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DIRECT AND INVERSE STRONG–TYPE INEQUALITIES FOR JACKSON–MATSUOKA POLYNOMIALS ON THE SPHERE

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Abstract. The direct and inverse strong-type inequalities are established for the best approximation by Jackson-Matsuoka polynomials in the L_p space on the unit sphere of $\mathbb{R}^d$ which with the help of the relation between K-functionals and modulus of smoothness of sphere.

1. Introduction and Notations

Let $\mathbb{S} := \mathbb{S}^{d-1} = \{x : \|x\| = 1\}$ denote the unit sphere in $\mathbb{R}^d$ $(d \geqslant 3)$, $d \in \mathbb{N}$, where $\|x\|$ denotes the usual Euclidean norm, $\mathbb{Z}_+$ the set of non-negative integers, and $\mathbb{N}$ the set of positive integers. We denote by $L_p := L_p(\mathbb{S})$, $1 \leqslant p \leqslant \infty$, the space of functions defined on $\mathbb{S}$ with the finite norm

$$\|f\|_p := \left(\int_{\mathbb{S}} |f(\varpi)|^p d\varpi \right)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty,$$

and for $p = \infty$ we assume that L_∞ is replaced by $C(\mathbb{S})$ the space of continuous functions on $\mathbb{S}$ with the usual uniform norm $\|f\|_\infty$, where $\varpi \in \mathbb{S}$, and $d\varpi$ is the measure element on $\mathbb{S}$, and $|\mathbb{S}^{d-1}| = \int_{\mathbb{S}} d\varpi = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$ is the surface area of $\mathbb{S}$.

Denote by $\mathscr{H}_k^d$ the space of spherical harmonics of degree k. The Laplace-Beltrami operator on the sphere denotes by

$$\mathfrak{D} f(\varpi) := \Delta f\left(\frac{\varpi}{|\varpi|}\right)|_{\varpi \in \mathbb{S}}, \tag{1.1}$$

which has eigenvalues $\lambda_k := -k(k+d-2)$ corresponding to the eigenspaces $\mathscr{H}_k^d$ with $k \in \mathbb{Z}_+$, that is, $\mathscr{H}_k^d = \{\Psi \in C(\mathbb{S}) : \mathfrak{D}\Psi = -k(k+d-2)\Psi\}$. For the properties of the space of spherical harmonics and the Laplace-Beltrami operators reference see [6, 7, 9].

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The standard Hilbert space theory shows that $L_2(\mathbb{S}) = \sum_{k=0}^{\infty} \bigoplus \mathscr{H}_k^d$. That is, with each $f \in L_2(\mathbb{S})$, we can associate its harmonics expansion

$$f(\varpi) = \sum_{k=0}^{\infty} Y_k(f,\varpi), \quad \varpi \in \mathbb{S},$$

in $L_2(\mathbb{S})$ norm. The orthogonal projection $Y_k : L_2(\mathbb{S}) \mapsto \mathscr{H}_k^d$ take the form

$$Y_k(f;\varpi) := \frac{\Gamma(\lambda)(k+\lambda)}{2\pi^{\lambda+1}} \int_{\mathbb{S}} P_k^{\lambda}(\varpi,\vartheta) f(\vartheta) d\vartheta, \tag{1.2}$$

where $2\lambda = d-2$, P_k^{λ} denotes hyperspherical polynomials of degree k which satisfies $(1-2r\cos\theta + r^2)^{-r} = \sum_{k=0}^{\infty} r^k P_k^{\lambda}(\cos\theta)$, $0 \leqslant \theta \leqslant \pi$. The properties of hyperspherical polynomials see in [12] for more details.

The spherical means denotes by

$$T_\theta(f) := T_\theta(f;\varpi) := \frac{1}{|\mathbb{S}^{d-2}|(\sin\theta)^{d-2}} \int_{\langle \varpi,\vartheta\rangle = \cos\theta} f(\vartheta) d\vartheta,$$

where $|\mathbb{S}^{d-2}|$ is the surface area of $\mathbb{S}^{d-2}$, $\langle x,y\rangle$ denote the usual Euclidean inner product. The properties of the spherical means are well known (see [1, 10]). In particular, the function $T_\theta(f;\varpi)$ has the expansion

$$T_\theta(f;\varpi) \sim \sum_{k=0}^{\infty} \frac{P_k^{\lambda}(\cos\theta)}{P_k^{\lambda}(1)} Y_k(f;\varpi) := \sum_{k=0}^{\infty} Q_k^{\lambda}(\cos\theta) Y_k(f;\varpi). \tag{1.3}$$

Simultaneously, they lead to the following definition of an analog of the modulus of smoothness:

DEFINITION 1.1. [4, 7] For $f \in L_p$, $1 \leqslant p < \infty$, or $f \in C(\mathbb{S})$, the modulus of smoothness on the sphere is given by

$$\omega(f;t)_p := \sup_{0<\theta\leqslant t} \|f - T_\theta(f)\|_p. \tag{1.4}$$

The K-functional on the sphere is given by

$$K(f;t^2)_p = \inf_{g \in W_p(\mathbb{S})} \left\{ \|f-g\|_p + t^2 \|\mathfrak{D} g\|_p \right\}, \tag{1.5}$$

where $W_p(\mathbb{S}) := \left\{ f : f \in L_p, \mathfrak{D} f \in L_p \right\}$, $0 < t < t_0$, t_0 is a positive constants, $\mathfrak{D} f$ denote the Laplace-Beltrami operator on the sphere.

In [4, 11], Z. Ditzian, and S. Riemenschneider, K. Y. Wang proved the weak equivalence relation

$$C^{-1} K(f;t^2)_p \leqslant \omega(f;t)_p \leqslant C K(f;t^2)_p. \tag{1.6}$$

Throughout this paper, C denotes a positive constant independent on n and f and $C(a)$ denotes a positive constant dependent on a, which may be different in different places.

Based on the classical Jackson-Matsuoka kernel (see in [8]) we define a new kernel

$$M_{n;j,i,s}(\theta) := \frac{1}{\Omega_{n;j,i,s}} \left(\frac{\sin^{2j} n\theta/2}{\sin^{2i} \theta/2} \right)^{2s}, \quad n = 1, 2, \ldots, \theta \in \mathbb{R},$$

where $j, i, s \in \mathbb{N}$, $\Omega_{n;j,i,s}$ is chosen such that $\int_0^\pi M_{n;j,i,s}(\theta) \sin^{2\lambda} \theta d\theta = 1$. It is known that $M_{n;j,i,s}(\theta)$ is an even nonnegative operator. In particular, it is an even and nonnegative trigonometric polynomial of degree at most $2s(nj + 2j - 2i)$ for $j \geqslant i$ and the Jackson polynomials for $j = i$. Using $M_{n;j,i,s}(\theta)$ we consider spherical convolution:

$$J_{n;j,i,s}(f;\varpi) := (f * M_{n;j,i,s})(\varpi) := \int_0^\pi T_\theta(f;\varpi) M_{n;j,i,s}(\theta)(\theta) \sin^{2\lambda} \theta d\theta. \tag{1.7}$$

It is called the Jackson-Matsuoka polynomials on the sphere based on the Jackson-Matsuoka kernel. In particular, $(f_0 * M_{n;j,i,s})(\varpi) = 1$ for $f_0(\varpi) = 1$. The classical Jackson-Matsuoka polynomials in classical L_p space that has been studied by many authors see in [3, 8].

In this paper, we consider the approximation of the Jackson-Matsuoka polynomials on the sphere. With the help of the relation between K-functionals and modulus of smoothness on the sphere and the properties of the spherical means, we obtain the direct and inverse estimate for the best approximation by Jackson-Matsuoka polynomials in the L_p space on the unit sphere of $\mathbb{R}^d$.

2. Auxiliary Lemmas

We need the following Lemmas.

LEMMA 2.1. *Let* $\Omega_{n;,j,i,s} = \int_0^\pi \left(\frac{\sin^{2j} \frac{n\theta}{2}}{\sin^{2i} \frac{\theta}{2}} \right)^{2s} \sin^{d-2} \theta d\theta$. *Then, the weak equivalence*

$$\Omega_{n;j,i,s} \asymp n^{4is-d+1}, \tag{2.1}$$

holds true for $4si > d-1$, $j \geqslant i$, $2\lambda = d-2$, $d \geqslant 3$, *where the weak equivalence relation* $A(n) \asymp B(n)$ *means that* $A(n) \ll B(n)$ *and* $B(n) \ll A(n)$, *and relation* $A_n \ll B_n$ *means that there is a positive constant* C *independent on* n *such that* $A(n) \leqslant CB(n)$.

Proof. Since $\frac{\theta}{\pi} \leqslant \sin\frac{\theta}{2} \leqslant \frac{\theta}{2}$ and $\sin\theta \leqslant \theta$ for $0 \leqslant \theta \leqslant \pi$, we have

$$\begin{aligned}
\Omega_{n;,j,i,s} &= \int_0^\pi \left(\frac{\sin^{2j} \frac{n\theta}{2}}{\sin^{2i} \frac{\theta}{2}} \right)^{2s} \sin^{d-2} \theta d\theta \\
&\asymp n^{4is-d+1} \int_0^{n\pi/2} t^{d-2} \left(\frac{\sin^{2j} t}{t^{2i}} \right)^{2s} dt \\
&\asymp n^{4is-d+1} \left(\int_0^{\pi/2} t^{d-2} \left(\frac{\sin^{2j} t}{t^{2i}} \right)^{2s} dt + \int_{\pi/2}^\infty t^{d-2} \left(\frac{\sin^{2j} t}{t^{2i}} \right)^{2s} dt \right) \\
&\asymp n^{4is-d+1},
\end{aligned} \tag{2.2}$$

since $4si > d-1$. The Lemma 2.1 has been proved. $\square$

LEMMA 2.2. *For $4is > r+d-1,\ j \geqslant i,\ r \in \mathbb{R}$, there is a constant $C(d,j,i,s)$ such that*

$$\int_0^\pi \theta^r M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta \leqslant C(d,j,i,s)n^{-r}. \tag{2.3}$$

Proof. Since $\frac{\theta}{\pi} \leqslant \sin\frac{\theta}{2} \leqslant \frac{\theta}{2}$ and $\sin\theta \leqslant \theta$ for $0 \leqslant \theta \leqslant \pi$, by $\Omega_{n;j,i,s} \asymp n^{4is-d+1}$, we have

$$\begin{aligned}
&\int_0^\pi \theta^r M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta \\
&\leqslant C(d,j,i,s)n^{-4is+d-1}\int_0^\pi \theta^r \left(\frac{\sin^{2j}\frac{n\theta}{2}}{\sin^{2i}\frac{\theta}{2}}\right)^{2s}\sin^{2\lambda}\theta d\theta \\
&\leqslant C(d,j,i,s)n^{-4is+d-1}n^{4is-r-d+1}\int_0^{n\pi/2} t^{r+d-2}\left(\frac{\sin^{2j}t}{t^{2i}}\right)^{2s}dt \\
&\leqslant C(d,j,i,s)n^{-r}\left(\int_0^{\pi/2} t^{r+d-2}\left(\frac{\sin^{2j}t}{t^{2i}}\right)^{2s}dt + \int_{\pi/2}^\infty t^{r+d-2}\left(\frac{\sin^{2j}t}{t^{2i}}\right)^{2s}dt\right) \\
&\leqslant C(d,j,i,s)C_2 n^\lambda \leqslant C(d,j,i,s)n^{-r},
\end{aligned}$$

where

$$C_2 = \int_0^{\pi/2} t^{r+d-2}\left(\frac{\sin^{2j}t}{t^{2i}}\right)^{2s}dt + \int_{\pi/2}^\infty t^{r+d-2}\left(\frac{\sin^{2j}t}{t^{2i}}\right)^{2s}dt,$$

since $4is > r+d-1,\ j \geqslant i,\ r \in \mathbb{R}$. The Lemma 2.2 has been proved. $\square$

LEMMA 2.3. *For $t \geqslant 0$, there is a constant C such that*

$$\omega(f;t\delta)_p \leqslant C\max\{1,t^2\}\omega(f;\delta)_p. \tag{2.4}$$

Proof. By the equivalence relation between the modulus of smoothness and K-functional, and the definition of $K(f;t^2)_p$, we have

$$\begin{aligned}
\omega(f;t\delta)_p &\leqslant CK(f;(t\delta)^2)_p \leqslant C(\|f-g\|_p + t^2\delta^2\|\mathfrak{D}g\|_p) \\
&\leqslant C\max\{1,t^2\}(\|f-g\|_p + \delta^2\|\mathfrak{D}g\|_p) \\
&\leqslant C\max\{1,t^2\}K(f;\delta^2)_p \leqslant C\max\{1,t^2\}\omega(f;\delta)_p
\end{aligned}$$

The Lemma 2.3 has been proved. $\square$

LEMMA 2.4. [5] *Suppose $g \in C^2(\mathbb{S})$. Then, for $\varpi \in (S)$ and $0 < t < \frac{\pi}{2}$, we have*

$$B_t(g,\varpi) - g(\varpi) = \frac{1}{\Phi(t)}\int_0^t \sin^{d-2}\theta d\theta \int_0^\theta \frac{1}{\sin^{d-2}u}\Phi(u)B_u(\mathfrak{D}g,\varpi)du. \tag{2.5}$$

$$T_\theta(g;\varpi)-g(\varpi)=\frac{\Gamma(\frac{d-1}{2})}{2\pi^{\frac{d-1}{2}}}\int_0^\theta \frac{\Phi(t)}{\sin^{d-2}t}B_t(\mathfrak{D}g,\varpi)dt, \tag{2.6}$$

where

$$B_t(f,\varpi)=\frac{1}{\Phi(t)}\int_{cost\leqslant\langle\varpi,\vartheta\rangle\leqslant 1} f(\vartheta)d\vartheta, \quad t>0, \varpi,\vartheta\in\mathbb{S}^{d-1},$$

$\Phi(t)=\frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}\int_0^t \sin^{d-2}udu,$

LEMMA 2.5. *Let* g, $\mathfrak{D}g$, $\mathfrak{D}^2g\in L_p(\mathbb{S})$, $1\leqslant p\leqslant\infty$, $J_{n;j,i,s}(f;\varpi)$ *be the Jackson-Matsuoka polynomials on the sphere based on the Jackson-Matsuoka kernel,* $4is>d+3$. *Then, there is a constant* $C(d,j,i,s)$ *such that*

$$\|J_{n;j,i,s}g-g-\alpha(n)\mathfrak{D}g\|_p\leqslant C(d,j,i,s)n^{-4}\|\mathfrak{D}^2g\|_p, \tag{2.7}$$

where $\alpha(n)\asymp n^{-2}$.

Proof. By (2.6), we have

$$\begin{aligned}
&J_{n;j,i,s}(g;\varpi)-g(\varpi) \quad (2.8)\\
&=\int_0^\pi M_{n;j,i,s}(\theta)(T_\theta(g;\varpi)-g(\varpi))\sin^{d-2}\theta d\theta\\
&=\int_0^\pi M_{n;j,i,s}(\theta)\sin^{d-2}\theta d\theta\frac{\Gamma(\frac{d-1}{2})}{2\pi^{\frac{d-1}{2}}}\int_0^\theta\frac{\Phi(t)}{\sin^{d-2}t}B_t(\mathfrak{D}g,\varpi)dt\\
&=\mathfrak{D}g(\varpi)\int_0^\pi M_{n;j,i,s}(\theta)\sin^{d-2}\theta d\theta\frac{\Gamma(\frac{d-1}{2})}{2\pi^{\frac{d-1}{2}}}\int_0^\theta\frac{\Phi(t)}{\sin^{d-2}t}dt\\
&\quad+\int_0^\pi M_{n;j,i,s}(\theta)\sin^{d-2}\theta d\theta\frac{\Gamma(\frac{d-1}{2})}{2\pi^{\frac{d-1}{2}}}\int_0^\theta\frac{\Phi(t)}{\sin^{d-2}t}(B_t(\mathfrak{D}g,\varpi)-\mathfrak{D}g(\varpi))dt\\
&\quad\times\int_0^t\sin^{d-2}u(B_t(\mathfrak{D}g,\varpi)-\mathfrak{D}g(\varpi))du\\
&=\mathfrak{D}g(\varpi)\int_0^\pi M_{n;j,i,s}(\theta)\sin^{d-2}\theta d\theta\int_0^\theta\frac{dt}{\sin^{d-2}t}\int_0^t\sin^{d-2}udu\\
&\quad+\int_0^\pi M_{n;j,i,s}(\theta)\sin^{d-2}\theta d\theta\int_0^\theta\frac{dt}{\sin^{d-2}t}\int_0^t\sin^{d-2}u(B_t(\mathfrak{D}g,\varpi)-\mathfrak{D}g(\varpi))du\\
&:=\alpha(n)\mathfrak{D}g(\varpi)+\int_0^\pi M_{n;j,i,s}(\theta)\sin^{d-2}\theta\Psi_\theta(g,\varpi)d\theta, \quad (2.9)
\end{aligned}$$

where

$$\alpha(n):=\int_0^\pi M_{n;j,i,s}(\theta)\sin^{d-2}\theta d\theta\int_0^\theta\frac{dt}{\sin^{d-2}t}\int_0^t\sin^{d-2}udu,$$

and

$$\Psi_\theta(g,\varpi):=\int_0^\theta\frac{dt}{\sin^{d-2}t}\int_0^t\sin^{d-2}u(B_t(\mathfrak{D}g,\varpi)-\mathfrak{D}g(\varpi))du.$$

$$\alpha(n) = \int_0^\pi M_{n;j,i,s}(\theta)\sin^{d-2}\theta d\theta \int_0^\theta \frac{dt}{\sin^{d-2}t}\int_0^t \sin^{d-2}u du$$
$$\asymp \int_0^\pi M_{n;j,i,s}(\theta)\sin^{d-2}\theta d\theta \int_0^\theta \frac{t\sin^{d-2}\xi}{\sin^{d-2}t}dt$$
$$\asymp \int_0^\pi \theta^2 M_{n;j,i,s}(\theta)\sin^{d-2}\theta d\theta \asymp n^{-2}, \quad (0<\xi<t). \tag{2.10}$$

We now estimate, using Lemma 2.4, the expression $B_t(\mathfrak{D}g,\varpi)-\mathfrak{D}g$, and obtain

$$\|\Psi_\theta(g)\|_p \leqslant C(d,j,i,s)\theta^4\|\mathfrak{D}^2 g\|_p.$$

By Lemma2.2, and Hölder-Minkowski inequality shows that

$$\|\int_0^\pi M_{n;j,i,s}(\theta)\sin^{d-2}\theta\Psi_\theta(g,\varpi)d\theta\|_p$$
$$\leqslant C(d,j,i,s)\|\mathfrak{D}^2 g\|_p\int_0^\pi \theta^4 M_{n;j,i,s}(\theta)\sin^{d-2}\theta d\theta$$
$$\leqslant C(d,j,i,s)n^{-4}\|\mathfrak{D}^2 g\|_p. \tag{2.11}$$

Consequently, by (2.8), (2.10) and (2.11) completes the proof of this Lemma. □

3. Main Results

THEOREM 3.1. *Suppose that $f\in L_p(\mathbb{S})$, $1\leqslant p\leqslant\infty$, $J_{n;j,i,s}(f;\varpi)$ be the Jackson-Matsuoka polynomials on the sphere based on the Jackson-Matsuoka kernel, $4is>d+3$, $2\lambda=d-2$, $j\geqslant i$. Then*

$$\|J_{n;j,i,s}(f)-f\|_p \leqslant C(d,j,i,s)\omega(f;n^{-1})_p. \tag{3.1}$$

Proof. Since $(f_0 * M_{n;j,i,s})(\varpi)=1$ for $f_0(\varpi)=1$, Therefore, we have that

$$\|J_{n;j,i,s}(f)-f\|_p = \|\int_0^\pi M_{n;j,i,s}(\theta)(f(\varpi)-T_\theta(f;\varpi)\sin^{2\lambda}\theta d\theta\|_p$$
$$\leqslant \int_0^\pi \|f-T_\theta(f)\|_p M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta.$$

Splitting the integral over $[0,\pi]$ into two integrals over $[0,1/n]$ and $[1/n,\pi]$, respectively, and using the definition of $\omega(f;t)_p$, we conclude that

$$\|f-T_\theta(f)\|_p \leqslant \omega(f;n^{-1})_p + \int_{1/n}^\pi \omega(f;\theta)_p M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta.$$

From the Lemma 2.3 it follows that, for $\theta\geqslant n^{-1}$,

$$\omega(f;\theta)_p = \omega(f;n\theta/n)_p \leqslant C\max\{1.n^2\theta^2\}\omega(f;\theta)_p \leqslant Cn^2\theta^2\omega(f;\theta)_p.$$

Therefore, it follows that

$$\|J_{n;j,i,s}(f)-f\|_p \leqslant \omega(f;\theta)_p(1+Cn^2\int_{1/n}^{\pi}\theta^2 M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta).$$

By the Lemma 2.2, we get $\|J_{n;j,i,s}(f)-f\|_p \leqslant C(d,j,i,s)\omega(f;n^{-1})_p$. □

THEOREM 3.2. *Suppose that* $f\in L_p(h_\kappa^2)$, $1\leqslant p\leqslant\infty$, $J_{n;j,i,s}(f;x)$ *be the Jackson-Matsuoka polynomials on the sphere based on the Jackson-Matsuoka kernel,* $4is> d+3$, $2\lambda=d-2$, $j\geqslant i$, $0<\alpha<1$. *Then the following statements are equivalent:*

$$(1)\quad \|J_{n;j,i,s}(f)-f\|_p = O(n^{-\alpha}),\ n\geqslant 2; \tag{3.2}$$

$$(2)\quad \omega(f;n^{-1})_p = O(t^{\alpha}),\ 0<t<1. \tag{3.3}$$

Proof. By Theorem 3.1, we have $(2)\Rightarrow(1)$. Now, we will prove $(1)\Rightarrow(2)$. Let m be a fixed positive integer, Denote by

$$J^m_{n;j,i,s}(f;\varpi) := \sum_{k=0}^{m}\left(\int_0^{\pi} M_{n;j,i,s}(\theta)Q_k^{\lambda}(\cos\theta)\sin^{2\lambda}\theta d\theta\right)^m Y_k(f;\varpi).$$

By orthogonality of the orthogonal projector Y_k, we have that

$$\begin{aligned}J^{m+l}(f) &= \sum_{k=0}^{m}\left(\int_0^{\pi} M_{n;j,i,s}(\theta)Q_k^{\lambda}(\cos\theta)\sin^{2\lambda}\theta d\theta\right)^m \\ &\quad \times Y_k\left(\sum_{v=0}^{m}\left(\int_0^{\pi} M_{n;j,i,s}(\theta)Q_v^{\lambda}(\cos\theta)\sin^{2\lambda}\theta d\theta\right)^l Y_v(f)\right)\\ &= J^m_{n;j,i,s}(J^l_{n;j,i,s}(f)).\end{aligned}$$

Let $g=J^m_{n;j,i,s}(f)$, we get

$$\|f-g\|_p = \|f-J^m_{n;j,i,s}(f)\|_p \leqslant \sum_{k=1}^{m}\|J^{k-1}_{n;j,i,s}(f)-J^k_{n;j,i,s}(f)\|_p \leqslant m\|f-J_{n;j,i,s}(f)\|_p, \tag{3.4}$$

where $J^0_{n;j,i,s}(f)=f$.

On the other hand,

$$\|\mathfrak{D}J^m_{n;j,i,s}(f)\|_p \leqslant \|\sum_{k=0}^{m}k(k+d-2)\left(\int_0^{\pi} M_{n;j,i,s}(\theta)|Q_k^{\lambda}(\cos\theta)|\sin^{2\lambda}\theta d\theta\right)^m Y_k(f)\|_p.$$

Note that [2]

$$|Q_k^{\lambda}(\cos\theta)| \equiv \left|\frac{P_k^{\lambda}(\cos\theta)}{P_k^{\lambda}(1)}\right| \leqslant C\min\{(k\theta)^{-1},1\}.$$

For $k\theta \geqslant 1$, from the (2.3) it follows that

$$\begin{aligned}
&\|\mathfrak{D}J^m_{n;j,i,s}(f)\|_p \\
&\leqslant C(d,j,i,s)\|\sum_{k=0}^{m} k(k+d-2)k^{-m\frac{d-2}{2}}\left(\int_0^\pi M_{n;j,i,s}(\theta)\theta^{-\lambda}\sin^{2\lambda}\theta d\theta\right)^m Y_k(f)\|_p \\
&\leqslant C(d,j,i,s)n^{m\frac{d-2}{2}}\|f\|_p \sum_{k=0}^{\infty} k^{2-m\frac{d-2}{2}} \leqslant C(d,j,i,s)n^{m\frac{d-2}{2}}\|f\|_p \qquad (3.5)
\end{aligned}$$

holds for $m > \frac{6}{d-2}$. For $k\theta < 1$, by Lemma 2.2, we get

$$\begin{aligned}
&\|\mathfrak{D}J^m_{n;j,i,s}(f)\|_p \\
&\leqslant \|\sum_{k=0}^{m}\left(\int_0^\pi M_{n;j,i,s}(\theta)\theta^{-\frac{2}{m}}(\theta^2 k(k+d-2))^{\frac{1}{m}}|Q_k^\lambda(\cos\theta)|\sin^{2\lambda}\theta d\theta\right)^m Y_k(f)\|_p \\
&\leqslant C(d,j,i,s)\|\sum_{k=0}^{m}\left(\int_0^\pi M_{n;j,i,s}(\theta)\theta^{-\frac{2}{m}}((k\theta)^2)^{\frac{2}{m}}\sin^{2\lambda}\theta d\theta\right)^m Y_k(f)\|_p \\
&\leqslant C(d,j,i,s)\|\sum_{k=0}^{m}\left(\int_0^\pi M_{n;j,i,s}(\theta)\theta^{-\frac{2}{m}}\sin^{2\lambda}\theta d\theta\right)^m Y_k(f)\|_p \\
&\leqslant C(d,j,i,s)n^2\|\sum_{k=0}^{\infty} Y_k(f)\|_p \leqslant C(d,j,i,s)n^2\|f\|_p. \qquad (3.6)
\end{aligned}$$

Consequently, the inequality

$$\|\mathfrak{D}J^m_{n;j,i,s}(f)\|_p \leqslant C(d,j,i,s)n^2\|f\|_p \qquad (3.7)$$

holds uniformly for $m > \frac{6}{d-2}$.

Without loss of generality, we may assume $m_1 > \frac{6}{d-2}$, $m > m_1 + \frac{6}{d-2}$. Using Lemma 2.5, we have

$$\begin{aligned}
&\alpha(n)\|\mathfrak{D}J^m_{n;j,i,s}(f)\|_p \\
&= \|\alpha(n)\mathfrak{D}J^m_{n;j,i,s}(f)\|_p \leqslant \|J^m_{n;j,i,s}(f) - f\|_p + C(d,j,i,s)n^{-4}\|\mathfrak{D}^2 J^m_{n;j,i,s}(f)\|_p \\
&\leqslant m\|J_{n;j,i,s}(f) - f\|_p + C(d,j,i,s)n^{-2}\|\mathfrak{D}^2 J^{m-m_1}_{n;j,i,s}(f)\|_p \\
&\leqslant m\|J_{n;j,i,s}(f) - f\|_p + C(d,j,i,s)\left(n^{-2}\|\mathfrak{D}J^m_{n;j,i,s}(f)\|_p + n^{-2}\|J^m_{n;j,i,s}(f) - J^{m-m_1}_{n;j,i,s}(f)\|_p\right) \\
&\leqslant m\|J_{n;j,i,s}(f) - f\|_p + C(d,j,i,s)\left(n^{-2}\|\mathfrak{D}J^m_{n;j,i,s}(f)\|_p + \|J^{m_1}_{n;j,i,s}(f) - f\|_p\right) \\
&\leqslant C(d,j,i,s)\left(\|J_{n;j,i,s}(f) - f\|_p + n^{-2}\|\mathfrak{D}J^m_{n;j,i,s}(f)\|_p\right) \\
&\leqslant C(d,j,i,s)\left(\|J_{n;j,i,s}(f) - f\|_p + \|(f)\|_p\right) \qquad (3.8)
\end{aligned}$$

Consequently, $n^{-2}\|\mathfrak{D}J^m_{n;j,i,s}(f)\|_p \leqslant C(d,j,i,s)\|f - J_{n;j,i,s}(f)\|_p$, by the definition of $K(f;t^2)_p$ and (1.6) shows that

$$\begin{aligned}
\omega(f;n^{-1})_p &\leqslant C(d,j,i,s)K(f;n^{-2})_p \\
&\leqslant C(d,j,i,s)\|f - J^m_{n;j,i,s}(f)\|_p + n^{-2}\|\mathfrak{D}J^m_{n;j,i,s}(f)\|_p \\
&\leqslant C(d,j,i,s)\|f - J_{n;j,i,s}(f)\|_p. \qquad (3.9)
\end{aligned}$$

In view of (1), we get

$$\omega(f;n^{-1})_p \leqslant C(d,j,i,s)n^{-\alpha}. \tag{3.10}$$

Letting $(n+1)^{-1} < t \leqslant n^{-1}$, we have

$$\begin{aligned} \omega(f;t)_p \leqslant \omega(f;n^{-1})_p \leqslant C(d,j,i,s)\left(\frac{n}{n+1}\right)^{-\alpha}(n+1)^{-\alpha} \\ \leqslant C(d,j,i,s)(n+1)^{-\alpha} \leqslant C(d,j,i,s)t^{\alpha}. \end{aligned} \tag{3.11}$$

The proof is completed. □

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