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MORE ABOUT JENSEN'S INEQUALITY AND CAUCHY'S MEANS FOR SUPERQUADRATIC FUNCTIONS

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Abstract. One of the many inequalities that superquadratic functions satisfy is the parallelogram inequality

$$f(u+v)+f(u-v) \geqslant 2f(u)+2f(v).$$

In this paper, we present Cauchy means for superquadratic functions and other mean value theorems. We show also positive semi-definiteness, log-convexity, exponential convexity of certain set of functions.

1. Introduction

Let E be a nonempty set and L be a linear class of real valued functions $f : E \to \mathbb{R}$ having the properties:

$$f, g \in L \Longrightarrow (af+bg) \in L, \ \forall a, b \in \mathbb{R}, \tag{L1}$$

$$1 \in L \text{ i.e., } f(t) = 1 \ \text{ for all } t \in E \Longrightarrow f \in L. \tag{L2}$$

An isotonic linear functional is a functional $A : L \to \mathbb{R}$ having the properties:

$$A(\alpha f + \beta g) = \alpha A(f) + \beta A(g) \ \text{ for } \ f, g \in L, \alpha, \beta \in \mathbb{R}, \ (A \text{ is linear}). \tag{A1}$$

$$f \in L, f(t) \geqslant 0 \ \text{ on } \ E \Longrightarrow A(f) \geqslant 0 \ \ (A \ \text{ is isotonic}). \tag{A2}$$

$$\text{If } \ A(1) = 1 \text{ we say that } A \text{ is normalized functional.} \tag{A3}$$

Definition A. [2, Definition 2.1] A function $\varphi : [0, \infty) \to \mathbb{R}$ is superquadratic provided that for all $x \geqslant 0$ there exists a constant $C(x)$ such that

$$\varphi(y) - \varphi(x) - \varphi(|y-x|) \geqslant C(x)(y-x) \tag{1.1}$$

for all $y \geqslant 0$. We say that φ is subquadratic if $-\varphi$ is a superquadratic function. We say that φ is strictly superquadratic function if for $x \neq y$, $x, y \neq 0$, there is strict inequality in (1.1). We say that φ is strictly subquadratic if $-\varphi$ is a strictly superquadratic function (see [7]).

The following lemma is proved in [2, Lemma 3.1].

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LEMMA A. *Suppose* $\varphi:[0,\infty)\to\mathbb{R}$ *is continuously differentiable and* $\varphi(0)\leqslant 0$. *If* φ' *is superadditive or* $x\mapsto\frac{\varphi'(x)}{x}$ *is increasing, then* φ *is superquadratic.*

We want to emphasize that using Definition A, it was proved in [1] and [2] that the parallelogram inequality holds for superquadratic functions and that if the superquadratic function is also positive, then it is also convex. Therefore in such a case we get refinements of Jensen inequality, Jensen Steffensen inequality and many other inequalities that hold for convex functions. A very important set of superquadratic functions is the set of the power functions $f(x)=x^p,\ \ p\geqslant 2,\ \ x\geqslant 0$. Thus we get refinements of Hölder inequality, Hardy inequality and many more inequalities related to power functions.

The following Jensen type inequality is given in [8, Theorem 10].

THEOREM A. *Let* L *satisfy* $L1$, $L2$ *on a non-empty set* E, *and let* $\varphi:[0,\infty)\to\mathbb{R}$ *be a continuous superquadratic function. Assume that* A *is an isotonic linear functional on* L *with* $A(1)=1$. *If* $f\in L$ *is non-negative and such that* $\varphi(f),\varphi(|f-A(f)|)\in L$, *then we have*

$$\varphi(A(f))\leqslant A(\varphi(f))-A(\varphi(|f-A(f)|)). \tag{1.2}$$

If the function φ *is subquadratic, then the inequality above is reversed.*

DEFINITION B. [3, Definition 1] A function $h:(a,b)\to\mathbb{R}$ is exponentially convex if it is continuous and

$$\sum_{i,j=1}^{n} u_iu_jh(x_i+x_j)\geqslant 0$$

for all $n\in\mathbb{N}$ and all choices $u_i\in\mathbb{R}, i=1,2,\ldots,n$ and $x_i\in(a,b)$, such that $x_i+x_j\in(a,b)$, $1\leqslant i,j\leqslant n$.

PROPOSITION A. [3, Proposition 1] *Let* $h:(a,b)\to\mathbb{R}$. *The following are equivalent.*

(i) h *is exponentially convex.*

(ii) h *is continuous and*

$$\sum_{i,j=1}^{n} u_iu_jh\left(\frac{x_i+x_j}{2}\right)\geqslant 0,$$

for every $u_i\in\mathbb{R}$ *and every* $x_i,x_j\in(a,b), 1\leqslant i,j\leqslant n$.

(iii) h *is continuous and for every* $x_i\in(a,b), i=1,2,\ldots,n$,

$$\det\left[h\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{n}\geqslant 0.$$

COROLLARY A. [3, 10] *If* $h:(a,b)\to(0,\infty)$ *is exponentially convex function, then for all* $x,y\in(a,b)$ h *is a log-convex function:*

$$h\Big(\frac{x+y}{2}\Big)\leqslant\sqrt{h(x)h(y)}.$$

REMARK A. In Definition B and Proposition A it is sufficient to require measurability and finiteness almost everywhere in place of continuity because of the following theorem (see [9, 13, p.105, Th.9.1b]): If the function $h:(a,b)\to\mathbb{R}$ is measurable and finite almost everywhere, h is continuous if also $-\infty < h(x) < \infty$ and $h\left(\frac{x+y}{2}\right)\leqslant\frac{h(x)+h(y)}{2}$, $a<x,y<b$. In this paper we follow the reasoning and the techniques of [3, 4, 5, 6, 12] but here we do it for superquadratic functions. We present here mean value theorems, show positive semi-definiteness, log-convexity and exponential convexity of $A_\varphi = A(\varphi(f)) - \varphi(A(f)) - A(\varphi(|f-A(f)|))$. We also define Cauchy mean for superquadratic function and show its monotonicity.

2. Mean value theorems

We start this section by defining the linear functional A_φ, then using the classical Cauchy mean value and some properties that lead to superquadracity, we build new means. In the rest of the section we will use $C^1(I)$ to denote the class of functions having first order continuous derivatives on an interval I and $C^2(I)$ to denote the class of functions having second order continuous derivatives on I.

DEFINITION 1. Let L satisfy properties $L1, L2$ on a non empty set E and $A: L\to\mathbb{R}$ be a functional having properties $(A1)-(A3)$. Let $\varphi:[0,\infty)\to\mathbb{R}$, $f\in L$ be a non-negative function with $\varphi(f), \varphi(|f-A(f)|)\in L$. We define the functional $A_\varphi: L\to\mathbb{R}$ as:

$$A_\varphi = A(\varphi(f)) - \varphi(A(f)) - A(\varphi(|f-A(f)|)). \tag{2.1}$$

If φ is continuous superquadratic function then by (1.2), $A_\varphi \geqslant 0$.

In the following we assume that a function f satisfies conditions, which imply that if A is strictly positive functional and φ is strictly superquadratic function, then $A_\varphi > 0$. The following lemma is important to prove mean value theorems.

LEMMA 1. *Suppose* $\varphi\in C^2([0,\infty))$, $-\infty<m\leqslant M<\infty$ *be such that*

$$m\leqslant\left(\frac{\varphi'(\xi)}{\xi}\right)' = \frac{\xi\varphi''(\xi)-\varphi'(\xi)}{\xi^2}\leqslant M \qquad \text{for all } \xi>0. \tag{2.2}$$

Consider the functions $\varphi_1,\varphi_2:[0,\infty)\to\mathbb{R}$ *defined as:*

$$\varphi_1(x)=\frac{Mx^3}{3}-\varphi(x),\quad \varphi_2(x)=\varphi(x)-\frac{mx^3}{3}.$$

Then the functions $x\mapsto\frac{\varphi_1'(x)}{x}$, $x\mapsto\frac{\varphi_2'(x)}{x}$ *are increasing. If also* $\varphi(0)=0$, *then* φ_i, $i=1,2$ *are superquadratic.*

Proof. By using inequality (2.2) it is easy to see that the functions $x\mapsto\frac{\varphi_1'(x)}{x}$, $x\mapsto\frac{\varphi_2'(x)}{x}$ are increasing. Also if $\varphi(0)=0$, then by Lemma A we have that φ_i, $i=1,2$ are superquadratic. □

By using Lemma 1 we prove the following theorem.

THEOREM 1. *Let $\varphi:[0,\infty)\to\mathbb{R}$, $\varphi(0)=0$ be a function with the assumptions of Theorem 1, and assume that A is strictly positive functional. If $\frac{\varphi'(x)}{x}\in C^1(0,\infty)$, then there exists $\xi\in(0,\infty)$ such that the following equality holds*

$$A_\varphi=\frac{1}{3}\frac{\xi\varphi''(\xi)-\varphi'(\xi)}{\xi^2}\left(A(f^3)-(A(f))^3-A(|f-A(f)|^3)\right). \tag{2.3}$$

Proof. *Case 1.* Suppose that $m=\min_{x\in(0,\infty)}\left(\frac{\varphi'(x)}{x}\right)'$ and $M=\max_{x\in(0,\infty)}\left(\frac{\varphi'(x)}{x}\right)'$ exist. Using φ_1 instead of φ in (1.2) we get

$$\begin{aligned}&A(\varphi(f))-\varphi(A(f))-A(\varphi(|f-A(f)|))\\&\qquad\leqslant\frac{M}{3}\left(A(f^3)-(A(f))^3-A(|f-A(f)|^3)\right).\end{aligned} \tag{2.4}$$

Similarly, using φ_2 instead of φ in (1.2) we get

$$\begin{aligned}&A(\varphi(f))-\varphi(A(f))-A(\varphi(|f-A(f)|))\\&\qquad\geqslant\frac{m}{3}(A(f^3)-(A(f))^3-A(|f-A(f)|^3)).\end{aligned} \tag{2.5}$$

As $\varphi=x^3$ is strictly superquadratic, and A is strictly positive therefore

$$A(f^3)-(A(f))^3-A(|f-A(f)|^3>0.$$

By combining inequalities (2.4), (2.5) and using the fact that

$$m\leqslant\frac{x\varphi''(x)-\varphi'(x)}{x^2}\leqslant M$$

there exists $\xi\in(0,\infty)$ such that we get (2.3).

Case 2. Suppose that $m=\min_{x\in(0,\infty)}\left(\frac{\varphi'(x)}{x}\right)'$ and $M=\sup_{x\in(0,\infty)}\left(\frac{\varphi'(x)}{x}\right)'$, and assume that M is not a maximum. In this case φ_1 is strictly superquadratic. Using φ_1 instead of φ in (1.2) we get

$$\begin{aligned}&A(\varphi(f))-\varphi(A(f))-A(\varphi(|f-A(f)|))\\&\qquad<\frac{M}{3}\left(A(f^3)-(A(f))^3-A(|f-A(f)|^3)\right).\end{aligned} \tag{2.6}$$

Using φ_2 instead of φ in (1.2) we get

$$\begin{aligned}&A(\varphi(f))-\varphi(A(f))-A(\varphi(|f-A(f)|))\\&\qquad\geqslant\frac{m}{3}(A(f^3)-(A(f))^3-A(|f-A(f)|^3)).\end{aligned} \tag{2.7}$$

By combining inequalities (2.6), (2.7) and using the fact that

$$m \leqslant \frac{x\varphi''(x) - \varphi'(x)}{x^2} < M$$

there exists $\xi \in (0,\infty)$ such that we get (2.3).

Case 3. Suppose that $m = \inf_{x\in(0,\infty)} \left(\frac{\varphi'(x)}{x}\right)'$ and $M = \max_{x\in(0,\infty)} \left(\frac{\varphi'(x)}{x}\right)'$. The proof is analogous to the proof in Case 2.

Case 4. Suppose that $m = \inf_{x\in(0,\infty)} \left(\frac{\varphi'(x)}{x}\right)'$ and $M = \sup_{x\in(0,\infty)} \left(\frac{\varphi'(x)}{x}\right)'$. The proof is analogous to the proof in Case 2.

In the case where $M = \infty$ (that is $\left(\frac{\varphi'(x)}{x}\right)'$ is not bounded above) and m exists, using just φ_2, we obtain

$$m \leqslant \frac{x\varphi''(x) - \varphi'(x)}{x^2}$$

in the case of minimum, and strong inequality in the case where m is infimum. The rest of the proof is as above.

The remaining cases could be treated analogously. □

THEOREM 2. *Let $\varphi,\psi : [0,\infty) \to \mathbb{R}$, $\varphi(0) = \psi(0) = 0$ be functions with the assumptions of Theorem 1, and assume that A is strictly positive functional. If $\frac{\varphi'(x)}{x}$, $\frac{\psi'(x)}{x} \in C^1(0,\infty)$, then there exists $\xi \in (0,\infty)$ such that we have*

$$\frac{A_\varphi}{A_\psi} = \frac{\xi\varphi''(\xi) - \varphi'(\xi)}{\xi\psi''(\xi) - \psi'(\xi)}, \tag{2.8}$$

provided the denominators are not equal to zero.

Proof. We consider a function h defined as $h = c_1\varphi - c_2\psi$, where c_1, c_2 are defined by

$$c_1 = A_\psi,\ c_2 = A_\varphi.$$

Then

$$\frac{h'}{x} = c_1\frac{\varphi'}{x} - c_2\frac{\psi'}{x} \in C^1(0,\infty),$$

after a short calculation we obtain that $A_h = 0$ and using Theorem 1 with $\varphi = h$ we have

$$\begin{aligned}\big(c_1(\xi\varphi''(\xi) - \varphi'(\xi)) - c_2(\xi\psi''(\xi) - \psi'(\xi))\big) \times \\ \times \big(A(f^3) - (A(f))^3 - A(|f - A(f)|^3)\big) = 0.\end{aligned} \tag{2.9}$$

As $\varphi = x^3$ is strictly superquadratic, and A is strictly positive therefore

$$A(f^3) - (A(f))^3 - A(|f - A(f)|^3 > 0.$$

We conclude that

$$\frac{c_2}{c_1} = \frac{\xi\varphi''(\xi) - \varphi'(\xi)}{\xi\psi''(\xi) - \psi'(\xi)} = \frac{A_\varphi}{A_\psi},$$

provided that denominator is not zero. This completes the proof. □

Theorem 2 enables us to define new means. Set

$$K(\xi) = \frac{\xi\varphi''(\xi) - \varphi'(\xi)}{\xi\psi''(\xi) - \psi'(\xi)}$$

and suppose that K is invertible, then

$$\xi = K^{-1}\left(\frac{A_\varphi}{A_\psi}\right)$$

is a new mean.

By using the following definition of generalized mean we give the next result.

DEFINITION 2. [11, p. 107] Let L satisfy properties $L1, L2$ on a non-empty set E, A be positive linear and normalized functional (that is, A satisfies conditions $(A1)-(A3)$) on L, and let $g \in L$. Then for strictly monotone continuous function α such that $\alpha \circ g \in L$, the generalized mean of g with respect to the positive functional A and the function α is defined by

$$M_\alpha(g,A) = \alpha^{-1}(A(\alpha \circ g)). \tag{2.10}$$

Using Theorem 2 we get that the following theorem holds for $M_\alpha(g,A)$:

THEOREM 3. *Let* $\alpha, \beta, \gamma : (0,\infty) \to \mathbb{R}$ *be strictly monotone functions. Let* $f : E \to \mathbb{R}$ *be a function such that* $f(t) \in (0,\infty)$, *forall* $t \in E$. *Let* $\alpha, \beta, \gamma \in C^2(0,\infty)$, *and* $\frac{(\alpha\circ\gamma^{-1})'(x)}{x}, \frac{(\beta\circ\gamma^{-1})'(x)}{x} \in C^1(0,\infty)$ *with* $\alpha \circ \gamma^{-1}(0) = 0$, $\beta \circ \gamma^{-1}(0) = 0$, *then*

$$\frac{\alpha(M_\alpha(f,A)) - \alpha(M_\gamma(f,A)) - \alpha(M_\alpha(\gamma^{-1}(|\gamma\circ f - \gamma(M_\gamma(f,A))|),A))}{\beta(M_\alpha(f,A)) - \beta(M_\gamma(f,A)) - \beta(M_\beta(\gamma^{-1}(|\gamma\circ f - \gamma(M_\gamma(f,A))|),A))}$$

$$= \frac{\gamma(\eta)(\alpha''(\eta)\gamma'(\eta) - \alpha'(\eta)\gamma''(\eta)) - \alpha'(\eta)(\gamma'(\eta))^2}{\gamma(\eta)(\beta''(\eta)\gamma'(\eta) - \beta'(\eta)\gamma''(\eta)) - \beta'(\eta)(\gamma'(\eta))^2}$$

holds for some η *in the image of* f *provided that denominators are not zero.*

Proof. We select the functions φ and ψ so that $\varphi = \alpha \circ \gamma^{-1}$, $\psi = \beta \circ \gamma^{-1}$ and $f = \gamma \circ f$, by making these substitutions in equation (2.8) we have that there exists an ξ such that

$$\frac{\alpha(M_\alpha(f,A)) - \alpha(M_\gamma(f,A)) - \alpha(M_\alpha(\gamma^{-1}(|\gamma\circ f - \gamma(M_\gamma(f,A))|),A))}{\beta(M_\alpha(f,A)) - \beta(M_\gamma(f,A)) - \beta(M_\beta(\gamma^{-1}(|\gamma\circ f - \gamma(M_\gamma(f,A))|),A))}$$

$$= \frac{\xi(\alpha''(\gamma^{-1}(\xi))\gamma'(\gamma^{-1}(\xi)) - \alpha'(\gamma^{-1}(\xi))\gamma''(\gamma^{-1}(\xi))) - \alpha'(\gamma^{-1}(\xi))(\gamma'(\gamma^{-1}(\xi)))^2}{\xi(\beta''(\gamma^{-1}(\xi))\gamma'(\gamma^{-1}(\xi)) - \beta'(\gamma^{-1}(\xi))\gamma''(\gamma^{-1}(\xi))) - \beta'(\gamma^{-1}(\xi))(\gamma'(\gamma^{-1}(\xi)))^2}.$$

Hence by setting $\gamma^{-1}(\xi) = \eta$, we have that there exists an η in the image of f such that

$$\frac{\alpha(M_\alpha(f,A)) - \alpha(M_\gamma(f,A)) - \alpha(M_\alpha(\gamma^{-1}(|\gamma\circ f - \gamma(M_\gamma(f,A))|),A))}{\beta(M_\alpha(f,A)) - \beta(M_\gamma(f,A)) - \beta(M_\beta(\gamma^{-1}(|\gamma\circ f - \gamma(M_\gamma(f,A))|),A))}$$

$$= \frac{\gamma(\eta)(\alpha''(\eta)\gamma'(\eta) - \alpha'(\eta)\gamma''(\eta)) - \alpha'(\eta)(\gamma'(\eta))^2}{\gamma(\eta)(\beta''(\eta)\gamma'(\eta) - \beta'(\eta)\gamma''(\eta)) - \beta'(\eta)(\gamma'(\eta))^2}. \quad \square$$

Theorem 3 enable us to define new means. Set

$$F(\eta) = \frac{\gamma(\eta)(\alpha''(\eta)\gamma'(\eta) - \alpha'(\eta)\gamma''(\eta)) - \alpha'(\eta)(\gamma'(\eta))^2}{\gamma(\eta)(\beta''(\eta)\gamma'(\eta) - \beta'(\eta)\gamma''(\eta)) - \beta'(\eta)(\gamma'(\eta))^2}$$

and when F is invertible, then

$$\eta = F^{-1}\left(\frac{\alpha(M_\alpha(f,A)) - \alpha(M_\gamma(f,A)) - \alpha(M_\alpha(\gamma^{-1}(|\gamma\circ f - \gamma(M_\gamma(f,A))|),A))}{\beta(M_\alpha(f,A)) - \beta(M_\gamma(f,A)) - \beta(M_\beta(\gamma^{-1}(|\gamma\circ f - \gamma(M_\gamma(f,A))|),A))}\right).$$

Since η is in the image of f, then we have $\min_{t\in E} f(t) \leqslant \eta \leqslant \max_{t\in E} f(t)$, i.e. we get a new mean.

Now we give definition of generalized power mean and use in the above theorem to give new mean.

DEFINITION 3. [11, p. 108] The generalized power mean of g with respect to the normalized positive linear functional A is defined by

$$M_r(g,A) = \begin{cases} (A(g^r))^{\frac{1}{r}}, & r \neq 0 \\ \exp(A(\log g)), & r = 0 \end{cases}. \tag{2.11}$$

Now we can deduce corresponding results for the generalized power mean.

COROLLARY 1. *Suppose that all the conditions of Theorem 3 are satisfied. Then for $r,l,s \in \mathbb{R}_+$ such that $r \neq l; r,l \neq 2s$, we have that*

$$\frac{M_r^r(f,A) - M_s^r(f,A) - M_r^r(|f^s - M_s^s(f,A)|^{\frac{1}{s}},A)}{M_l^l(f,A) - M_s^l(f,A) - M_l^l(|f^s - M_s^s(f,A)|^{\frac{1}{s}},A)} = \frac{r(r-2s)}{l(l-2s)}\eta^{r-l} \tag{2.12}$$

holds for some η in the image of f, provided that the denominators are not zero. Therefore we have a new mean.

Proof. If we set $\alpha(t)=t^r, \beta(t)=t^l, \gamma(t)=t^s$ in Theorem 3, we get assertion (2.12). Since η is in the image of f, (2.12) suggests a new mean which satisfies

$$\min f(t) \leqslant \left(\frac{l(l-2s)}{r(r-2s)} \frac{M_r^r(f,A)-M_s^r(f,A)-M_r^r(|f^s-M_s^s(f,A)|^{\frac{1}{s}},A)}{M_l^l(f,A)-M_s^l(f,A)-M_l^l(|f^s-M_s^s(f,A)|^{\frac{1}{s}},A)} \right)^{\frac{1}{r-l}}$$
$$\leqslant \max f(t), \quad r,l,s \in \mathbb{R}_+, r \neq l, 2s. \quad \square$$

From (2.12) it follows that we can define a new mean $M_{r,l}^{[s]}(f,A)$ as:

$$M_{r,l}^{[s]}(f,A) = \left(\frac{l(l-2s)}{r(r-2s)} \frac{M_r^r(f,A)-M_s^r(f,A)-M_r^r(|f^s-M_s^s(f,A)|^{\frac{1}{s}},A)}{M_l^l(f,A)-M_s^l(f,A)-M_l^l(|f^s-M_s^s(f,A)|^{\frac{1}{s}},A)} \right)^{\frac{1}{r-l}}, \quad (2.13)$$

for $r,l,s \in \mathbb{R}_+, \ \ r \neq l; \ r,l \neq 2s$.

For some other cases of r,l,s we apply the following conditions that we call $(A4)$:

$$\lim_{t \to t_0} A(f^t) = A(f^{t_0}), \quad \text{(A4)}$$

$$\lim_{\Delta t \to 0} \frac{A\left(f^{t+\Delta t}\right) - A\left(f^t\right)}{\Delta t} = A(f^t \log f)$$

and we get the following definition of $M_{r,l}^{[s]}$ for more cases of $r,l,s \in \mathbb{R}_+$.

DEFINITION 4. Let L satisfy properties $L1, L2$ on a non-empty set E, A be a positive linear functional on L with $(A3)$ and let $(A4)$ be valid. Then for $r,l,s \in \mathbb{R}_+$ we define Cauchy-type mean of f with respect to the positive linear functional A for $r \neq l, \ \ r \neq 2s, \ l \neq 2s$ by (2.13).

Denoting $d = |f^s - M_s^s(f,A)|^{\frac{1}{s}}$, we get in the limiting cases:

$$M_{l,l}^{[s]}(f,A) = \exp\left(\frac{A(f^l \log f) - M_s^l(f,A)\log M_s(f,A) - A(d^l \log d)}{M_l^l(f,A)-M_s^l(f,A)-M_l^l(d,A)} - \frac{2(l-s)}{l(l-2s)} \right),$$

$$M_{l,2s}^{[s]}(f,A) = M_{2s,l}^{[s]}(f,A)$$
$$= \left(\frac{2s(M_l^l(f,A)-M_s^l(f,A)-M_l^l(d,A))}{l(l-2s)(A(f^{2s}\log f) - M_s^{2s}(f,A)\log M_s(f,A) - A(d^{2s}\log d))} \right)^{\frac{1}{l-2s}},$$

$$M_{2s,2s}^{[s]}(f,A) = \exp\left(\frac{A(f^{2s}(\log f)^2) - M_s^{2s}(f,A)(\log M_s(f,A))^2 - A(d^{2s}(\log d)^2)}{2(A(f^{2s}\log f) - M_s^{2s}(f,A)\log M_s(f,A) - A(d^{2s}\log d))} - \frac{1}{2s} \right).$$

3. Positive-semidefiniteness, exponential convexity and log-convexity

In the following we assume that the linear functional A satisfies $(A4)$.

LEMMA 2. *Consider the function φ_k for $k > 0$ defined as*

$$\varphi_k(x) = \begin{cases} \frac{x^k}{k(k-2)}, & k \neq 2, \\ \frac{x^2}{2}\log x, & k = 2. \end{cases} \tag{3.1}$$

Then, with the convention $0\log 0 = 0$, φ_k is superquadratic.

Proof. Since $\varphi_k(0) = 0$ and $\left(\frac{\varphi_k'(x)}{x}\right)' = x^{k-3} > 0$ for $x > 0$, by Lemma A, φ_k is superquadratic. $\square$

THEOREM 4. *For A_{φ_s} defined by using Lemma 2 in (2.1) we have*

a) for $n \in \mathbb{N}$, $p_i \in \mathbb{R}_+$, $i = 1,\dots,n$, the matrix $A = \left[A_{\varphi_{\frac{p_i+p_j}{2}}}\right]_{i,j=1}^{n}$, is a positive semi-definite matrix.

b) the function $s \mapsto A_{\varphi_s}$, $s \in \mathbb{R}_+$, is exponentially convex.

c) if A_{φ_s} is positive then the function $s \mapsto A_{\varphi_s}$ is log-convex that is for $r < s < t$ where $r,s,t \in \mathbb{R}_+$ we have

$$(A_{\varphi_s})^{t-r} \leqslant (A_{\varphi_r})^{t-s}(A_{\varphi_t})^{s-r}. \tag{3.2}$$

Proof. Define the function $F(x) = \sum_{i,j=1}^{n} u_i u_j \varphi_{p_{ij}}(x)$, where $p_{ij} = \frac{p_i+p_j}{2}$. Then,

$$\left(\frac{F'(x)}{x}\right)' = \sum_{i,j=1}^{n} u_i u_j \left(\frac{\varphi_{p_{ij}}'(x)}{x}\right)' = \left(\sum_{i=1}^{n} u_i x^{\frac{p_i-3}{2}}\right)^2 \geqslant 0$$

and $F(0) = 0$. This implies that F is superquadratic, so using this F in the place of φ in (2.1) we have

$$A_F = \sum_{i,j=1}^{n} u_i u_j A_{\varphi_{p_{ij}}} \geqslant 0. \tag{3.3}$$

Hence the matrix $A = \left[A_{\varphi_{\frac{p_i+p_j}{2}}}\right]_{i,j=1}^{n}$ is positive semi-definite.

$b)$ Since after some computation we have $\lim_{s\to 2} A_{\varphi_s} = A_{\varphi_2}$ so the function $s \mapsto A_{\varphi_s}$, $s \in \mathbb{R}_+$ is continuous on $\mathbb{R}_+$, then by (3.3) and Proposition 1, we have that the function $s \mapsto A_{\varphi_s}$, $s \in \mathbb{R}_+$ is exponentially convex.

$c)$ As A_{φ_s} is positive and $s \mapsto A_{\varphi_s}$, $s \in \mathbb{R}_+$ is continuous, then by Corollary 1 we have $s \mapsto A_{\varphi_s}$ is log-convex and we get (3.2). $\square$

In the next corollary and two theorems we suppose that the functional A_{φ_s} is such that continuity property of Theorem 4. b) is satisfied on an appropriate interval.

COROLLARY 2. *Let* $d_k = A(f^k) - (A(f))^k - A(|f - A(f)|^k), \quad k > 0,\ k \neq 2$ *, and* $d_2 = A(f^2 \log f) - (A(f))^2 \log A(f) - A(|f - A(f)|^2 \log |f - A(f)|)$.

By applying Theorem 4. c) we get

(i) *For* $s > 4$:

$$A(f^s) \geqslant (A(f))^s + A(|f - A(f)|^s) + \frac{s(s-2)}{3}(3d_4/8d_3)^{s-3} d_3, \tag{3.4}$$

(ii) *For* $0 < s < 2$:

$$A(f^s) \leqslant (A(f))^s + A(|f - A(f)|^s) - \frac{s(2-s)}{3}(8d_3/3d_4)^{3-s} d_3, \tag{3.5}$$

(iii) *For* $1 < s < 2$:

$$A(f^s) \geqslant (A(f))^s + A(|f-A(f)|^s) + s(s-2)A(|f-A(f)|)\left(\frac{d_2}{2A(|f-A(f)|)}\right)^{s-3}, \tag{3.6}$$

(iv) *For* $2 < s < 3$:

$$A(f^s) \leqslant (A(f))^s + A(|f - A(f)|^s) + \frac{s(s-2)}{3}(2d_3/3d_2)^{s-2} d_2, \tag{3.7}$$

(v) *For* $3 < s < 4$:

$$A(f^s) \leqslant (A(f))^s + A(|f - A(f)|^s) + \frac{s(s-2)}{3}(3d_4/8d_3)^{s-3} d_3. \tag{3.8}$$

EXAMPLE. Let $y \geqslant x \geqslant 0$ and

$$d_4 = \frac{x^4 + y^4}{2} - \left(\frac{x+y}{2}\right)^4 - \left(\frac{y-x}{2}\right)^4 = \frac{3}{8}(y-x)^2 (y+x)^2$$

$$d_3 = \frac{x^3 + y^3}{2} - \left(\frac{x+y}{2}\right)^3 - \left(\frac{y-x}{2}\right)^3 = \frac{(y-x)^2 (y+2x)}{4}$$

Then for $s > 4$

$$\begin{aligned} d_s &= \frac{x^s + y^s}{2} - \left(\frac{x+y}{2}\right)^s - \left(\frac{y-x}{2}\right)^s \\ &\geqslant \frac{s(s-2)}{3} d_3 \left(\frac{3d_4}{8d_3}\right)^{s-3} = \frac{s(s-2)}{3}\left(\frac{3^2 (y+x)^2}{4^2 (y+2x)}\right)^{s-3} \frac{(y-x)^2 (y+2x)}{4} \end{aligned}$$

By computing directly d_5 we get from the above inequality

$$\frac{7y^3 + 14y^2 x + 11yx^2 + 8x^3}{16} \geqslant 5\frac{3^4 (y+x)^4}{2^{10}(y+x)}.$$

If $0<s<2$ and $y \geqslant x \geqslant 0$ then

$$\frac{x^s+y^s}{2}-\left(\frac{x+y}{2}\right)^s-\left(\frac{y-x}{2}\right)^s \leqslant \frac{-s(s-2)}{3}\left(\frac{8^2(y+2x)}{3^2(y+x)^2}\right)^{3-s}\frac{(y-x)^2(y+2x)}{4}.$$

Therefore for $s=1$ the following holds:

$$-(y-x) \leqslant -\frac{1}{3\cdot 4}\left(\frac{8^2}{3^2}\right)^2\frac{(y+2x)^3(y-x)^2}{(y+x)^4}.$$

In the following theorem we give result analogous to refinement of Holder's inequality for superquadratic functions given in [2].

THEOREM 5. *Let L satisfy conditions $L1, L2$ and A satisfy conditions $(A1),(A2)$ on a set E. Let $1<p<2$, $\frac{1}{p}+\frac{1}{q}=1$, if $f,g>0$ and f^p, g^q, fg, fg, $\log fg^{1-q}$, $f^2g^{2-q} \in L$ then we have*

$$\frac{1}{p(p-2)}\left(\left((A(g^q))^{\frac{1}{q}}(A(f^p)-A(|h|^p))\right)^{\frac{1}{p}}\right)^p-(A(fg))^p$$
$$\leqslant \frac{1}{2}(A(g|h|))^{2-p}\cdot\Bigg(A(f^2g^{2-q}\log(fg^{1-q}))A(g^q)$$
$$-(A(fg))^2\log\frac{A(fg)}{A(g^q)}-A(g^q)A(g^{2-q}|h|^2)\log(g^{1-q}|h|)\Bigg)^{p-1}, \quad (3.9)$$

where $h=|f-\frac{A(fg)}{A(g^q)}|$. For $p>2$ the above inequality is reversed.

Proof. In Theorem 4. c) take $r=1, s=p, t=2$ so that, $1<p<2$, then we have

$$(A_{\varphi_p})^1 \leqslant (A_{\varphi_1})^{2-p}(A_{\varphi_2})^{p-1}.$$

By setting $A(f)=\frac{A(wf)}{A(w)}$ we get

$$\frac{1}{p(p-2)}\left(\frac{A(wf^p)}{A(w)}-\left(\frac{A(wf)}{A(w)}\right)^p-\frac{A(w|f-\frac{A(wf)}{A(w)}|^p)}{A(w)}\right)$$
$$\leqslant \frac{1}{2}\left(\frac{A(w|f-\frac{A(wf)}{A(w)}|)}{A(w)}\right)^{2-p}\times$$
$$\times\left(\frac{A(wf^2\log f)}{A(w)}-\left(\frac{A(wf)}{A(w)}\right)^2\log\frac{A(wf)}{A(w)}-\frac{A(w|f-\frac{A(wf)}{A(w)}|^2)}{A(w)}\log\left(|f-\frac{A(wf)}{A(w)}|\right)\right)^{p-1}.$$

Putting $w=g^q$, $f=fg^{1-q}$, after some calculation we get the inequality (3.9). $\square$

When $\log f$ is convex we see that [6, Lemma 1.3]:

LEMMA 3. *Let f be log-convex function and $x_1 \leqslant y_1, x_2 \leqslant y_2, x_1 \neq x_2, y_1 \neq y_2$. Then the following inequality is valid*

$$\left(\frac{f(x_2)}{f(x_1)}\right)^{\frac{1}{x_2-x_1}} \leqslant \left(\frac{f(y_2)}{f(y_1)}\right)^{\frac{1}{y_2-y_1}}.$$

In the following we assume that A_ψ is continuous for $\psi = \varphi_k, \phi_t$.

THEOREM 6. *Let $t,r,u,v \in \mathbb{R}_+$ be such that $t \leqslant v, r \leqslant u$. Then we have*

$$M_{t,r}^{[s]} \leqslant M_{v,u}^{[s]}.$$

Proof. As A_{φ_t} is log-convex, Lemma 3 implies that for $t,r,u,v \in \mathbb{R}_+$ such that $t \leqslant v, r \leqslant u$, $t \neq r$, $v \neq u$ we have,

$$\left(\frac{A_{\varphi_t}}{A_{\varphi_r}}\right)^{\frac{1}{t-r}} \leqslant \left(\frac{A_{\varphi_v}}{A_{\varphi_u}}\right)^{\frac{1}{v-u}}. \tag{3.10}$$

By substitution $f = f^s$, $t = \frac{t}{s}$, $r = \frac{r}{s}$, $u = \frac{u}{s}$, $v = \frac{v}{s}$, and from the continuity of A_{φ_t} we get our result for $u = r$, $v = t$; $u = r$, $t \neq v$; $u \neq r$, $t = v$. □

LEMMA 4. *Let us define the function*

$$\phi_t(x) = \begin{cases} \frac{txe^{tx}-e^{tx}+1}{t^3}, & t \neq 0, \\ \frac{x^3}{3}, & t = 0. \end{cases}$$

Then $\left(\frac{\phi_t'(x)}{x}\right)' = e^{tx} > 0$, *and* $\phi_t(0) = 0$. *Therefore* ϕ_t *is superquadratic.*

THEOREM 7. *Theorem 4 is still valid if we set $\varphi_s = \phi_s$.*

Proof. See the proof of Theorem 4. □

DEFINITION 5. Let L satisfy properties $L1, L2$ on a non-empty set E, A be a positive linear functional on L with $(A3)$ and let $(A4)$ be valid. Let $t,r \in \mathbb{R}_+$ and f^t, f^r, $\log f$, $(\log f)^2$, $(\log f)^3, (\log f)^4 \in L$ for $t,r \in \mathbb{R}_+$. Then we define Cauchy-type mean of f with respect to the positive linear functional A, $\tilde{M}_{t,r}(f,A)$ by

$$\tilde{M}_{t,r}(f,A) = \left(\frac{r^3(tA(f^t\log f) - M_t^t - M_0^t\log(M_0^t) + M_0^t - tA(b\exp(tb)) + A(\exp(tb)) - 1)}{t^3(rA(f^r\log f) - M_r^r - M_0^r\log(M_0^r) + M_0^r - rA(b\exp(rb)) + A(\exp(rb)) - 1)}\right)^{\frac{1}{t-r}}$$

$t \neq r$ where $M_r(f,A) = M_r$, and $b = |\log(\frac{f}{M_0})|$.

In the limiting case when l goes to r we have

$$\tilde{M}_{r,r}(f,A) = \exp\left(\frac{r(A(f^r(\log f)^2) - M_0^r(\log(M_0))^2 - A(b^2\exp(rb)))}{(rA(f^r\log f) - M_r^r - M_0^r\log(M_0^r) + M_0^r - rA(b\exp(rb)) + A(\exp(rb)) - 1)} - \frac{3}{r}\right).$$

When r goes to 0 we have

$$\tilde{M}_{0,0}(f,A) = \exp\left(\frac{3(A\left((\log f)^4\right) - (\log(M_0))^4 - A(b^4))}{8(A\left((\log f)^3\right) - (\log(M_0))^3 - A(b^3))}\right).$$

THEOREM 8. *Let $t,r,u,v \in \mathbb{R}_+$ be such that $t \leqslant v,\ r \leqslant u$. Then*

$$\tilde{M}_{t,r} \leqslant \tilde{M}_{v,u}.$$

Proof. As A_{ϕ_t} is log-convex function, Lemma 3 implies that for $t,r,u,v \in \mathbb{R}_+$ such that $t \leqslant v, r \leqslant u$, $t \neq r$, $v \neq u$ we get,

$$\left(\frac{A_{\phi_t}}{A_{\phi_r}}\right)^{\frac{1}{t-r}} \leqslant \left(\frac{A_{\phi_v}}{A_{\phi_u}}\right)^{\frac{1}{v-u}}.$$

From the continuity of A_{ϕ_t} we get our result for $u = r,\ v = t$; $u = r,\ t \neq v$; $u \neq r, t = v$. □

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