frac{1}{w(Q)}\int_Q|B_2(x)-B_{2_Q}|^t wdx\right)^{\frac{1}{t}}\left(\frac{1}{w(Q)}\int_Q|\mathscr{T}^{\vec{A}}(f)(x)|^p wdx\right)^{\frac{1}{p}}\\ &\leqslant C\|B_1\|_{BMO}\|B_2\|_{BMO}\|\mathscr{T}^{\vec{A}}f\|_{B_p(w)}.\end{aligned}\tag{3.5}$$

and similarly,

$$\begin{aligned}&\frac{1}{w(Q)}\int_Q|\mathbb{T}_2f(x)+\mathbb{T}_3f(x)|\,w(x)dx\\ &\leqslant C\|B_1\|_{BMO}\|[B_2,\mathscr{T}^{\vec{A}}]f\|_{B_p(w)}+C\|B_2\|_{BMO}\|[B_1,\mathscr{T}^{\vec{A}}]f\|_{B_p(w)}\end{aligned}\tag{3.6}$$

Take $\frac{1}{p}+\frac{1}{q}+\frac{1}{t}=\frac{1}{p_0}$ for some $1<q,t<\infty$, then we have

$$\begin{aligned}\frac{1}{w(Q)}\int_Q|\mathbb{T}_4f_1(x)|\,w(x)dx &\leqslant \left(\frac{1}{w(Q)}\int_Q\left|\mathscr{T}^{\vec{A}}\left(B_1^QB_2^Qf_1\right)(x)\right|^{p_0}wdx\right)^{\frac{1}{p_0}}\\ &\leqslant C\left(\frac{1}{w(Q)}\int_{\widetilde{Q}}\left|B_1^QB_2^Qf_1(x)\right|^{p_0}wdx\right)^{\frac{1}{p_0}}\\ &\leqslant C\|B_1\|_{BMO}\|B_2\|_{BMO}\|f\|_{B_p(w)}.\end{aligned}\tag{3.7}$$

Choose a point $x_0\in\partial(2Q)$ and recall the proof of the estimates (3.2), we have

$$\begin{aligned}|T_4(f_2)(x)-\mathbb{T}_4(f_2)(x_0)| &= \left|\mathscr{T}^{\vec{A}}\left(B_1^QB_2^Qf_2\right)(x)-\mathscr{T}^{\vec{A}}\left(B_1^QB_2^Qf_2\right)(x_0)\right|\\ &\leqslant C\prod_{j=1}^k\sum_{|\gamma|=m_j}\|D^\gamma A_j\|_{BMO}\|B_1\|_{BMO}\|B_2\|_{BMO}\|f\|_{B_p(w)}\end{aligned}\tag{3.8}$$

Combing the inequalities (3.4), (3.5), (3.6), (3.7) and (3.8), we gain that

$$\begin{aligned}&\frac{1}{w(Q)}\int_Q\left|\mathbb{T}^{\vec{A},\vec{B}}(f)(x)-\mathbb{T}_4(f_2)(x_0)\right|w(x)dx\\ &\leqslant C\|B_1\|_{BMO}\|B_2\|_{BMO}\prod_{j=1}^k\sum_{|\gamma|=m_j}\|D^\gamma A_j\|_{BMO}\|f\|_{B_p(w)}\\ &\quad+C\|B_1\|_{BMO}\|[B_2,\mathscr{T}^{\vec{A}}]f\|_{B_p(w)}+C\|B_2\|_{BMO}\|[B_1,\mathscr{T}^{\vec{A}}]f\|_{B_p(w)}\\ &\quad+C\|B_1\|_{BMO}\|B_2\|_{BMO}\|\mathscr{T}^{\vec{A}}f\|_{B_p(w)}\end{aligned}\tag{3.9}$$

This and the induction principle give the inequality (1.11). The proof of Theorem 1.3 is complete.

We now turn to *the proof of Corollary* 1.4. If we check the proof of Theorem 1.3, we in fact get for any $x \in \mathbb{R}^n$ that

$$\begin{aligned} M^{\sharp}\left(\mathbb{T}^{\vec{A},\vec{B}}(f)\right)(x) \leqslant C \prod_{j=1}^{k} \sum_{|\gamma|=m_j} \|D^{\gamma} A_j\|_{BMO} \prod_{i=1}^{l} \|B_i\|_{BMO} M(f)(x) \\ + C \prod_{i=1}^{l} \|B_i\|_{BMO} M\left(\mathscr{T}^{\vec{A}}(f)\right)(x) \\ + C \sum_{i=1}^{l-1} \sum_{\sigma \in C_i^l} \prod_{j \in \sigma'} \|B_j\|_{BMO} M\left(\mathbb{T}^{\vec{A},\vec{B_\sigma}}(f)\right)(x) \end{aligned} \tag{3.10}$$

Applying the Stein-Fefferman's inequality

$$\|f\|_{L^p(\mathbb{R}^n)} \leqslant \|M(f)\|_{L^p(\mathbb{R}^n)} \leqslant C \|M^{\sharp}(f)\|_{L^p(\mathbb{R}^n)} \tag{3.11}$$

and the well-known inequality $\|M(f)\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{L^p(\mathbb{R}^n)}$ whenever $1 < p < \infty$, where $M(f)$ denotes the Hardy-Littlewood maximal function of f, we can then deduce that

$$\begin{aligned} \left\|\mathbb{T}^{\vec{A},\vec{B}}(f)\right\|_{L^p(\mathbb{R}^n)} &\leqslant C \left\| M^{\sharp}\left(\mathbb{T}^{\vec{A},\vec{B}}(f)\right)\right\|_{L^p(\mathbb{R}^n)} \\ &\leqslant C \prod_{j=1}^{k} \sum_{|\gamma|=m_j} \|D^{\gamma} A_j\|_{BMO} \prod_{i=1}^{l} \|B_i\|_{BMO} \|M(f)\|_{L^p(\mathbb{R}^n)} \\ &\quad + C \prod_{i=1}^{l} \|B_i\|_{BMO} \left\| M\left(\mathscr{T}^{\vec{A}}(f)\right)\right\|_{L^p(\mathbb{R}^n)} \\ &\quad + C \sum_{i=1}^{l-1} \sum_{\sigma \in C_i^l} \prod_{j \in \sigma'} \|B_j\|_{BMO} \left\| M\left(\mathbb{T}^{\vec{A},\vec{B_\sigma}}(f)\right)\right\|_{L^p(\mathbb{R}^n)} \\ &\leqslant C \prod_{j=1}^{k} \sum_{|\gamma|=m_j} \|D^{\gamma} A_j\|_{BMO} \prod_{i=1}^{l} \|B_i\|_{BMO} \|f\|_{L^p(\mathbb{R}^n)} \\ &\quad + C \sum_{i=1}^{l-1} \sum_{\sigma \in C_i^l} \prod_{j \in \sigma'} \|B_j\|_{BMO} \left\|\mathbb{T}^{\vec{A},\vec{B_\sigma}}(f)\right\|_{L^p(\mathbb{R}^n)} \end{aligned}$$

where in the last inequality we have used the L^p-boundedness for the operator $\mathscr{T}^{\vec{A}}$ for $p_0 \leqslant p < \infty$ by Corollary 1.2. Finally, we make use of induction on l, we can derive the L^p-boundedness for the operator $\mathbb{T}^{\vec{A},\vec{B}}$ and the inequality (1.12). The proof is finished. □

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