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WILKER AND HUYGENS TYPE INEQUALITIES FOR THE LEMNISCATE FUNCTIONS

CHAO-PING CHEN

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Abstract. In this paper, we establish Wilker and Huygens type inequalities for the Lemniscate functions.

1. Introduction

The lemniscate, also called the lemniscate of Bernoulli, is the locus of points (x,y) in the plane satisfying the equation $(x^2+y^2)^2 = x^2+y^2$. In polar coordinates (r,θ), the equation becomes $r^2 = \cos(2\theta)$ and its arc length is given by the function

$$\operatorname{arcsl} x = \int_0^x \frac{\mathrm{d}\,t}{\sqrt{1-t^4}}, \qquad |x| \leqslant 1, \tag{1}$$

where $\operatorname{arcsl} x$ is called the arc lemniscate sine function studied by C.F. Gauss in 1797–1798. Another lemniscate function investigated by Gauss is the hyperbolic arc lemniscate sine function, defined as

$$\operatorname{arcslh} x = \int_0^x \frac{\mathrm{d}\,t}{\sqrt{1+t^4}}, \qquad x \in \mathbb{R}. \tag{2}$$

Functions (1) and (2) can be found (see [2, p. 259], [3, (2.5)–(2.6)], [10, 11] and [16, Ch. 1]).

Another pair of lemniscate functions, the arc lemniscate tangent arctl and the hyperbolic arc lemniscate tangent arctlh, have been introduced in [10, (3.1)–(3.2)]. Therein it has been proven that

$$\operatorname{arctl} x = \operatorname{arcsl}\left(\frac{x}{\sqrt[4]{1+x^4}}\right), \qquad x \in \mathbb{R} \tag{3}$$

and

$$\operatorname{arctlh} x = \operatorname{arcslh}\left(\frac{x}{\sqrt[4]{1-x^4}}\right), \qquad |x| < 1 \tag{4}$$

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(see [10, Prop. 3.1]). It is worth mentioning that all four lemniscate functions can be expressed in terms of the completely symmetric elliptic integral of the first kind

$$R_F(x,y,z)=\frac{1}{2}\int_0^{\infty}[(t+x)(t+y)(t+z)]^{-1/2}\,\mathrm{d}t,$$

where at most one of the nonnegative variables $x;y;z$ is 0 (see [4, (9.2-1)].

Wilker in [18] proposed two open problems:

(a) Prove that if $0<x<\pi/2$, then

$$\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}>2\,. \tag{5}$$

(b) Find the largest constant c such that

$$\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}>2+cx^3\tan x\,.$$

for $0<x<\pi/2$.

In [17], inequality (5) was proved, and the following inequality

$$2+\left(\frac{2}{\pi}\right)^4x^3\tan x<\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}<2+\frac{8}{45}x^3\tan x \qquad \text{for } 0<x<\frac{\pi}{2}, \tag{6}$$

where the constants $\left(\frac{2}{\pi}\right)^4$ and $\frac{8}{45}$ are best possible, was also established.

Wilker type inequalities (5) and (6) have attracted much interest of many mathematicians and have motivated a large number of research papers involving different proofs, various generalizations and improvements (cf. [6, 9, 12, 13, 14, 15, 17, 19, 20, 21, 22, 23, 24, 27, 28, 29] and the references cited therein). The inequality (5) is now known as the first Wilker inequality in the literature [13].

A related inequality which is of interest to us is Huygens inequality [7], which asserts that

$$2\left(\frac{\sin x}{x}\right)+\frac{\tan x}{x}>3 \qquad \text{for all} \quad 0<|x|<\frac{\pi}{2}\,. \tag{7}$$

In [26], Zhu established some new inequalities of the Huygens type for trigonometric and hyperbolic functions. Baricz and Sándor [1] pointed out that inequalities (5) and (7) are simple consequences of the arithmetic-geometric mean inequality together with the well-known Lazarević-type inequality [8, p. 238]

$$(\cos x)^{1/3}<\frac{\sin x}{x} \qquad \text{for all} \quad 0<|x|<\frac{\pi}{2}\,,$$

or equivalently,

$$\left(\frac{\sin x}{x}\right)^2\frac{\tan x}{x}>1 \qquad \text{for all} \quad 0<|x|<\frac{\pi}{2}\,. \tag{8}$$

Wu and Srivastava [19, Lemma 3] established another inequality

$$\left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}>2 \qquad \text{for all} \quad 0<|x|<\frac{\pi}{2}\,, \tag{9}$$

which is now known as the second Wilker inequality in the literature [13].

In [5], Chen and Cheung showed that the first Wilker inequality (5), Huygens inequality (7), Lazarević-type inequality (8) and the second Wilker inequality (9) can be grouped into the following inequality chain:

$$\begin{aligned}\frac{(\sin x/x)^2+\tan x/x}{2} &> \frac{2(\sin x/x)+\tan x/x}{3} > \sqrt[3]{\left(\frac{\sin x}{x}\right)^2\frac{\tan x}{x}} > 1 \\ &> \frac{2}{1/(\sin x/x)^2+1/(\tan x/x)}\,, \quad 0<|x|<\frac{\pi}{2}\,, \end{aligned} \tag{10}$$

in terms of the arithmetic, geometric and harmonic means.

Recently, Zhu [25] established a hyperbolic version of the first Wilker inequality

$$\left(\frac{\sinh x}{x}\right)^2+\frac{\tanh x}{x}>2\,, \qquad x\neq 0\,. \tag{11}$$

Neuman and Sándor [13] gave generalizations and extensions of the inequalities (5)–(9) to the case of hyperbolic functions. Chen and Cheung [5] showed the following inequality chain:

$$\begin{aligned}\frac{(\sinh x/x)^2+\tanh x/x}{2} &> \frac{2(\sinh x/x)+\tanh x/x}{3} > \sqrt[3]{\left(\frac{\sinh x}{x}\right)^2\frac{\tanh x}{x}} > 1 \\ &> \frac{2}{1/(\sinh x/x)^2+1/(\tanh x/x)}, \qquad x\neq 0\,, \end{aligned} \tag{12}$$

in terms of the arithmetic, geometric and harmonic means.

Very recently, Chen and Cheung [5] established Wilker and Huygens type inequalities for inverse trigonometric and inverse hyperbolic functions.

In this paper, we establish Wilker and Huygens type inequalities for the lemniscate functions.

2. Lemmas and Propositions

It is known that the binomial coefficients

$$\binom{a}{n}=\frac{a(a-1)\cdots(a-n+1)}{n!}=\frac{\Gamma(1+a)}{n!\cdot\Gamma(1+a-n)}=\frac{(-1)^n\Gamma(n-a)}{n!\cdot\Gamma(-a)},$$

where Γ denotes the gamma function.

LEMMA 1. (i) *For* $|x|<1$,

$$\operatorname{arcsl} x=\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}(4n+1)\cdot n!}x^{4n+1}. \tag{13}$$

(ii) *Let* $p\geqslant 0$ *be an integer. Then for* $0<x<1$,

$$\sum_{k=0}^{2p-1}(-1)^k u_k(x)<\operatorname{arctl} x<\sum_{k=0}^{2p}(-1)^k u_k(x), \tag{14}$$

where

$$u_k(x)=\frac{\Gamma(k+\frac{3}{4})}{\Gamma(\frac{3}{4})\cdot(4k+1)\cdot k!}x^{4k+1}.$$

Proof. We note that $1/\sqrt{1-t^4}$ can be expressed in series form as follows:

$$\frac{1}{\sqrt{1-t^4}}=\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-1)^n t^{4n}=\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\cdot n!}t^{4n},\quad |t|<1.$$

Consequently, for $|x|<1$,

$$\operatorname{arcsl} x=\int_0^x\frac{1}{\sqrt{1-t^4}}\,\mathrm{d}t=\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}(4n+1)\cdot n!}x^{4n+1}.$$

Elementary calculations reveal that for $|x|<1$,

$$\begin{aligned}(\operatorname{arctl} x)'&=\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x/\sqrt[4]{1+x^4}}\frac{1}{\sqrt{1-t^4}}\,\mathrm{d}t=\frac{1}{(1+x^4)^{3/4}}\\&=\sum_{n=0}^{\infty}\binom{-\frac{3}{4}}{n}x^{4n}=\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(n+\frac{3}{4})}{\Gamma(\frac{3}{4})\cdot n!}x^{4n}.\end{aligned}$$

Consequently, for $|x|<1$,

$$\operatorname{arctl} x=\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(n+\frac{3}{4})}{\Gamma(\frac{3}{4})\cdot(4n+1)\cdot n!}x^{4n+1}=\sum_{n=0}^{\infty}(-1)^n u_n(x), \tag{15}$$

where

$$u_n(x)=\frac{\Gamma(n+\frac{3}{4})}{\Gamma(\frac{3}{4})\cdot(4n+1)\cdot n!}x^{4n+1}.$$

We find that, for $0 < x < 1$ and $m \geqslant 0$,

$$u_{2m}(x) - u_{2m+1}(x) = \left(\frac{1}{8m+1} - \frac{8m+3}{4(8m+5)(2m+1)}x^4\right)\frac{\Gamma(2m+\frac{3}{4})}{\Gamma(\frac{3}{4})\cdot(2m)!}x^{8m+1} > 0,$$

since

$$\frac{1}{8m+1} > \frac{8m+3}{4(8m+5)(2m+1)} > \frac{8m+3}{4(8m+5)(2m+1)}x^4$$

for $0 < x < 1$ and $m \geqslant 0$. Hence, it follows that for $0 < x < 1$ and $p \geqslant 1$,

$$\operatorname{arctl} x = \big(u_0(x) - u_1(x)\big) + \big(u_2(x) - u_3(x)\big) + \cdots > \sum_{k=0}^{2p-1}(-1)^k u_k(x).$$

We find that, for $0 < x < 1$ and $m \geqslant 1$,

$$u_{2m-1}(x) - u_{2m}(x) = \left(\frac{1}{8m-3} - \frac{8m-1}{8m(8m+1)}x^4\right)\frac{\Gamma(2m-\frac{1}{4})}{\Gamma(\frac{3}{4})\cdot(2m-1)!}x^{8m-3} > 0,$$

since

$$\frac{1}{8m-3} > \frac{8m-1}{8m(8m+1)} > \frac{8m-1}{8m(8m+1)}x^4$$

for $0 < x < 1$ and $m \geqslant 1$. Hence, it follows that for $0 < x < 1$ and $p \geqslant 0$,

$$\operatorname{arctl} x = u_0(x) - \big(u_1(x) - u_2(x)\big) - \big(u_3(x) - u_4(x)\big) - \cdots < \sum_{k=0}^{2p}(-1)^k u_k(x).$$

This completes the proof of Lemma 1. $\square$

PROPOSITION 1. *For $0 < |x| < 1$, we have*

$$\left(\frac{x}{\operatorname{arcsl} x}\right)^2 + \frac{x}{\operatorname{arctl} x} < 2. \tag{16}$$

Proof. It follows from (13) that for $0 < |x| < 1$,

$$\begin{aligned}\left(\frac{\operatorname{arcsl} x}{x}\right)^2 &= \left(1 + \frac{1}{10}x^4 + \frac{1}{24}x^8 + \frac{5}{208}x^{12} + \cdots\right)^2\\ &= 1 + \frac{1}{5}x^4 + \frac{7}{75}x^8 + \frac{11}{195}x^{12} + \cdots\\ &> 1 + \frac{1}{5}x^4 + \frac{7}{75}x^8.\end{aligned} \tag{17}$$

It follows from (14) that for $0 < |x| < 1$,

$$1 - \frac{3}{20}x^4 < \frac{\operatorname{arctl} x}{x} < 1. \tag{18}$$

By using inequalities (17) and (18), we find for $0 < |x| < 1$,

$$\begin{aligned}\left(\frac{x}{\operatorname{arcsl} x}\right)^2 + \frac{x}{\operatorname{arctl} x} - 2 &< \frac{1}{1+\frac{1}{5}x^4+\frac{7}{75}x^8} + \frac{1}{1-\frac{3}{20}x^4} - 2\\ &= \frac{x^4(-75-50x^4+42x^8)}{(75+15x^4+7x^8)(20-3x^4)} < 0,\end{aligned}$$

since

$$-75 - 50t + 42t^2 < 0 \qquad \text{for} \quad 0 < t < 1.$$

The proof is complete. □

LEMMA 2. (i) *Let $p \geqslant 0$ be an integer. Then for $0 < x < 1$,*

$$\sum_{k=0}^{2p-1} (-1)^k v_k(x) < \operatorname{arcslh} x < \sum_{k=0}^{2p} (-1)^k v_k(x), \tag{19}$$

where

$$v_k(x) = \frac{\Gamma(k+\frac{1}{2})}{\sqrt{\pi}(4k+1)\cdot n!} x^{4k+1}.$$

(ii) *For $|x| < 1$, we have*

$$\operatorname{arctlh} x = \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{3}{4})}{\Gamma(\frac{3}{4})\cdot(4n+1)\cdot n!} x^{4n+1}. \tag{20}$$

Proof. We note that $1/\sqrt{1+t^4}$ can be expressed in series form as follows:

$$\frac{1}{\sqrt{1+t^4}} = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\cdot n!} t^{4n}, \quad |t| < 1.$$

Consequently, for $|x| < 1$,

$$\operatorname{arcslh} x = \int_0^x \frac{1}{\sqrt{1+t^4}}\,\mathrm{d}t = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}(4n+1)\cdot n!} x^{4n+1} = \sum_{n=0}^{\infty} (-1)^n v_n(x),$$

where

$$v_n(x) = \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}(4n+1)\cdot n!} x^{4n+1}.$$

We find that, for $0<x<1$ and $m\geqslant 0$,

$$v_{2m}(x)-v_{2m+1}(x)=\left(\frac{1}{8m+1}-\frac{4m+1}{2(8m+5)(2m+1)}x^4\right)\frac{\Gamma(2m+\frac{1}{2})}{\sqrt{\pi}\cdot(2m)!}x^{8m+1}>0,$$

since

$$\frac{1}{8m+1}>\frac{4m+1}{2(8m+5)(2m+1)}>\frac{4m+1}{2(8m+5)(2m+1)}x^4$$

for $0<x<1$ and $m\geqslant 0$. Hence, it follows that for $0<x<1$ and $p\geqslant 1$,

$$\operatorname{arcslh} x=\big(v_0(x)-v_1(x)\big)+\big(v_2(x)-v_3(x)\big)+\cdots>\sum_{k=0}^{2p-1}(-1)^k v_k(x).$$

We find that, for $0<x<1$ and $m\geqslant 1$,

$$v_{2m-1}(x)-v_{2m}(x)=\left(\frac{1}{8m-3}-\frac{4m-1}{4m(8m+1)}x^4\right)\frac{\Gamma(2m-\frac{1}{2})}{\sqrt{\pi}\cdot(2m-1)!}x^{8m-3}>0,$$

since

$$\frac{1}{8m-3}>\frac{4m-1}{4m(8m+1)}>\frac{4m-1}{4m(8m+1)}x^4$$

for $0<x<1$ and $m\geqslant 1$. Hence, it follows that for $0<x<1$ and $p\geqslant 0$,

$$\operatorname{arcslh} x=v_0(x)-\big(v_1(x)-v_2(x)\big)-\big(v_3(x)-v_4(x)\big)-\cdots<\sum_{k=0}^{2p}(-1)^k v_k(x).$$

Elementary calculations reveal that for $|x|<1$,

$$(\operatorname{arctlh} x)'=\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x/\sqrt[4]{1-x^4}}\frac{1}{\sqrt{1+t^4}}\,\mathrm{d}t=\frac{1}{(1-x^4)^{3/4}}=\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{3}{4})}{\Gamma(\frac{3}{4})\cdot n!}x^{4n}.$$

Consequently, for $|x|<1$,

$$\operatorname{arctlh} x=\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{3}{4})}{\Gamma(\frac{3}{4})\cdot(4n+1)\cdot n!}x^{4n+1}.$$

This completes the proof of Lemma 2. □

PROPOSITION 2. *For* $0 < |x| < 1$, *we have*

$$\frac{x}{\operatorname{arcslh} x} + \left(\frac{x}{\operatorname{arctlh} x}\right)^2 < 2. \tag{21}$$

Proof. It follows from (19) for $0 < x < 1$,

$$1 - \frac{1}{10}x^4 < \frac{\operatorname{arcslh} x}{x} < 1 - \frac{1}{10}x^4 + \frac{1}{24}x^8. \tag{22}$$

It follows from (20) for $0 < x < 1$,

$$\begin{aligned}\left(\frac{\operatorname{arctlh} x}{x}\right)^2 &= \left(1 + \frac{3}{20}x^4 + \frac{7}{96}x^8 + \cdots\right)^2 = 1 + \frac{3}{10}x^4 + \frac{101}{600}x^8 + \cdots \\ &> 1 + \frac{3}{10}x^4. \end{aligned} \tag{23}$$

By using inequalities (22) and (23), we find for $0 < |x| < 1$,

$$\begin{aligned}\frac{x}{\operatorname{arcslh} x} + \left(\frac{x}{\operatorname{arctlh} x}\right)^2 - 2 &< \frac{1}{1 - \frac{1}{10}x^4} + \frac{1}{1 + \frac{3}{10}x^4} - 2 \\ &= -\frac{2x^4(10 - 3x^4)}{(10 - x^4)(10 + 3x^4)} < 0. \end{aligned}$$

The proof is complete. □

3. Main results

THEOREM 1. *For* $0 < |x| < 1$, *we have*

$$\left(\frac{\operatorname{arcsl} x}{x}\right)^2 + \frac{\operatorname{arctl} x}{x} > 2 \tag{24}$$

and

$$2\left(\frac{\operatorname{arcsl} x}{x}\right) + \frac{\operatorname{arctl} x}{x} > 3. \tag{25}$$

Proof. Inequality (16) can be rewritten as

$$\frac{2}{1/(\operatorname{arcsl} x/x)^2 + 1/(\operatorname{arctl} x/x)} > 1, \qquad 0 < |x| < 1,$$

that is to say, the harmonic mean of $\left(\frac{\operatorname{arcsl} x}{x}\right)^2$ and $\frac{\operatorname{arctl} x}{x}$ is greater than 1. By using the arithmetic–geometric–harmonic mean inequality, we get, for $0 < |x| < 1$,

$$\begin{aligned}\frac{(\operatorname{arcsl} x/x)^2 + \operatorname{arctl} x/x}{2} &> \sqrt{\left(\frac{\operatorname{arcsl} x}{x}\right)^2 \left(\frac{\operatorname{arctl} x}{x}\right)} \\ &> \frac{2}{1/(\operatorname{arcsl} x/x)^2 + 1/(\operatorname{arctl} x/x)} > 1, \end{aligned} \tag{26}$$

and

$$\frac{2\,(\operatorname{arcsl}x/x)+\operatorname{arctl}x/x}{3}>\sqrt[3]{\left(\frac{\operatorname{arcsl}x}{x}\right)^2\left(\frac{\operatorname{arctl}x}{x}\right)}>1\,. \quad \square \tag{27}$$

REMARK 1. For $0<|x|<1$, we have, by (26),

$$\begin{aligned}
&\frac{(\operatorname{arcsl}x/x)^2+\operatorname{arctl}x/x}{2}-\frac{2\,(\operatorname{arcsl}x/x)+\operatorname{arctl}x/x}{3}\\
&=\frac{1}{6}\left[2\left(\frac{\operatorname{arcsl}x}{x}\right)^2+\left(\frac{\operatorname{arcsl}x}{x}\right)^2+\frac{\operatorname{arctl}x}{x}-4\left(\frac{\operatorname{arcsl}x}{x}\right)\right]\\
&>\frac{1}{6}\left[2\left(\frac{\operatorname{arcsl}x}{x}\right)^2+2-4\left(\frac{\operatorname{arcsl}x}{x}\right)\right]\\
&=\frac{1}{3}\left(1-\frac{\operatorname{arcsl}x}{x}\right)^2>0,
\end{aligned}$$

which shows that inequality (25) is sharper than inequality (24).

Theorem 2 below establishes a sharp result of inequality (24), which presents an analogue of the first inequality in (6).

THEOREM 2. *For* $0<|x|<1$*, we have*

$$2+\frac{1}{20}x^3\operatorname{arctl}x<\left(\frac{\operatorname{arcsl}x}{x}\right)^2+\frac{\operatorname{arctl}x}{x}. \tag{28}$$

The constant $\frac{1}{20}$ *is best possible.*

Proof. By (17) and (18), we have for $0<|x|<1$,

$$\left(\frac{\operatorname{arcsl}x}{x}\right)^2+\frac{\operatorname{arctl}x}{x}-2-\frac{1}{20}x^3\operatorname{arctl}x>\frac{1}{20}x^3(x-\operatorname{arctl}x)+\frac{7}{75}x^8>0.$$

Elementary calculations reveal that

$$\lim_{x\to 0^+}\frac{\left(\frac{\operatorname{arcsl}x}{x}\right)^2+\frac{\operatorname{arctl}x}{x}-2}{x^3\operatorname{arctl}x}=\frac{1}{20}.$$

Hence, inequality (28) holds with best possible constant $\frac{1}{20}$. $\square$

There is no strict comparison between the representation $\dfrac{(\operatorname{arcslh}x/x)^2+\operatorname{arctlh}x/x}{2}$ and constant 1. Now we ask: Can the arithmetic mean of $\operatorname{arcslh}x/x$ and $(\operatorname{arctlh}x/x)^2$ be compared with constant 1? Theorem 3 gives an affirmative answer.

THEOREM 3. *For* $0<|x|<1$, *we have*

$$\frac{\operatorname{arcslh} x}{x}+\left(\frac{\operatorname{arctlh} x}{x}\right)^2>2 \tag{29}$$

and

$$\frac{\operatorname{arcslh} x}{x}+2\left(\frac{\operatorname{arctlh} x}{x}\right)>3. \tag{30}$$

Proof. Inequality (21) can be rewritten as

$$\frac{2}{1/(\operatorname{arcslh} x/x)+1/(\operatorname{arctlh} x/x)^2}>1\,, \qquad 0<|x|<1\,,$$

that is to say, the harmonic mean of $\frac{\operatorname{arcslh} x}{x}$ and $\left(\frac{\operatorname{arctlh} x}{x}\right)^2$ is greater than 1. By using the arithmetic–geometric–harmonic mean inequality, we get, for $0<|x|<1$,

$$\begin{aligned}\frac{\operatorname{arcslh} x/x+(\operatorname{arctlh} x/x)^2}{2}&>\sqrt{\left(\frac{\operatorname{arcslh} x}{x}\right)\left(\frac{\operatorname{arctlh} x}{x}\right)^2}\\&>\frac{2}{1/(\operatorname{arcslh} x/x)+1/(\operatorname{arctlh} x/x)^2}>1\,,\end{aligned} \tag{31}$$

and

$$\frac{\operatorname{arcslh} x/x+2(\operatorname{arctlh} x/x)}{3}>\sqrt[3]{\left(\frac{\operatorname{arcslh} x}{x}\right)\left(\frac{\operatorname{arctlh} x}{x}\right)^2}>1\,. \qquad \square \tag{32}$$

REMARK 2. For $0<|x|<1$, we have, by (31),

$$\begin{aligned}&\frac{\operatorname{arcslh} x/x+(\operatorname{arctlh} x/x)^2}{2}-\frac{\operatorname{arcslh} x/x+2(\operatorname{arctlh} x/x)}{3}\\&=\frac{1}{6}\left[\frac{\operatorname{arcslh} x}{x}+\left(\frac{\operatorname{arctlh} x}{x}\right)^2+2\left(\frac{\operatorname{arctlh} x}{x}\right)^2-4\left(\frac{\operatorname{arctlh} x}{x}\right)\right]\\&>\frac{1}{6}\left[2+2\left(\frac{\operatorname{arctlh} x}{x}\right)^2-4\left(\frac{\operatorname{arctlh} x}{x}\right)\right]\\&=\frac{1}{3}\left(1-\frac{\operatorname{arctlh} x}{x}\right)^2>0,\end{aligned}$$

which shows that inequality (30) is sharper than inequality (29).

Finally, we propose the following conjecture.

CONJECTURE 1. *For* $0 < |x| < 1$*, we have*

$$2+\frac{1}{5}x^3 \operatorname{arctlh} x < \frac{\operatorname{arcslh} x}{x} + \left(\frac{\operatorname{arctlh} x}{x}\right)^2. \tag{33}$$

The constant $\frac{1}{5}$ *is best possible.*

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Chao-Ping Chen
School of Mathematics and Informatics
Henan Polytechnic University
Jiaozuo City 454003
Henan Province
People's Republic of China
e-mail: chenchaoping@sohu.com