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MORE INEQUALITIES FOR POSITIVE LINEAR MAPS

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Abstract. We derive inequalities for the norm of the variance of matrices. It is shown that unital linear maps on 2×2 matrices preserve the commutativity properties of matrices. This feature allows us to generalize several inequalities for such maps. We show by way of an example that this technique cannot be extended to the case of $n\times n$, $n\geqslant 3$, matrices.

1. Introduction

Let $M(n)$ be the C^*-algebra of all $n\times n$ complex matrices and let $\Phi : M(n)\to M(k)$ be a positive linear map. For an element A of $M(n)$ the variance of $\Phi(A)$ is defined as [1, pp. 74]

$$\operatorname{Var}(\Phi(A)) = \Phi(A^*A) - \Phi(A)^*\Phi(A). \tag{1.1}$$

The $\operatorname{Var}(\Phi(A))$ will be abbreviated by $\operatorname{Var}A$, whenever there is no danger of confusion. Kadison's inequality [2] says that if A is Hermitian, then $\operatorname{Var}A\geqslant 0$. The complementary Bhatia-Davis [3] inequality asserts that for $m\leqslant A\leqslant M$,

$$0\leqslant \operatorname{Var}A = \Phi\left(A^2\right) - \Phi(A)^2 \leqslant (MI-\Phi(A))(\Phi(A)-mI)\leqslant \left(\frac{M-m}{2}\right)^2 I. \tag{1.2}$$

In this paper we assume that "$\leqslant$" denotes the Loewner order relation among Hermitian matrices. If A is normal, Choi's generalization [4] says that $\operatorname{Var}A\geqslant 0$. Bhatia and Sharma [5] have proved

$$\operatorname{Var}(\Phi(A))\leqslant \triangle(A)^2 I, \tag{1.3}$$

for all positive unital maps Φ. Here

$$\triangle(A) = \inf_{z\in\mathbb{C}} \|A-zI\| \tag{1.4}$$

is the distance of A from scalar matrices zI, $z\in\mathbb{C}$. In the special case when A is normal, $\triangle(A)=r_A$, where r_A is the radius of the smallest disc containing the spectrum of A. The inequality (1.3) implies several norm estimates for positive linear maps between finite dimensional C^*-algebras, see Theorems 2.1–2.2, below.

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The second inequality (1.2) can be written as

$$\Phi\left(A^{2}\right) \leqslant (m+M)\,\Phi(A) - mMI. \tag{1.5}$$

A more generalized inequality says that if f is a convex function on $[m,M]$, then [1, pp. 56]

$$\Phi(f(A)) \leqslant L(\Phi(A)), \tag{1.6}$$

where L is a linear interpolant

$$L(t) = \frac{1}{M-m}\left\{(t-m)\,f(M) + (M-t)\,f(m)\right\}. \tag{1.7}$$

The upper bound for $\Phi(A^{r})$ in terms of $\Phi(A)^{r}$ follows as a special case of (1.6), $r = 2,3,\ldots$. But the lower bounds such as $\Phi\left(A^{3}\right) \geqslant \Phi(A)^{3}$ and $\Phi\left(A^{4}\right) \geqslant \Phi(A)^{4}$ are not always true. For instance, the well known Choi's example shows that $\Phi\left(A^{4}\right) \ngeqslant \Phi(A)^{4}$, for Φ the compression map projecting the 3×3 matrix onto its leading 2×2 submatrix and $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. For a reference see [4, pp. 568]. Bhatia and Sharma [5] have shown that in fact such inequalities are true for maps $\Phi : M(2) \to M(k)$, [5]

$$f(\Phi(A)) \leqslant \Phi(f(A)), \tag{1.8}$$

where f is a convex function on an open interval containing the eigenvalues of a Hermitian element A of $M(2)$. In this paper we show that a similar type of situation arises in other standard inequalities. The crucial point here is that the unital linear maps of two commuting matrices in $M(2)$ are commuting (Lemma 2.3, below). We use this Lemma and derive in the subsequent theorems the several inequalities for such maps. In this way $M(2)$ is the exceptional case which is non commutative, but still the same strong properties as for the commutative case $M(1)$ holds. For example, Bourin and Ricard [6] have recently proved a non-commutative version of the Chebychev's inequality which says that for a positive definite matrix A

$$\left|\Phi(A^{r})\,\Phi(A^{s})\right| \leqslant \Phi\left(A^{r+s}\right), \tag{1.9}$$

where $0 \leqslant r \leqslant s$, $\left|\Phi(A^{r})\,\Phi(A^{s})\right| = \left(\Phi(A^{s})\,\Phi(A^{r})^{2}\,\Phi(A^{s})\right)^{\frac{1}{2}}$ and A^{r}, $r \in R$, is meant in the sense of spectral calculus. We show that for maps on $M(2)$ the lower bound in (1.9) can be replaced by $\Phi(A^{s})\,\Phi(A^{r})$, and also prove a complementary bound (Theorem 2.7 , below). An example of a linear map on $M(3)$ is given to show that the complementary inequality is not always true (Remark 2.8 and Example 2.9, below).

2. Main results

THEOREM 2.1. *Let $\Phi : M(n) \to M(k)$ be a positive unital linear map. Then*

$$\|VarA\| = \left\|\Phi(A^*A) - \Phi(A)^*\Phi(A)\right\| \leqslant \triangle(A)^2 \tag{2.1}$$

for all $A \in M(n)$.

Proof. For $0 \leqslant C \leqslant kI$ we have $\|C\| \leqslant k$. So, (1.3) implies (2.1) when A is normal. If A is arbitrary (not necessarily normal) then $\mathrm{Var}A \geqslant 0$ provided Φ is a 2-positive map and (2.1) follows at once from (1.3). If Φ is just positive then we know that $\|\Phi\| = 1$ and [5]

$$\Phi(A^*A) - \Phi(A)^*\Phi(A) \leqslant \|A\|^2 I. \tag{2.2}$$

For any operator X we have $X^*X \leqslant \|X\|^2 I$. Therefore,

$$\Phi(A)^*\Phi(A) \leqslant \|\Phi(A)\|^2 I \leqslant \|A\|^2 I. \tag{2.3}$$

So,

$$\Phi(A)^*\Phi(A) - \Phi(A^*A) \leqslant \|\Phi(A)\|^2 I \leqslant \|A\|^2 I. \tag{2.4}$$

If X is Hermitian and $\pm X \leqslant kI$ then $\|X\| \leqslant k$. We find from (2.2) and (2.4) that

$$\left\|\Phi(A^*A) - \Phi(A)^*\Phi(A)\right\| \leqslant \|A\|^2. \tag{2.5}$$

On replacing A by $A - zI$ in (2.5) we conclude that

$$\left\|\Phi(A^*A) - \Phi(A)^*\Phi(A)\right\| \leqslant \inf_{z\in\mathbb{C}} \|A - zI\|^2 = \triangle(A)^2. \quad \square \tag{2.6}$$

THEOREM 2.2. *Let the singular values $s_i(A)$ of the variance of a matrix $A \in M(n)$ be arranged in the decreasing order. Then, for $k = 1, 2, \ldots, n$*

$$\prod_{i=1}^{k} s_i(A) \leqslant \triangle(A)^{2k} \tag{2.7}$$

and

$$\sum_{i=1}^{k} s_i(A) \leqslant k\triangle(A)^2. \tag{2.8}$$

Let $|||\cdot|||$ denote a unitarily invariant norm on $M(n)$. Then for any matrix A in $M(n)$

$$|||VarA||| \leqslant \triangle(A)^2\, |||I|||. \tag{2.9}$$

Proof. Let X and Y be Hermitian with respective singular values $s_i(X)$ and $s_i(Y)$, $i = 1, 2, \ldots, n$. We know if $\pm X \leqslant Y$ then $\{s_i(X)\}$ is weak-log majorized by $\{s_i(Y)\}$. The inequality (2.7) then follows from (2.2) and (2.4). The weak-log majorization implies majorization, [7]. So (2.8) follows from (2.7). Also if (2.8) holds then by Fan dominance principle (2.9) also holds, [8, Lemma 2.1]. $\square$

When $n = 2$ the situation is rather simple and more special results can be derived. Many of them depend on the preservation of commutativity proved in the following lemma.

LEMMA 2.3. *Let $\Phi : M(2) \to M(k)$ be a linear and unital map. Let A and B be Hermitian matrices in $M(2)$. Then $AB = BA$ implies $\Phi(A)\Phi(B) = \Phi(B)\Phi(A)$.*

Proof. Let λ_i and μ_i be the eigenvalues of A and B respectively, $i = 1,2$. By the spectral theorem,

$$A = \lambda_1 P_1 + \lambda_2 P_2, \tag{2.10}$$

$$B = \mu_1 P_1 + \mu_2 P_2, \tag{2.11}$$

P_1 and P_2 are corresponding projections with

$$P_1 + P_2 = I. \tag{2.12}$$

Apply Φ to (2.12)

$$\Phi(P_1) + \Phi(P_2) = I. \tag{2.13}$$

Therefore, $\Phi(P_1)$ and $\Phi(P_2) = I - \Phi(P_1)$ commute. On applying Φ to (2.10) and (2.11), we respectively get

$$\Phi(A) = \lambda_1 \Phi(P_1) + \lambda_2 \Phi(P_2) \tag{2.14}$$

and

$$\Phi(B) = \mu_1 \Phi(P_1) + \mu_2 \Phi(P_2). \tag{2.15}$$

A simple computation shows that $\Phi(A)$ and $\Phi(B)$ commute. $\square$

The Lemma 2.3 shows that unital linear maps on 2×2 matrices preserve commutativity property of matrices. We use this fact to derive several inequalities for such maps. This technique however cannot be extended to the case of $n \times n$, $n \geqslant 3$, matrices. For example, if $A = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$, $B = A^2$ and Φ is a compression map that takes A to its top left 2×2 submatrix, then $AB = BA$ but $\Phi(A)\Phi(B) \neq \Phi(B)\Phi(A)$.

Let A and B be Hermitian with spectral resolution (2.10) and (2.11), respectively. Then, we say that the spectra of A and B are similarly ordered if $\lambda_1 \leqslant \lambda_2$ and $\mu_1 \leqslant \mu_2$ or $\lambda_1 \geqslant \lambda_2$ and $\mu_1 \geqslant \mu_2$. The spectra are oppositely ordered if $\lambda_1 \leqslant \lambda_2$ and $\mu_1 \geqslant \mu_2$ or $\lambda_1 \geqslant \lambda_2$ and $\mu_1 \leqslant \mu_2$. We now prove Chebyshev's inequality for 2×2 matrices.

THEOREM 2.4. *Let $\Phi : M(2) \to M(k)$ be a positive unital linear map. Let A and B be commuting Hermitian matrices in $M(2)$. If the spectra of A and B are similarly ordered. Then*

$$\Phi(AB) \geqslant \Phi(A)\Phi(B). \tag{2.16}$$

If the spectra A and B are oppositely ordered. Then

$$\Phi(AB) \leqslant \Phi(A)\Phi(B). \tag{2.17}$$

Proof. Let A and B be commuting Hermitian matrices with eigenvalues λ_i, $i = 1,2$ and μ_i, $i = 1,2$ respectively. The spectral theorem implies

$$\Phi(AB) = \lambda_1 \mu_1 \Phi(P_1) + \lambda_2 \mu_2 \Phi(P_2). \tag{2.18}$$

Since $\Phi(P_1)$ and $\Phi(P_2)$ commute, we find from (2.14), (2.15) and (2.18) that

$$\Phi(AB) - \Phi(A)\Phi(B) = (\lambda_2 - \lambda_1)(\mu_2 - \mu_1)\Phi(P_1)\Phi(P_2). \tag{2.19}$$

From (2.19) we conclude that the inequality (2.16) holds when $(\lambda_2 - \lambda_1)(\mu_2 - \mu_1) \geqslant 0$ and reverses its order when $(\lambda_2 - \lambda_1)(\mu_2 - \mu_1) \leqslant 0$. □

COROLLARY 2.5. *Let A be a positive definite matrix and let Φ be as in Theorem 2.4. If r and s are real numbers such that $rs > 0$, then*

$$\Phi\left(A^{r+s}\right) \geqslant \Phi(A^r)\Phi(A^s). \tag{2.20}$$

If $rs < 0$, the reverse inequality holds. Also,

$$\Phi(A\log A) \geqslant \Phi(A)\Phi(\log A) \tag{2.21}$$

and

$$\Phi\left(A^{-1}\log A\right) \leqslant \Phi\left(A^{-1}\right)\Phi(\log A). \tag{2.22}$$

Proof. It is clear that the spectra of A^r and A^s are similarly ordered when $rs > 0$, and oppositely ordered when $rs < 0$. Therefore, the assertions for inequality (2.20) follow from Theorem 2.4. Since A and $\log A$ have similarly ordered spectra, (2.21) follows from (2.16). Likewise, (2.17) implies (2.22). □

COROLLARY 2.6.. *Let the spectra of commuting positive definite matrices A_1, $A_2, \ldots, A_n$ be similarly ordered. Then, for a positive unital linear map $\Phi : M(2) \to M(k)$*

$$\Phi\left(\prod_{i=1}^{n} A_i\right) \geqslant \prod_{i=1}^{n} \Phi(A_i). \tag{2.23}$$

Proof. The proof follows by the principle of the mathematical induction. By Theorem 2.4, the inequality (2.23) holds good for $n = 2$. Assume that (2.23) is true for $n = k$. It is clear that the matrices $A_1A_2\ldots A_k$ and A_{k+1} are similarly ordered positive definite matrices. Therefore,

$$\Phi\left(\prod_{i=1}^{k+1} A_i\right) \geqslant \Phi\left(\prod_{i=1}^{k} A_i\right)\Phi(A_{k+1}) \geqslant \prod_{i=1}^{k+1} \Phi(A_i).$$

This proves the inequality (2.23). □

Let A be a positive definite matrix in $M(2)$. Let Φ be a positive linear map. Then, by Kadison inequality (1.1), we have

$$\Phi\left(A^{\frac{r+s}{2}}\right)^2 \leqslant \Phi\left(A^{r+s}\right). \tag{2.24}$$

We show that this inequality can be strengthened for positive unital linear maps $\Phi : M(2) \to M(k)$.

THEOREM 2.7. *For a positive unital linear map $\Phi : M(2) \to M(k)$ and positive definite matrix A in $M(2)$, we have*

$$\Phi\left(A^{\frac{r+s}{2}}\right)^2 \leqslant \Phi(A^r)\Phi(A^s) \leqslant \Phi\left(A^{r+s}\right). \tag{2.25}$$

Proof. The proof follows by the spectral calculus for Hermitian operators. Since $A \in M(2)$ is positive definite, the first inequality in (2.25) is equivalent to

$$\left(\lambda_1^{\frac{r+s}{2}}\Phi(P_1)+\lambda_2^{\frac{r+s}{2}}\Phi(P_2)\right)^2 \leqslant (\lambda_1^r\Phi(P_1)+\lambda_2^r\Phi(P_2))(\lambda_1^s\Phi(P_1)+\lambda_2^s\Phi(P_2)). \tag{2.26}$$

Since $\Phi(P_1)$ and $\Phi(P_2)$ commute, the inequality (2.26) holds if and only if

$$\left(\lambda_1^{\frac{r}{2}}\lambda_2^{\frac{s}{2}} - \lambda_1^{\frac{s}{2}}\lambda_2^{\frac{r}{2}}\right)^2\Phi(P_1)\Phi(P_2) \geqslant 0.$$

This proves the first inequality in (2.25). The second inequality in (2.25) follows from Corollary 2.5. □

REMARK 2.8. As mentioned earlier, Bourin and Ricard [6] have proved that for $\Phi : M(n) \to M(k)$ and $0 \leqslant r \leqslant s$,

$$|\Phi(A^r)\Phi(A^s)| \leqslant \Phi\left(A^{r+s}\right). \tag{2.27}$$

We give an example to show that

$$\Phi\left(A^{\frac{r+s}{2}}\right)^2 \nleqslant |\Phi(A^r)\Phi(A^s)|. \tag{2.28}$$

EXAMPLE 2.9. Let

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

and

$$\Phi(A) = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \ \Phi\left(A^2\right) = \begin{bmatrix} 2 & -3 \\ -3 & 6 \end{bmatrix}, \ \Phi\left(A^3\right) = \begin{bmatrix} 5 & -9 \\ -9 & 18 \end{bmatrix}.$$

Therefore,

$$\Phi\left(A^2\right)^2 = \begin{bmatrix} 13 & -24 \\ -24 & 45 \end{bmatrix}$$

and

$$\left|\Phi(A)\Phi\left(A^3\right)\right| = \begin{bmatrix} 725 & -1413 \\ -1413 & 2754 \end{bmatrix}^{\frac{1}{2}} = \begin{bmatrix} 12.42185 & -23.894302 \\ -23.894302 & 4.723253 \end{bmatrix}.$$

We note that $(1,1)$ entry of $|\Phi(A)\Phi\left(A^3\right)| - \Phi\left(A^2\right)^2$ is negative. So in general it is not true that $\left|\Phi(A)\Phi\left(A^3\right)\right| \geqslant \Phi\left(A^2\right)^2$.

It is well known that the Grüss inequality is a complementary inequality to Chebyshev's inequality. We prove another complementary bound for Chebyshev's inequality which contains the Kantorovich inequality as one of its special case.

THEOREM 2.10. *Let $\Phi : M(2) \to M(k)$ be a positive unital linear map. Let A and B be two commuting positive definite matrices with respective eigenvalues λ_i and μ_i, $i = 1,2$. Then, for $\lambda_1 \leqslant \lambda_2$ and $\mu_1 \leqslant \mu_2$*

$$\left(\frac{\sqrt{\lambda_1\mu_2}+\sqrt{\lambda_2\mu_1}}{\sqrt{\lambda_1\mu_1}+\sqrt{\lambda_2\mu_2}}\right)^2 \Phi(A)\Phi(B) \leqslant \Phi(AB) \leqslant \left(\frac{\sqrt{\lambda_1\mu_1}+\sqrt{\lambda_2\mu_2}}{\sqrt{\lambda_1\mu_2}+\sqrt{\lambda_2\mu_1}}\right)^2 \Phi(A)\Phi(B). \tag{2.29}$$

Proof. From (2.14), (2.15) and (2.18), we find that

$$\Phi(AB) - \Phi(A)\Phi(B) = \frac{\mu_2-\mu_1}{\lambda_2-\lambda_1}(\lambda_2 - \Phi(A))(\Phi(A)-\lambda_1) \tag{2.30}$$

and

$$\Phi(B) = \frac{\mu_2-\mu_1}{\lambda_2-\lambda_1}\Phi(A) + \frac{\mu_1\lambda_2-\lambda_1\mu_2}{\lambda_2-\lambda_1}. \tag{2.31}$$

Combine (2.30) and (2.31), we get

$$\Phi(AB) = \frac{\mu_2-\mu_1}{\lambda_2-\lambda_1}(\lambda_2-\Phi(A))(\Phi(A)-\lambda_1) + \frac{\Phi(A)}{\lambda_2-\lambda_1}((\mu_2-\mu_1)\Phi(A)+\mu_1\lambda_2-\lambda_1\mu_2). \tag{2.32}$$

So the second inequality in (2.29) holds if and only if

$$\begin{aligned}&(\lambda_2-\Phi(A))(\Phi(A)-\lambda_1) + \Phi(A)\left[\Phi(A)+\frac{\mu_1\lambda_2-\lambda_1\mu_2}{\mu_2-\mu_1}\right]\\ &\leqslant \alpha^2\Phi(A)\left(\Phi(A)+\frac{\mu_1\lambda_2-\lambda_1\mu_2}{\mu_2-\mu_1}\right)\end{aligned} \tag{2.33}$$

where

$$\alpha = \frac{\sqrt{\lambda_1\mu_1}+\sqrt{\lambda_2\mu_2}}{\sqrt{\lambda_1\mu_2}+\sqrt{\lambda_2\mu_1}}. \tag{2.34}$$

On simplification we see that (2.33) holds if and only if

$$\Phi(A)^2 + \left[\left(1-\frac{1}{\alpha^2}\right)\frac{\mu_1\lambda_2-\lambda_1\mu_2}{\mu_2-\mu_1} - \frac{\lambda_1+\lambda_2}{\alpha^2}\right]\Phi(A) + \frac{\lambda_1\lambda_2}{\alpha^2} \geqslant 0. \tag{2.35}$$

Also,

$$\frac{\lambda_1+\lambda_2}{\alpha^2} - \left(1-\frac{1}{\alpha^2}\right)\frac{\mu_1\lambda_2-\lambda_1\mu_2}{\mu_2-\mu_1} = 2\frac{\sqrt{\lambda_1\lambda_2}}{\alpha}. \tag{2.36}$$

So (2.35) holds if and only if

$$\left(\Phi(A) - \frac{\sqrt{\lambda_1\lambda_2}}{\alpha}\right)^2 \geqslant 0.$$

This proves the claim of the theorem.

To prove the first inequality in (2.29) we note that if the spectra of A and B are oppositely ordered,

$$\Phi(B) = \mu_2\Phi(P_1) + \mu_1\Phi(P_2),$$
$$\Phi(AB) = \lambda_1\mu_2\Phi(P_1) + \lambda_2\mu_1\Phi(P_2) \tag{2.37}$$

and $\Phi(A)$ is given by (2.14). We then have

$$\Phi(AB) - \Phi(A)\Phi(B) = \frac{\mu_1 - \mu_2}{\lambda_2 - \lambda_1}(\lambda_2 - \Phi(A))(\Phi(A) - \lambda_1) \tag{2.38}$$

and

$$\Phi(B) = \frac{\lambda_2\mu_2 - \lambda_1\mu_1}{\lambda_2 - \lambda_1} - \frac{\mu_2 - \mu_1}{\lambda_2 - \lambda_1}\Phi(A). \tag{2.39}$$

From (2.38) and (2.39) we see that first inequality in (2.29) holds if and only if

$$(\Phi(A) - \lambda_1)(\Phi(A) - \lambda_2) + \Phi(A)\left[\frac{\lambda_2\mu_2 - \lambda_1\mu_1}{\mu_2 - \mu_1} - \Phi(A)\right]$$
$$\geqslant \frac{1}{\alpha^2}\Phi(A)\left[\frac{\lambda_2\mu_2 - \lambda_1\mu_1}{\mu_2 - \mu_1} - \Phi(A)\right]. \tag{2.40}$$

A simple calculation shows that (2.40) is true. □

REMARK 2.11. We must indicate here that if $\Phi : M(2) \to M(n)$ is a positive unital linear map then for two commuting Hermitian matrices A and B the following Schwarz's type inequalities hold:

$$\Phi(A^2)\Phi(B^2) \geqslant \Phi(AB)^2, \tag{2.41}$$

$$\Phi(A^2)\Phi(B^2) - \Phi(AB)^2 \leqslant \frac{(\lambda_1\mu_2 - \lambda_2\mu_1)^2}{4} \tag{2.42}$$

and

$$\Phi(A^2)\Phi(B^2) \leqslant \frac{(\lambda_1\mu_2 + \lambda_2\mu_1)^2}{4\lambda_1\lambda_2\mu_1\mu_2}\Phi(AB)^2, \tag{2.43}$$

where λ_i and μ_i are the eigenvalues of A and B respectively, $i = 1, 2$. If in addition A and B are positive definite, then for $0 \leqslant r \leqslant 2$

$$\Phi(A^2)\Phi(B^2) \geqslant \Phi(A^rB^{2-r})\Phi(A^{2-r}B^r). \tag{2.44}$$

The reverse inequality holds when r lies outside $(0,2)$. Also

$$\Phi(A^rB^{2-r})\Phi(A^{2-r}B^r) \geqslant \Phi(AB)^2 \tag{2.45}$$

for every real r.

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