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MIXED SYMMETRIC MEANS RELATED TO THE CLASSICAL JENSEN'S INEQUALITY

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Abstract. In this paper, we define some new mixed symmetric means corresponding to various refinements of classical Jensen's inequality. A new refinement of classical Jensen's inequality is given. We also prove the n-exponential convexity for the functionals constructed from the refinement results. In the end some applications are discussed.

1. Introduction and preliminary results

Let $n \in \mathbb{N} = \{1,2,\ldots\}$, $\mathbf{t} = (t_1,\ldots,t_{n-1})$ where $t_i \in [0,1]$ and $\mathbf{p} = (p_1,\ldots,p_n)$ be a positive n-tuple, such that $\sum_{i=1}^{n} p_i = 1$ (throughout in this paper). For $\mathbf{x} := (x_1,\ldots,x_n) \in I^n$ $(n \geqslant 2)$, we consider

(H_1) $I \subset \mathbb{R}$ be an interval and $q : I \to \mathbb{R}$ be a convex function.

Then the classical discrete Jensen's inequality states [9]:

$$q\left(\sum_{i=1}^{n} p_i x_i\right) \leqslant \sum_{i=1}^{n} p_i q(x_i). \tag{1}$$

The following interpolation of (1) is given in [10]:

THEOREM 1.1. *Assume (H_1) and define*

$$\begin{aligned} q_{n,k} &= q_{n,k}(\mathbf{x},\mathbf{p},\mathbf{t},q) \\ &:= \sum_{i_1=1}^{n} \ldots \sum_{i_k=1}^{n} p_{i_1} \ldots p_{i_k} q\left(x_{i_1}(1-t_1) + \sum_{j=1}^{k-1} x_{i_j}(1-t_{j+1}) t_1 \ldots t_j + \bar{x} t_1 \ldots t_k\right); \end{aligned}$$

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for $k=1,\ldots,n-1$, *where* $\bar{x}=\sum_{i=1}^{n} p_i x_i$. *Then*

$$\begin{aligned} q\left(\sum_{i=1}^{n} p_i x_i\right) &\leqslant q_{n,1}\leqslant q_{n,2}\leqslant \ldots \leqslant q_{n,n-1} \\ &\leqslant \sum_{i_1=1}^{n}\ldots\sum_{i_n=1}^{n} p_{i_1}\ldots p_{i_n} q\left(x_{i_1}(1-t_1)+\sum_{j=1}^{n-2} x_{i_j}(1-t_{j+1})t_1\ldots t_j + x_{i_n}t_1\ldots t_{n-1}\right) \\ &\leqslant \sum_{i=1}^{n} p_i q(x_i). \end{aligned}$$

Let the index set T be either $\{1,\ldots,n\}$ with $n\in\mathbb{N}$ or $\mathbb{N}$. We need some facts from measure and integration theory are as follows ([5],see also[4]):

Let $(Y_i,\mathscr{B}_i,v_i)$, $i\in T$ be probability spaces, where either $T:=\{1,\ldots,n\}$ with $n\in\mathbb{N}$, or $T:=\mathbb{N}$. The product of these spaces is denoted by $(Y^T,\mathscr{B}^T,v^T)$, i.e $Y^T:=\underset{i\in T}{\times} Y_i$ and $\mathscr{B}^T$ is the smallest $\sigma-$algebra in Y such that each $pr^T_{\{i\}}$ is $\mathscr{B}^T-\mathscr{B}_i$ measureable ($i\in T$). If $T=\{1,\ldots,n\}$, then v^T is the only measure on $\mathscr{B}^T$ which satisfies

$$v^T(B_1\times\ldots\times B_n)=v_1(B_1)\ldots v_n(B_n)$$

for every $B_i\in\mathscr{B}_i$. If $T=\mathbb{N}$, then v^T is the unique measure on $\mathscr{B}^T$ such that the image measure of v^T under the projection mapping $pr^{\infty}_{1\ldots k}$ is the product of the measures $v_1,\ldots,v_k$ $(k\in\mathbb{N})$.

We observe that $\left(Y^T,\mathscr{B}^T,v^T\right)$ is also a probability space.

The n-fold ($n\geqslant 1\, or\, n=\infty$) product of the probability spaces $(X,\mathscr{A},\mu)$ is denoted by $(X^n,\mathscr{A}^n,\mu^n)$. We suppose that the $\mu-integrability$ of a function $g:X\to\mathbb{R}$ over X implies the measurability of g.

(H_2) Let (Y_i,B_i,v_i) be probability spaces and $f_i:Y_i\to I$ be a v_i-integrable function over $Y_i(i=1,\ldots,n)$.

THEOREM 1.2. ([5]) *Assume* (H_1) *and* (H_2) *and let* $q\circ f_i$ *be* $v_i-integrable$ *over* $Y_i(i=1,\ldots,n)$. *Then*

$$q\left(\sum_{i=1}^{n} p_i\int_{Y_i} f_i dv_i\right)\leqslant \int_{Y^T} q\left(\sum_{i=1}^{n} p_i\int_{Y_i} f_i(y_i)\right)dv^T(y_1,\ldots,y_n)\leqslant \sum_{i=1}^{n} p_i\int_{Y_i} q\circ f_i dv_i, \tag{2}$$

where $T=\{1,\ldots,n\}$.

The next theorem corresponds the asymptotic behavior of the core term in (2).

THEOREM 1.3. ([5]) *Let* $I\subset\mathbb{R}$ *be an interval, and let* $q:I\to\mathbb{R}$ *be a convex and bounded function on* I. *Let* (Y_i,B_i,v_i), $i\in\mathbb{N}$ *be probability spaces and* $f_i:Y_i\to I$ *be a square* v_i*-integrable function over* $Y_i(i\in\mathbb{N})$ *such that*

$$\int_{Y_i} f_i dv_i=\int_{Y_1} f_1 dv_1,\ i\in\mathbb{N}, \tag{3}$$

and

$$\sum_{i=1}^{\infty}\frac{1}{i^2}\int_{Y_i} f_i^2 d\nu_i < \infty. \tag{4}$$

Then

$$\lim_{n\to\infty}\int_{Y^{\{1,\dots,n\}}} q\left(\frac{1}{n}\sum_{i=1}^{n} f_i(y_i)\right) d\nu^{\{1,\dots,n\}}(y_1,\dots,y_n) = q\left(\int_{Y_1} f_1 d\nu_1\right).$$

Consider the following hypothesis:

(H_3) (X, A, μ) be probability space and $f : X \to I$ be a μ-integrable function over X.

The next theorem is also followed by [5].

THEOREM 1.4. *Assume (H_1) and (H_3) such that $q\circ f$ is $\mu-$integrable over X. Then*

(a)

$$q\left(\int_X f d\mu\right) \leqslant \int_{X^n} q\left(\sum_{i=1}^{n} p_i f(x_i)\right) d\mu^n(x_1,\dots,x_n) \leqslant \int_X q\circ f d\nu,$$

(b)

$$\int_{X^{n+1}} q\left(\frac{1}{n+1}\sum_{i=1}^{n+1} f(x_i)\right) d\mu^{n+1}(x_1,\dots,x_{n+1})$$

$$\leqslant \int_{X^n} q\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right) d\mu^n(x_1,\dots,x_n) \leqslant \int_{X^n} q\left(\sum_{i=1}^{n} p_i f(x_i)\right) d\mu^n(x_1,\dots,x_n).$$

(c) If q is bounded, then

$$\lim_{n\to\infty}\int_{X^n} q\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right) d\mu^n(x_1,\dots,x_n) = q\left(\int_X f d\mu\right).$$

REMARK 1.5. From Theorem 1.1, we write

$$\Psi^1(q) := \Psi^1(\mathbf{x},\mathbf{p},\mathbf{t},q) := \sum_{i=1}^{n} p_i q(x_i)$$
$$-\sum_{i_1=1}^{n}\dots\sum_{i_n=1}^{n} p_{i_1}\dots p_{i_n} q\left(x_{i_1}(1-t_1)+\sum_{j=1}^{n-2} x_{i_j}(1-t_{j+1})t_1\dots t_j + x_{i_n}t_1\dots t_{n-1}\right) \geqslant 0,$$

$$\Psi^2(q) := \Psi^2(\mathbf{x},\mathbf{p},\mathbf{t},q) := \sum_{i=1}^{n} p_i q(x_i) - q_{n,k} \geqslant 0; \quad k = 1,\dots,n-1,$$

$$\Psi^3(q) := \Psi^3(\mathbf{x},\mathbf{p},\mathbf{t},q)$$
$$:= \sum_{i_1=1}^{n} \cdots \sum_{i_n=1}^{n} p_{i_1}\cdots p_{i_n} q\left(x_{i_1}(1-t_1)+\sum_{j=1}^{n-2} x_{i_j}(1-t_{j+1})t_1\ldots t_j + x_{i_n}t_1\ldots t_{n-1}\right) - q_{n,k} \geqslant 0,$$
$$\Psi^4(q) := \Psi^4(\mathbf{x},\mathbf{p},\mathbf{t},q) := q_{n,k} - q_{n,m} \geqslant 0; \quad 1 \leqslant m < k \leqslant n-1,$$
$$\Psi^5(q) := \Psi^5(\mathbf{x},\mathbf{p},\mathbf{t},q) := q_{n,k} - q\left(\sum_{i=1}^{n} p_i x_i\right) \geqslant 0; \quad k = 1,\ldots,n-1,$$
$$\Psi^6(q) := \Psi^6(\mathbf{x},\mathbf{p},\mathbf{t},q)$$
$$:= \sum_{i_1=1}^{n} \cdots \sum_{i_n=1}^{n} p_{i_1}\cdots p_{i_n} q\left(x_{i_1}(1-t_1)+\sum_{j=1}^{n-2} x_{i_j}(1-t_{j+1})t_1\ldots t_j + x_{i_n}t_1\ldots t_{n-1}\right)$$
$$- q\left(\sum_{i=1}^{n} p_i x_i\right) \geqslant 0,$$
$$\Psi^7(q) := \Psi^7(\mathbf{x},\mathbf{p},q) := \sum_{i=1}^{n} p_i q(x_i) - q\left(\sum_{i=1}^{n} p_i x_i\right) \geqslant 0.$$

From Theorem 1.2, we write

$$\Psi^8(q) := \Psi^8(\mathbf{f},\mathbf{p},q) := \sum_{i=1}^{n} p_i \int_{Y_i} q\circ f_i dv_i - \int_{Y^T} q\left(\sum_{i=1}^{n} p_i \int_{Y_i} f_i(y_i)\right) dv^T(y_1,\ldots,y_n) \geqslant 0,$$
$$\Psi^9(q) := \Psi^9(\mathbf{f},\mathbf{p},q) := \int_{Y^T} q\left(\sum_{i=1}^{n} p_i \int_{Y_i} f_i(y_i)\right) dv^T(y_1,\ldots,y_n) - q\left(\sum_{i=1}^{n} p_i \int_{Y_i} f_i dv_i\right) \geqslant 0,$$
$$\Psi^{10}(q) := \Psi^{10}(\mathbf{f},\mathbf{p},q) := \sum_{i=1}^{n} p_i \int_{Y_i} q\circ f_i dv_i - q\left(\sum_{i=1}^{n} p_i \int_{Y_i} f_i dv_i\right) \geqslant 0,$$

where $\mathbf{f} := (f_1,\ldots,f_n)$.

From Theorem 1.4 (a), we write

$$\Psi^{11}(q) := \Psi^{11}(\mathbf{p},f,q) := \int_{X^n} q\left(\sum_{i=1}^{n} p_i f(x_i)\right) d\mu^n(x_1,.,x_n) - q\left(\int_X f d\mu\right) \geqslant 0,$$
$$\Psi^{12}(q) := \Psi^{12}(\mathbf{p},f,q) := \int_X q\circ f dv - \int_{X^n} q\left(\sum_{i=1}^{n} p_i f(x_i)\right) d\mu^n(x_1,.,x_n) \geqslant 0,$$
$$\Psi^{13}(q) := \Psi^{13}(f,q) := \int_X q\circ f dv - q\left(\int_X f d\mu\right) \geqslant 0,$$

and from Theorem 1.4 (b)

$$\Psi^{14}(q) := \Psi^{14}(\mathbf{p}, f, q) := \int_{X^n} q\left(\sum_{i=1}^{n} p_i f(x_i)\right) d\mu^n(x_1, ., x_n)$$
$$- \int_{X^n} q\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right) d\mu^n(x_1, ., x_n) \geqslant 0,$$
$$\Psi^{15}(q) := \Psi^{15}(\mathbf{p}, f, q) := \int_{X^n} q\left(\sum_{i=1}^{n} p_i f(x_i)\right) d\mu^n(x_1, ., x_n)$$
$$- \int_{X^{n+1}} q\left(\frac{1}{n+1}\sum_{i=1}^{n+1} f(x_i)\right) d\mu^{n+1}(x_1, ., x_{n+1}) \geqslant 0.$$

REMARK 1.6. The first inequality in Theorem 1.4 (b) provides the generalization of result given in [3] and [13] (see also Theorem 3.36 in [12] page 97). That result is utilized in [2] to give the log-convexity for a class of convex functions and is also used in [8] to give the exponential convexity for the same class.

In this paper we refine the first inequality in Theorem 1.4 (b). Mixed symmetric means are defined and their monotonicity is presented. The notion of n-exponential convexity is introduced in [11]. The class of n-exponential convex functions is more general than the class of log-convex functions. We follow the method illustrated in [11] to give the n-exponential convexity and exponential convexity for the family of functionals $\Psi^i(q)$ $(i = 1, ..., 21)$. Therefore our results related to Theorem 2.1 are more general than the corresponding results in [2] and in [8].

2. New refinement of Jensen's inequality

THEOREM 2.1. *Consider the assumptions of Theorem 1.4. We define*

$$Q_{n,k} := \int_{X^k} q\left((1-t_1)f(x_1) + \sum_{j=1}^{k-1}(1-t_{j+1})t_1...t_j f(x_j) + t_1...t_k \bar{f}\right) d\mu^k(x_1, ...x_k),$$

where $\bar{f} = \int_X f(x)d\mu(x)$ *and* $t_i \in [0,1]$ $i = 1, ...n-1$. *Then*

$$q\left(\int_X f\, d\mu\right) \leqslant Q_{n,1} \leqslant Q_{n,2} \leqslant ... \leqslant Q_{n,n-1}$$
$$\leqslant \int_{X^n} q\left((1-t_1)f(x_1) + \sum_{j=1}^{n-2}(1-t_{j+1})t_1...t_j f(x_j) + t_1...t_{n-1} f(x_n)\right) d\mu^n(x_1, ...x_n)$$
$$\leqslant \int_X q \circ f\, d\mu.$$

Proof. By use of Jensen's inequality and integration with respect to μ, we have the following:

$$\begin{aligned}
&\int_X q\circ f\,d\mu\\
&=\int_{X^n}\left((1-t_1)q\circ f(x_1)+\sum_{j=1}^{n-2}(1-t_{j+1})t_1\ldots t_j q\circ f(x_j)+t_1\ldots t_{n-1}q\circ f(x_n)\right)d\mu^n(x_1,\ldots x_n)\\
&\geqslant\int_{X^n} q\left((1-t_1)f(x_1)+\sum_{j=1}^{n-2}(1-t_{j+1})t_1\ldots t_j f(x_j)+t_1\ldots t_{n-1}f(x_n)\right)d\mu^n(x_1,\ldots x_n)\\
&\geqslant\int_{X^{n-1}} q\left((1-t_1)f(x_1)+\sum_{j=1}^{n-2}(1-t_{j+1})t_1\ldots t_j f(x_j)+t_1\ldots t_{n-1}\overline{f}\right)d\mu^{n-1}(x_1,\ldots x_{n-1})\\
&\ \vdots\\
&\geqslant\int_X q\left((1-t_1)f(x_1)+t_1\overline{f}\right)d\mu(x_1)\geqslant q\left(\int_X f\,d\mu\right). \quad\square
\end{aligned}$$

REMARK 2.2. From previous theorem we write

$$\begin{aligned}
\Psi^{16}(q) &:= \Psi^{16}(\mathbf{t},f,q) := \int_X q\circ f\,d\mu\\
&\qquad -\int_{X^n} q\left((1-t_1)f(x_1)+\sum_{j=1}^{n-2}(1-t_{j+1})t_1\ldots t_j f(x_j)+t_1\ldots t_{n-1}f(x_n)\right)d\mu^n(x_1,\ldots x_n)\\
&\quad\geqslant 0,\\
\Psi^{17}(q) &:= \Psi^{18}(\mathbf{t},f,q)\\
&\quad := \int_{X^n} q\left((1-t_1)f(x_1)+\sum_{j=1}^{n-2}(1-t_{j+1})t_1\ldots t_j f(x_j)+t_1\ldots t_{n-1}f(x_n)\right)d\mu^n(x_1,\ldots x_n)\\
&\qquad -Q_{n,k}\geqslant 0;\quad k=1,\ldots,n-1,\\
\Psi^{18}(q) &:= \Psi^{17}(\mathbf{t},f,q) := \int_X q\circ f\,d\mu - Q_{n,k}\geqslant 0;\quad k=1,\ldots,n-1,\\
\Psi^{19}(q) &:= \Psi^{19}(\mathbf{t},f,q) := Q_{n,k}-Q_{n,m}\geqslant 0;\quad 1\leqslant m<k\leqslant n-1,\\
\Psi^{20}(q) &:= \Psi^{20}(\mathbf{t},f,q) := Q_{n,k}-q\left(\int_X f\,d\mu\right)\geqslant 0;\quad k=1,\ldots,n-1,
\end{aligned}$$

$$\Psi^{21}(q) := \Psi^{21}(\mathbf{t}, f, q)$$
$$:= \int_{X^n} q\left((1-t_1)f(x_1) + \sum_{j=1}^{n-2}(1-t_{j+1})t_1...t_j f(x_j) + t_1...t_{n-1}f(x_n)\right) d\mu^n(x_1,...x_n)$$
$$-q\left(\int_X f\, d\mu\right) \geqslant 0.$$

Hence for any convex function q, we have

$$\Psi^i(q) \geqslant 0; \quad i = 1,...,21. \tag{5}$$

REMARK 2.3. In this way the results for $\Psi^i(q)$, $i = 18,...,20$ are more general than the results given for Theorem 1.14 in [8].

Now we formulate the mean value theorems for $\Psi^i(q)$, $i = 1,...,21$.

THEOREM 2.4. *Consider the functionals as defined in (5) and let $g \in C^2[a,b]$. Then there exists $\xi_i \in [a,b]$ such that*

$$\Psi^i(g) = \frac{1}{2}g''(\xi_i)\Psi^i\left(x^2\right), \quad i = 1,...,21.$$

Proof. Since $g \in C^2[a,b]$ therefore there exist real numbers $m = \min_{x\in[a,b]} g''(x)$ and $M = \max_{x\in[a,b]} g''(x)$. It is easy to show that the functions $\phi_1(x)$, $\phi_2(x)$ defined as

$$\phi_1(x) = \frac{M}{2}x^2 - g(x),$$

and

$$\phi_2(x) = g(x) - \frac{m}{2}x^2,$$

are convex. Fix $1 \leqslant i \leqslant 21$ and put $q = \phi_1$ in (5) to get

$$\Psi^i\left(\frac{M}{2}x^2 - g(x)\right) \geqslant 0,$$

$$\Rightarrow \Psi^i(g(x)) \leqslant \frac{M}{2}\Psi^i\left(x^2\right). \tag{6}$$

Similarly, put $q = \phi_2$ in (5) to get

$$\Psi^i\left(g(x) - \frac{m}{2}x^2\right) \geqslant 0$$

$$\Rightarrow \frac{m}{2}\Psi^i\left(x^2\right) \leqslant \Psi^i(g(x)). \tag{7}$$

From (6) and (7), we get

$$\frac{m}{2}\Psi^i\left(x^2\right) \leqslant \Psi^i\left(g\left(x\right)\right) \leqslant \frac{M}{2}\Psi^i\left(x^2\right).$$

If $\Psi^i\left(x^2\right) = 0$ then nothing to prove. If $\Psi^i\left(x^2\right) \neq 0$, then

$$m \leqslant \frac{2\Psi^i\left(g\left(x\right)\right)}{\Psi^i\left(x^2\right)} \leqslant M.$$

Consequently

$$\Psi^i\left(g\right) = \frac{1}{2}g''\left(\xi_i\right)\Psi^i\left(x^2\right). \quad \square$$

THEOREM 2.5. *Consider the functionals as defined in (5) and let* $g,h \in C^2[a,b]$. *Then there exists* $\xi_i \in [a,b]$ *such that*

$$\frac{\Psi^i\left(g\right)}{\Psi^i\left(h\right)} = \frac{g''\left(\xi_i\right)}{h''\left(\xi_i\right)}, \quad i = 1,\ldots,21,$$

provided that the denominators are non zero.

Proof. Fix $1 \leqslant i \leqslant 21$ and define $L \in C^2[a,b]$ in the way that

$$L = c_1 g - c_2 h,$$

where c_1 and c_2 are as follow;

$$c_1 = \Psi^i\left(h\right)$$

and

$$c_2 = \Psi^i\left(g\right).$$

Now using Theorem 2.4 for the function L, we have

$$\left(c_1\frac{g''\left(\xi_i\right)}{2} - c_2\frac{h''\left(\xi_i\right)}{2}\right)\Psi^i\left(x^2\right) = 0. \tag{8}$$

Since $\Psi^i\left(x^2\right) \neq 0$, therefore (8) gives

$$\frac{\Psi^i\left(g\right)}{\Psi^i\left(h\right)} = \frac{g''\left(\xi_i\right)}{h''\left(\xi_i\right)}. \quad \square$$

3. Mixed symmetric means

Let us consider a convex set Y together with a probability measure ν and f be a positive continuous function. Then the integral power means of order $s \in \mathbb{R}$ are defined as follows [1]:

$$\widetilde{M}_s\left(f\right) = \begin{cases} \left(\int\limits_Y \left(f\left(y\right)\right)^s d\nu\left(y\right)\right)^{\frac{1}{s}}; & s \neq 0, \\ exp\left(\int\limits_Y \log\left(f\left(y\right)\right) d\nu\left(y\right)\right); & s = 0. \end{cases}$$

Assume (H_2). We define the weighted power means and mixed symmetric means as follows:

$$M_s(f_1,...,f_n;\mathbf{p}) = \begin{cases} \left(\sum\limits_{i=1}^{n} p_i(f_i(y_i))^s\right)^{\frac{1}{s}}; & s \neq 0, \\ \prod\limits_{i=1}^{n} (f_i(y_i))^{p_i}; & s = 0. \end{cases}$$

$$M_{r,s}(f_1,...,f_n;\mathbf{p}) = \begin{cases} \left(\sum\limits_{i=1}^{n} p_i \widetilde{M}_s^r(f_i)\right)^{\frac{1}{r}}; & r \neq 0, \\ \prod\limits_{i=1}^{n} \widetilde{M}_s^{p_i}(f_i); & r = 0. \end{cases}$$

$$\widetilde{M}_{r,s}(f_1,...,f_n;\mathbf{p}) = \left\{ \left(\int\limits_{Y^T} {M_s}^r(f_1,...,f_n;\mathbf{p})\, dv^T(y_1,...,y_n)\right)^{\frac{1}{r}}; \quad r \neq 0, \right.$$

We can establish the relations among these means as an application of Theorem 1.2.

COROLLARY 3.1. *Let $r,s \in \mathbb{R}$ such that $s \leqslant r$ and assume (H_2), then we have*

$$M_{s,s}(f) \leqslant \widetilde{M}_{r,s}(f;\mathbf{p}) \leqslant \widetilde{M}_r(f) \tag{9}$$

$$M_{r,r}(f) \geqslant \widetilde{M}_{s,r}(f;\mathbf{p}) \geqslant \widetilde{M}_s(f) \tag{10}$$

Proof. Let $r,s \in \mathbb{R}$ such that $s \leqslant r$, if $r,s \neq 0$, then we set $q(x) = x^{\frac{r}{s}}$, $f_i = {f_i}^s$ in (2) and raising the power $\frac{1}{r}$, we get (9). Similarly we set $q(x) = x^{\frac{s}{r}}$, $f_i = {f_i}^r$ in (2) and raising the power $\frac{1}{s}$, we get (10).

When $s = 0$ or $r = 0$, we get the required results by taking limit. □

We also need the following hypothesis:

(H_4) $h, g : I \to \mathbb{R}$ be continuous and strictly monotone functions.

Assume (H_2) and (H_4), then quasi-arithmetic are defined as follows:

$$\widetilde{M}_{h,g}(f_1,...,f_n;\mathbf{p}) = h^{-1}\left(\int\limits_{Y^T} h(M_g(f_1,...,f_n;\mathbf{p}))\, dv^T(y_1,...y_n)\right),$$

where $T = \{1,...,n\}$,

$$M_g(f_1,...,f_n;\mathbf{p}) = g^{-1}\left(\sum_{i=1}^{n} p_i(g \circ f_i(y_i))\right),$$

and

$$\widetilde{M}_g(f_1,...,f_n;\mathbf{p}) = g^{-1}\left(\sum_{i=1}^{n} p_i \int g \circ f_i(y_i)\, dv_i\right).$$

We describe the monotonicity of these means as follows:

COROLLARY 3.2. *Assume (H_2) and (H_4). If $h\circ g^{-1}$ is convex and h is increasing, or $h\circ g^{-1}$ is concave and h is decreasing. Then*

$$\widetilde{M}_g(f_1,...,f_n;\mathbf{p}) \leqslant \widetilde{M}_{h,g}(f_1,...,f_n;\mathbf{p}) \leqslant \widetilde{M}_h(f_1,...,f_n;\mathbf{p}), \tag{11}$$

and if $g\circ h^{-1}$ is convex and g is decreasing, or $g\circ h^{-1}$ is concave and g is increasing.Then

$$\widetilde{M}_h(f_1,...,f_n;\mathbf{p}) \geqslant \widetilde{M}_{g,h}(f_1,...,f_n;\mathbf{p}) \geqslant \widetilde{M}_g(f_1,...,f_n;\mathbf{p}). \tag{12}$$

Proof. We set $q = h\circ g^{-1}$, $f_i = g\circ f_i$ in (2) and applying h^{-1}, we obtain (11). We also set $q = g\circ h^{-1}$, $f_i = h\circ f_i$ again in (2) and applying g^{-1}, we get (12). □

COROLLARY 3.3. *Let $r,s\in\mathbb{R}$ such that $s\leqslant r$, (Y_i,B_i,v_i), $i\in\mathbb{N}$ be probability spaces, $f_i : Y_i \to I$ be a square v_i-integrable function over $Y_i (i\in\mathbb{N})$ such that (3) and (4) are valid. Then*

$$\lim_{n\to\infty}\widetilde{M}_{r,s}(f_1,...,f_n) = \widetilde{M}_s(f_1),$$

and

$$\lim_{n\to\infty}\widetilde{M}_{s,r}(f_1,...,f_n) = \widetilde{M}_r(f_1).$$

Proof. Apply Theorem 1.3 and follow the proof of Corollary 3.1. □

COROLLARY 3.4. *Suppose (H_4) and let (Y_i,B_i,v_i), $i\in\mathbb{N}$ be probability spaces, $f_i : Y_i \to I$ be a square v_i-integrable function over $Y_i (i\in\mathbb{N})$ such that (3) and (4) are valid. Now, if $h\circ g^{-1}$ is bounded convex and h is increasing, or $h\circ g^{-1}$ is bounded concave and h is decreasing. Then*

$$\lim_{n\to\infty}\widetilde{M}_{h,g}(f_1,...,f_n) = \widetilde{M}_g(f_1),$$

and if $g\circ h^{-1}$ is bounded convex and g is decreasing, or $g\circ h^{-1}$ is bounded concave and g is increasing.Then

$$\lim_{n\to\infty}\widetilde{M}_{g,h}(f_1,...,f_n) = \widetilde{M}_h(f_1).$$

Proof. Apply Theorem 1.3 and follow the proof of Corollary 3.2. □

COROLLARY 3.5. *Let $r,s\in\mathbb{R}$ such that $s\leqslant r$ and assume (H_3), then we have*

$$\widetilde{M}_s(f) \leqslant \widetilde{M}_{r,s}(f;\mathbf{p}) \leqslant \widetilde{M}_r(f),$$

and

$$\widetilde{M}_r(f) \geqslant \widetilde{M}_{s,r}(f;\mathbf{p}) \geqslant \widetilde{M}_s(f).$$

Proof. Apply Theorem 1.4 (a) and follow the proof of Corollary 3.1. □

COROLLARY 3.6. *Let $r,s \in \mathbb{R}$ such that $s \leqslant r$ and assume (H_3), then we have*

$$\widetilde{M}_{r,s}(f;n+1) \leqslant \widetilde{M}_{r,s}(f;n) \leqslant \widetilde{M}_{r,s}(f;\mathbf{p}),$$

$$\widetilde{M}_{s,r}(f;n+1) \geqslant \widetilde{M}_{s,r}(f;n) \geqslant \widetilde{M}_{s,r}(f;\mathbf{p}).$$

Proof. Apply Theorem 1.4 (b) and follow the proof of Corollary 3.1. □

COROLLARY 3.7. *Let $r,s \in \mathbb{R}$ such that $s \leqslant r$ and assume (H_3), then we have*

$$\lim_{n\to\infty} \widetilde{M}_{r,s}(f;n) = \widetilde{M}_s(f),$$

$$\lim_{n\to\infty} \widetilde{M}_{s,r}(f;n) = \widetilde{M}_r(f).$$

Proof. Apply Theorem 1.4 (c) and follow the proof of Corollary 3.1. □

COROLLARY 3.8. *Assume (H_3) and (H_4). Now, if $h \circ g^{-1}$ is convex and h is increasing, or $h \circ g^{-1}$ is concave and h is decreasing. Then*

$$\widetilde{M}_g(f) \leqslant \widetilde{M}_{h,g}(f;\mathbf{p}) \leqslant \widetilde{M}_h(f),$$

and if $g \circ h^{-1}$ is convex and g is decreasing, or $g \circ h^{-1}$ is concave and g is increasing.Then

$$\widetilde{M}_h(f) \geqslant \widetilde{M}_{g,h}(f;\mathbf{p}) \geqslant \widetilde{M}_g(f).$$

Proof. Apply Theorem 1.4 (a) and follow the proof of Corollary 3.2. □

COROLLARY 3.9. *Assume (H_3) and (H_4). Now, if $h \circ g^{-1}$ is convex and h is increasing, or $h \circ g^{-1}$ is concave and h is decreasing. Then*

$$\widetilde{M}_{h,g}(f;n+1) \leqslant \widetilde{M}_{h,g}(f;n) \leqslant \widetilde{M}_{h,g}(f;\mathbf{p}),$$

and if $g \circ h^{-1}$ is convex and g is decreasing, or $g \circ h^{-1}$ is concave and g is increasing.Then

$$\widetilde{M}_{g,h}(f;n+1) \geqslant \widetilde{M}_{g,h}(f;n) \geqslant \widetilde{M}_{g,h}(f;\mathbf{p}).$$

Proof. Apply Theorem 1.4 (b) and follow the proof of Corollary 3.2. □

COROLLARY 3.10. *Assume (H_3) and (H_4). Now, if $h \circ g^{-1}$ is convex and h is increasing, or $h \circ g^{-1}$ is concave and h is decreasing. Then*

$$\lim_{n\to\infty} \widetilde{M}_{h,g}(f;n) = \widetilde{M}_g(f),$$

and if $g \circ h^{-1}$ is convex and g is decreasing, or $g \circ h^{-1}$ is concave and g is increasing.Then

$$\lim_{n\to\infty} \widetilde{M}_{g,h}(f;n) = \widetilde{M}_h(f).$$

Proof. Apply Theorem 1.4 (c) and follow the proof of Corollary 3.2. □

Assume (H_3). Then associated to the core term of Theorem 5, we define the mixed means as follows:

$$\widetilde{M}_{r,s}(n,k,f;\mathbf{t}) = \begin{cases} \left(\int\limits_{X^k} M_s^r(f,\widetilde{M}_s;\mathbf{t};k) d\mu^k(x_1,\ldots,x_k) \right)^{\frac{1}{r}}; \ r \neq 0, \\ \exp\left(\int\limits_{X^k} \log M_s(f,\widetilde{M}_s;\mathbf{t};k) d\mu^k(x_1,\ldots,x_k) \right); r = 0, \end{cases}$$

where

$$M_s(f,\widetilde{M}_s;\mathbf{t};k) := \begin{cases} \left((1-t_1)f^s(x_1) + \sum\limits_{j=1}^{k-1} (1-t_{j+1})t_1\ldots t_j f^s(x_j) + t_1\ldots t_k \widetilde{M}_s^s(f) \right)^{\frac{1}{s}}; \ r \neq 0, \\ \exp\left((1-t_1)\log f(x_1) + \sum\limits_{j=1}^{k-1} (1-t_{j+1})t_1\ldots t_j \log f(x_j) + t_1\ldots t_k \log \widetilde{M}_0(f) \right)^{\frac{1}{s}}; \ r = 0. \end{cases}$$

COROLLARY 3.11. *Let $r,s \in \mathbb{R}$ such that $s \leqslant r$ and assume (H_3), then*

$$\widetilde{M}_s(f) \leqslant \widetilde{M}_{r,s}(n,1;f;\mathbf{t}) \leqslant \ldots \leqslant \widetilde{M}_{r,s}(n,n-1;f;\mathbf{t}) \leqslant \widetilde{M}_{r,s}(f;\mathbf{t}) \leqslant \widetilde{M}_r(f),$$

$$\widetilde{M}_r(f) \leqslant \widetilde{M}_{s,r}(n,1;f;\mathbf{t}) \leqslant \ldots \leqslant \widetilde{M}_{s,r}(n,n-1;f;\mathbf{t}) \leqslant \widetilde{M}_{s,r}(f;\mathbf{t}) \leqslant \widetilde{M}_s(f).$$

Proof. Apply Theorem 2.1 and follow the proof of Corollary 3.1. □

Assume (H_3) and (H_4). Then using (5) we define the generalized means as follows:

$$\widetilde{M}_{h,g}(n,k;f;\mathbf{t}) = h^{-1}\left(\int\limits_{X^k} h\left(M_g(f,\widetilde{M}_g;\mathbf{t};k) \right) d\mu^k(x_1,\ldots,x_k) \right),$$

where

$$M_g(f,\widetilde{M}_g;\mathbf{t};k) := g^{-1}\left((1-t_1) g\circ f(x_1) + \sum_{j=1}^{k-1} (1-t_{j+1})t_1\ldots t_j g\circ f(x_j) + t_1\ldots t_k g(\widetilde{M}_g(f)) \right).$$

COROLLARY 3.12. *Assume (H_3) and (H_4). Now, if $h\circ g^{-1}$ is convex and h is increasing, or $h\circ g^{-1}$ is concave and h is decreasing. Then*

$$\widetilde{M}_g(f) \leqslant \widetilde{M}_{h,g}(n,1;f;\mathbf{t}) \leqslant \ldots \leqslant \widetilde{M}_{h,g}(n,n-1;f;\mathbf{t}) \leqslant \widetilde{M}_{h,g}(f;\mathbf{t}) \leqslant \widetilde{M}_h(f),$$

and if $g\circ h^{-1}$ is convex and g is decreasing, or $g\circ h^{-1}$ is concave and g is increasing.Then

$$\widetilde{M}_h(f) \leqslant \widetilde{M}_{g,h}(n,1;f;\mathbf{t}) \leqslant \ldots \leqslant \widetilde{M}_{g,h}(n,n-1;f;\mathbf{t}) \leqslant \widetilde{M}_{g,h}(f;\mathbf{t}) \leqslant \widetilde{M}_g(f).$$

Proof. Apply Theorem 2.1 and follow the proof of Corollary 3.2. □

REMARK 3.13. Similar to Corollary 3.11 and Corollary 3.12, we can give the results for Theorem 1.1 and those will be the special cases of Corollary 3.11 and Corollary 3.12 with discrete measure.

4. Exponential convexity

DEFINITION 1. [11] A function $\phi : I \to \mathbb{R}$ is *n-exponentially convex* in the Jensen sense on I if

$$\sum_{k,l=1}^{n} \alpha_k \alpha_l \phi\left(\frac{x_k + x_l}{2}\right) \geqslant 0$$

holds for $\alpha_k \in \mathbb{R}$ and $x_k \in I$, $k = 1, 2, ..., n$.

A function $\phi : I \to \mathbb{R}$ is *n-exponentially convex* if it is *n-exponentially convex* in the Jensen sense and continuous on I.

REMARK 4.1. From the definition it is clear that 1-exponentially convex functions in the Jensen sense are in fact nonnegative functions. Also, n-exponentially convex functions in the Jensen sense are m-exponentially convex in the Jensen sense for every $m \in \mathbb{N}, m \leqslant n$.

PROPOSITION 4.2. *If $\phi : I \to \mathbb{R}$ is an n-exponentially convex function, then the matrix $\left[\phi\left(\frac{x_k + x_l}{2}\right)\right]_{k,l=1}^{m}$ is a positive semi-definite matrix for all $m \in \mathbb{N}, m \leqslant n$. Particularly,*

$$\det\left[\phi\left(\frac{x_k + x_l}{2}\right)\right]_{k,l=1}^{m} \geqslant 0$$

for all $m \in \mathbb{N}$, $m = 1, 2, ..., n$.

DEFINITION 2. A function $\phi : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.

A function $\phi : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

REMARK 4.3. It is easy to show that ϕ :I$\to \mathbb{R}$ is log-convex in the Jensen sense if and only if

$$\alpha^2 \phi(x) + 2\alpha\beta\phi\left(\frac{x+y}{2}\right) + \beta^2 \phi(y) \geqslant 0$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that any positive function is log-convex in the Jensen-sense if and only if it is 2-exponentially convex in the Jensen sense.

Also, using basic convexity theory it follows that a function is log-convex if and only if it is 2-exponentially convex.

When dealing with functions with different degree of smoothness divided differences are found to be very useful.

DEFINITION 3. The second order divided difference of a function $\phi : I \to \mathbb{R}$ at mutually different points $y_0, y_1, y_2 \in I$ is defined recursively by

$$[y_i;\phi] = \phi(y_i), \quad i = 0,1,2$$

$$[y_i, y_{i+1};\phi] = \frac{\phi(y_{i+1}) - \phi(y_i)}{y_{i+1} - y_i}, \quad i = 0,1$$

$$[y_0, y_1, y_2;\phi] = \frac{[y_1, y_2;\phi] - [y_0, y_1;\phi]}{y_2 - y_0}. \tag{13}$$

REMARK 4.4. The value $[y_0, y_1, y_2;\phi]$ is independent of the order of the points y_0, y_1, and y_2. By taking limits this definition may be extended to include the cases in which any two or all three points coincide as follows: $\forall y_0,\, y_1,\, y_2 \in I$

$$\lim_{y_1 \to y_0} [y_0, y_1, y_2;\phi] = [y_0, y_0, y_2;\phi] = \frac{\phi(y_2) - \phi(y_0) - \phi'(y_0)(y_2 - y_0)}{(y_2 - y_0)^2},\ y_2 \neq y_0$$

provided that ϕ' exists, and furthermore, taking the limits $y_i \to y_0, i = 1,2$ in (13), we get

$$[y_0, y_0, y_0;\phi] \ = \lim_{y_i \to y_0} [y_0, y_1, y_2;\phi] = \frac{\phi''(y_0)}{2} \text{ for } i = 1,2$$

provided that ϕ'' exist on I.

THEOREM 4.5. *Let $\Lambda = \{\phi_t : t \in J\}$, where J an interval in $\mathbb{R}$, be a family of functions defined on an interval $[a,b]$, such that the function $t \to [y_0, y_1, y_2;\phi_t]$ is n-exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in [a,b]$. Let Ψ^i $(i = 1,..,21)$ be be linear functionals defined as in (5). Then $t \to \Psi^i(\phi_t)$ is an n-exponentially convex function in the Jensen sense on J. If the function $t \to \Psi^i(\phi_t)$ is continuous on J, then it is n-exponentially convex on J.*

Proof. Consider any i such that $1 \leqslant i \leqslant 21$.
Let us define the function

$$\omega(y) = \sum_{k,l=1}^{n} b_k b_l \phi_{t_{kl}}(y),$$

where $t_{kl} = \frac{t_k + t_l}{2}$, $t_k \in J, b_k \in \mathbb{R}$, $k = 1,2,...,n$.

Since the function $t \to [y_0, y_1, y_2;\phi_t]$ is n-exponentially convex in the Jensen sense, we have

$$[y_0, y_1, y_2;\omega] = \sum_{k,l=1}^{n} b_k b_l [y_0, y_1, y_2;\phi_{t_{kl}}] \geqslant 0,$$

which implies that ω is convex function on $[a,b]$ and therefore we have $\Psi^i(\omega) \geqslant 0$, $i = 1,...,21$. Hence

$$\sum_{k,l=1}^{n} b_k b_l \Psi^i(\phi_{t_{kl}}) \geqslant 0.$$

We conclude that the function $t \to \Psi^i(\phi_t)$ is an n-exponentially convex function in the Jensen sense on J.

If the function $t \to \Psi^i(\phi_t)$ is continuous on J, then it is n-exponentially convex on J by definition. □

As a consequence of the above theorem we can give the following corollary.

COROLLARY 4.6. *Let $\Lambda = \{\phi_t : t \in J\}$, where J an interval in $\mathbb{R}$, be a family of functions defined on an interval $[a,b]$, such that the function $t \to [y_0, y_1, y_2; \phi_t]$ is exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in [a,b]$. Let Ψ^i $(i = 1,..,21)$ be be linear functionals defined as in (5). Then $t \to \Psi^i(\phi_t)$ is an exponentially convex function in the Jensen sense on J. If the function $t \to \Psi^i(\phi_t)$ is continuous on J, then it is exponentially convex on J.*

COROLLARY 4.7. *Let $\Lambda = \{\phi_t : t \in J\}$, where J an interval in $\mathbb{R}$, be a family of functions defined on an interval $[a,b]$, such that the function $t \to [y_0, y_1, y_2; \phi_t]$ is 2-exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in [a,b]$. Let Ψ^i $(i = 1,..,21)$ be be linear functionals defined in (5). Then the following statements hold:*

(i) If the function $t \to \Psi^i(\phi_t)$ is positive and continuous on J, then it is 2-exponentially convex on J, and thus log convex.

(ii) If the function $t \to \Psi^i(\phi_t)$ is positive then for every $s,t,u,v \in J$, such that $s \leqslant u$ and $t \leqslant v$, we have

$$\mathfrak{u}_{s,t}(\Psi^i, \Lambda) \leqslant \mathfrak{u}_{u,v}(\Psi^i, \Lambda) \tag{14}$$

where

$$\mathfrak{u}_{s,t}(\Psi^i, \Lambda) = \begin{cases} \left(\frac{\Psi^i(\phi_s)}{\Psi^i(\phi_t)} \right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(\frac{\frac{d}{ds}\Psi^i(\phi_s)}{\Psi^i(\phi_s)} \right), & s = t \end{cases} \tag{15}$$

for $\phi_s, \phi_t \in \Lambda$ and for the case $t = s$ we consider that the function $t \to \Psi^i(\phi_t)$ $(i = 1,...,21)$ is differentiable.

Proof. Consider i such that $1 \leqslant i \leqslant 21$.

(i) See Remark 4.3 and Theorem 4.5.

(ii) From the definition of convex function ϕ, we have the following inequality [12, page 2]

$$\frac{\phi(s)-\phi(t)}{s-t} \leqslant \frac{\phi(u)-\phi(v)}{u-v}, \tag{16}$$

$\forall s,t,u,v \in J$ such that $s \leqslant u, t \leqslant v, s \neq t, u \neq v$.

Since by (i), $\Psi^i(\phi_s)$ is log-convex, so set $\phi(x) = \log \Psi^i(\phi_s)$ in (16) we have

$$\frac{\log \Psi^i(\phi_s) - \log \Psi^i(\phi_t)}{s-t} \leqslant \frac{\log \Psi^i(\phi_u) - \log \Psi^i(\phi_v)}{u-v} \tag{17}$$

for $s \leqslant u, t \leqslant v, s \neq t, u \neq v$, which equivalent to (14). For $s=t, u=v$ follows from (17) by taking limit. □

EXAMPLE 1. We consider the class

$$\Lambda_1 = \{\phi_t : \mathbb{R} \to [0,\infty); t \in \mathbb{R}\}$$

where

$$\phi_t(x) = \begin{cases} \frac{1}{t^2} e^{tx}; & t \neq 0, \\ \frac{1}{2} x^2; & t = 0. \end{cases}$$

Then ϕ_t $(t \in \mathbb{R})$ is a convex function on $\mathbb{R}$ and $t \mapsto \phi_t''(x)$ is exponentially convex [7]. By similar reasoning as given in the proof of Theorem 4.5, we get the exponential convexity of $t \mapsto [y_0, y_1, y_2; \phi_t]$ (and hence the exponential convexity in Jensen sense). By using the Corollary 4.6, we get the exponential convexity of $t \mapsto \Psi^i(\phi_t)$, $(i = 1, \dots, 21)$ in Jensen sense. Also these mappings are continuous, therefore exponentially convex. Then from (15) we have

$$\mathfrak{u}_{s,t}(\Psi^i, \Lambda_1) = \begin{cases} \left(\frac{\Psi^i(\phi_s)}{\Psi^i(\phi_t)}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(\frac{\Psi^i(id\,\phi_s)}{\Psi^i(\phi_s)} - \frac{2}{s}\right), & s = t \neq 0, \\ \exp\left(\frac{\Psi^i(id\,\phi_0)}{3\Psi^i(\phi_0)}\right), & s = t = 0, \end{cases}$$

where $i = 1, \dots, 21$.

Also from (14) we have the monotonicity of functions $\mathfrak{u}_{s,t}(\Psi^i, \Lambda_1)$ in both parameters for $i = 1, \dots, 21$.

For positive $\Psi^i(\phi_t)$, $(i = 1, \dots, 21)$ Theorem 2.5 insures the existence of $m, M \in \mathbb{R}$ such that

$$m \leqslant \mathfrak{M}_{s,t}(\Psi^i, \Lambda_1) \leqslant M, \quad i = 1, \dots, 21,$$

where

$$\mathfrak{M}_{s,t}(\Psi^i, \Lambda_1) := \log \mathfrak{u}_{s,t}(\Psi^i, \Lambda_1), \quad i = 1, \dots, 21.$$

i.e $\mathfrak{M}_{s,t}(\Psi^i, \Lambda_1)$ for $i = 1, \dots, 21$ are means and the monotonicity of these means is evident from (14).

EXAMPLE 2. We consider the class

$$\Lambda_2 = \{\psi_t : (0,\infty) \to \mathbb{R}; t \in \mathbb{R}\}$$

where

$$\psi_t(x) := \begin{cases} \frac{x^t}{t(t-1)}; \ t \neq 0,1, \\ -\log x; \ t = 0, \\ x\log x; \ t = 1. \end{cases}$$

Then ψ_t $(t \in \mathbb{R})$ is a convex function for $x \in (0,\infty)$ and $t \mapsto \psi_t''(x)$ is exponentially convex. By similar arguments as given in Example 1 we get the exponential convexity of $t \mapsto \Psi^i(\psi_t)$, $(i = 1,...,21)$. Therefore from (15) we have

$$\mathfrak{u}_{s,t}(.,.,.,\Psi^i,\Lambda_2) = \begin{cases} \left(\frac{\Psi^i(\psi_s)}{\Psi^i(\psi_t)}\right)^{\frac{1}{s-t}}; \ s \neq t, \\ \exp\left(\frac{1-2t}{t(t-1)} - \frac{\Psi^i(\psi_t\psi_0)}{\Psi^i(\psi_t)}\right); \ s = t \neq 0,1, \\ \exp\left(1 - \frac{\Psi^i(\psi_0^2)}{2\Psi^i(\psi_0)}\right); \ s = t = 0, \\ \exp\left(-1 - \frac{\Psi^i(\psi_0\psi_1)}{2\Psi^i(\psi_1)}\right); \ s = t = 1. \end{cases}$$

where $i = 1,...,21$.

We consider $\Psi^i(\psi_t) > 0$, $(i = 1,...,21)$, then from Theorem 2.5, there exist $\bar{m}, \bar{M} \in \mathbb{R}$ such that

$$\bar{m} \leqslant \left(\frac{\Psi^i(\psi_s)}{\Psi^i(\psi_t)}\right)^{\frac{1}{s-t}} \leqslant \bar{M}; \quad (s \neq t), \quad i = 1,...,21. \tag{18}$$

The means $\mathfrak{u}_{s,t}(\Psi^i,\Lambda_2)$ are continuous, symmetric and monotone in both parameters (by use of (14)) for $i = 1,...,21$. Let $s,t,r \in \mathbb{R}$ then by substitutions $s \to \frac{s}{r}$, $t \to \frac{t}{r}$ $(r \neq 0, s \neq t)$, $x_j \to x_j^r$ for $i = 1,...,7$, $f_j \to f_j^r$ for $i = 8,9,10$ and $f \to f^r$ for $i = 11,...,21$ in (18) we get

$$\bar{m} \leqslant \left(\frac{\Psi^i(\mathbf{x}^r,\mathbf{p},\mathbf{t},\psi_{s/r})}{\Psi^i(\mathbf{x}^r,\mathbf{p},\mathbf{t},\psi_{t/r})}\right)^{\frac{1}{s-t}} \leqslant \bar{M}; \ i = 1,...,6,$$

$$\bar{m} \leqslant \left(\frac{\Psi^7(\mathbf{x}^r,\mathbf{p},\psi_{s/r})}{\Psi^7(\mathbf{x}^r,\mathbf{p},\psi_{t/r})}\right)^{\frac{1}{s-t}} \leqslant \bar{M}; \ \ \mathbf{x}^r := (x_1^r,...,x_n^r),$$

$$\bar{m} \leqslant \left(\frac{\Psi^i(\mathbf{f}^r,\mathbf{p},\psi_{s/r})}{\Psi^i(\mathbf{f}^r,\mathbf{p},\psi_{t/r})}\right)^{\frac{1}{s-t}} \leqslant \bar{M}; \ \ \mathbf{f}^r := (f_1^r,...,f_n^r), \ \ i = 8,9,10,$$

$$\bar{m} \leqslant \left(\frac{\Psi^i(\mathbf{p},f^r,\psi_{s/r})}{\Psi^i(\mathbf{p},f^r,\psi_{t/r})}\right)^{\frac{1}{s-t}} \leqslant \bar{M}; \ i = 11,...,15, \ i \neq 13,$$

$$\bar{m} \leqslant \left(\frac{\Psi^{13}(f^r,\psi_{s/r})}{\Psi^{13}(f^r,\psi_{t/r})}\right)^{\frac{1}{s-t}} \leqslant \bar{M},$$

$$\bar{m} \leqslant \left(\frac{\Psi^i(\mathbf{t}, f^r, \psi_{s/r})}{\Psi^i(\mathbf{t}, f^r, \psi_{t/r})}\right)^{\frac{1}{s-t}} \leqslant \bar{M}, \; i = 16, \ldots, 21.$$

We define means in three parameters as follows:

$$\mathfrak{u}_{s,t,r}(\mathbf{x}^r, \mathbf{p}, \mathbf{t}, \Psi^i, \Lambda_2) = \begin{cases} \left(\mathfrak{u}_{s/r,t/r}(\mathbf{x}^r, \mathbf{p}, \mathbf{t}, \Psi^i, \Lambda_2)\right)^{\frac{1}{r}}; & r \neq 0, \\ \mathfrak{u}_{s,t}(\log \mathbf{x}, \mathbf{p}, \mathbf{t}, \Psi^i, \Lambda_1); & r = 0, \end{cases}$$

where $\log \mathbf{x} = (\log x_1, \ldots, \log x_n)$ and i = 1,...,6.

$$\mathfrak{u}_{s,t,r}(\mathbf{x}^r, \mathbf{p}, \Psi^7, \Lambda_2) = \begin{cases} \left(\mathfrak{u}_{s/r,t/r}(\mathbf{x}^r, \mathbf{p}, \Psi^7, \Lambda_2)\right)^{\frac{1}{r}}; & r \neq 0, \\ \mathfrak{u}_{s,t}(\log \mathbf{x}, \mathbf{p}, \Psi^7, \Lambda_1); & r = 0. \end{cases}$$

The means $\mathfrak{u}_{s,t,r}(\mathbf{x}^r, \mathbf{p}, \Psi^7, \Lambda_2)$ are also given in [8] and [6].

$$\mathfrak{u}_{s,t,r}(\mathbf{f}^r, \mathbf{p}, \Psi^i, \Lambda_2) = \begin{cases} \left(\mathfrak{u}_{s/r,t/r}(\mathbf{f}^r, \mathbf{p}, \Psi^i, \Lambda_2)\right)^{\frac{1}{r}}; & r \neq 0, \\ \mathfrak{u}_{s,t}(\log \mathbf{f}, \mathbf{p}, \Psi^i, \Lambda_1); & r = 0, \end{cases}$$

where $\log \mathbf{f} = (\log f_1, \ldots, \log f_n)$ and $i = 8, 9, 10$.

$$\mathfrak{u}_{s,t,r}(\mathbf{p}, f^r, \Psi^i, \Lambda_2) = \begin{cases} \left(\mathfrak{u}_{s/r,t/r}(\mathbf{p}, f^r, \Psi^i, \Lambda_2)\right)^{\frac{1}{r}}; & r \neq 0, \\ \mathfrak{u}_{s,t}(\mathbf{p}, \log f, \Psi^i, \Lambda_1); & r = 0, \end{cases}$$

where $i = 11, \ldots, 15$ and $i \neq 13$.

$$\mathfrak{u}_{s,t,r}(f^r, \Psi^{13}, \Lambda_2) = \begin{cases} \left(\mathfrak{u}_{s/r,t/r}(f^r, \Psi^{13}, \Lambda_2)\right)^{\frac{1}{r}}; & r \neq 0, \\ \mathfrak{u}_{s,t}(\log f, \Psi^{13}, \Lambda_1); & r = 0. \end{cases}$$

$$\mathfrak{u}_{s,t,r}(\mathbf{t}, f^r, \Psi^i, \Lambda_2) = \begin{cases} \left(\mathfrak{u}_{s/r,t/r}(\mathbf{t}, f^r, \Psi^i, \Lambda_2)\right)^{\frac{1}{r}}; & r \neq 0, \\ \mathfrak{u}_{s,t}(\mathbf{t}, \log f, \Psi^i, \Lambda_1); & r = 0, \end{cases} \qquad i = 16, \ldots, 21.$$

The monotonicity of three parameter means is followed by the monotonicity and continuity of two parameter means.

EXAMPLE 3. We consider a class of convex functions

$$\Lambda_3 = \{\eta_t : (0, \infty) \to (0, \infty); t \in (0, \infty)\}$$

where

$$\eta_t(x) := \begin{cases} \frac{t^{-x}}{\log^2 t}; \; t \neq 1, \\ \frac{x^2}{2}; \; t = 1. \end{cases}$$

$t \mapsto \psi_t''(x)$ is exponentially convex, being the Laplace transform of a non-negative function (see [7], [14]). For the class Λ_3, (15) gives

$$\mathfrak{u}_{s,t}(\Psi^i,\Lambda_3) = \begin{cases} \left(\frac{\Psi^i(\eta_s)}{\Psi^i(\eta_t)}\right)^{\frac{1}{s-t}}; s \neq t, \\ \exp\left(-\frac{2}{t\log t} - \frac{\Psi^i(id\cdot\eta_t)}{t\Psi^i(\eta_t)}\right); \ s = t \neq 1, \\ \exp\left(-\frac{\Psi^i(id\cdot\eta_1)}{3\Psi^i(\eta_1)}\right); \ s = t = 1, \end{cases}$$

where $i = 1,...,21$ and monotonicity of $\mathfrak{u}_{s,t}(\Psi^i,\Lambda_3)$ is followed by (14).

We consider $\Psi^i(\eta_t) > 0$, $(i = 1,...,21)$, then for $\underline{\mathrm{m}},\underline{\mathrm{M}} \in \mathbb{R}$, Theorem 2.5 gives the means

$$\mathfrak{M}_{s,t}(\Psi^i,\Lambda_3) := -L(s,t)\log\mathfrak{u}_{s,t}(\Psi^i,\Lambda_3), \quad i = 1,...,21,$$

such that

$$\underline{\mathrm{m}} \leqslant \mathfrak{M}_{s,t}(\Psi^i,\Lambda_3) \leqslant \underline{\mathrm{M}}, \quad i = 1,...,21,$$

where $L(s,t)$ are logarithmic means

$$L(s,t) := \begin{cases} \frac{s-t}{\log s - \log t}; \ s \neq t, \\ t; \qquad\qquad s = t. \end{cases}$$

EXAMPLE 4. We consider a class of convex functions

$$\Lambda_4 = \{\gamma_t : (0,\infty) \to (0,\infty); t \in (0,\infty)\}$$

defined as

$$\gamma_t(x) := \frac{e^{-x\sqrt{t}}}{t}.$$

$t \mapsto \psi_t''(x) = e^{-x\sqrt{t}}$ is exponentially convex, being the Laplace transform of a non-negative function (see [7], [14]). For the class Λ_4, (15) becomes

$$\mathfrak{u}_{s,t}(\Psi^i,\Lambda_4) = \begin{cases} \left(\frac{\Psi^i(\gamma_s)}{\Psi^i(\gamma_t)}\right)^{\frac{1}{s-t}}; \ s \neq t, \\ \exp\left(-\frac{1}{t} - \frac{\Psi^i(id\cdot\gamma_t)}{2\sqrt{t}\Psi^i(\gamma_t)}\right); \ s = t. \end{cases}$$

where $i = 1,...,21$ and monotonicity of $\mathfrak{u}_{s,t}(\Psi^i,\Lambda_4)$ is followed by (14).

We consider $\Psi^i(\gamma_t) > 0$, $(i = 1,...,21)$, then for $\tilde{m},\tilde{M} \in \mathbb{R}$ Theorem 2.5 gives the means

$$\mathfrak{M}_{s,t}(\Psi^i,\Lambda_4) := -(\sqrt{s}+\sqrt{t})\log\mathfrak{u}_{s,t}(\Psi^i,\Lambda_4), \quad i = 1,...,21,$$

such that

$$\tilde{m} \leqslant \mathfrak{M}_{s,t}(\Psi^i,\Lambda_4) \leqslant \tilde{M}, \quad i = 1,...,21.$$

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