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MATHEMATICAL INEQUALITIES FOR BIPARAMETRIC EXTENDED INFORMATION MEASURES

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Abstract. In this paper we introduce quasilinear-type divergences defined by the two-parameter generalization of the logarithm. Jeffreys and Jensen-Shannon divergence are also extended to biparametric forms.

1. Preliminaries

The study of natural phenomena that deviate from standard statistical distributions increased the interest in alternative definitions of the information measures. In 1988, Tsallis [12] introduced one-parameter extension of Shannon entropy by

$$H_q(\mathbf{p}) \equiv -\sum_{j=1}^{n} p_j^q \ln_q p_j = \sum_{j=1}^{n} p_j \ln_q \frac{1}{p_j},\ (q \geqslant 0, q \neq 1) \tag{1}$$

where $\mathbf{p} = \{p_1, p_2, \cdots, p_n\}$ is a probability mass function with $p_j > 0$ for all $j = 1, 2, \cdots, n$.

Here the deformation of the logarithm ($q-$logarithm) is defined by $\ln_q(x) \equiv \frac{x^{1-q}-1}{1-q}$ for $x > 0$, where the parameter q is a measure of non-extensivity of the system. Tsallis entropy includes Shannon entropy in the limiting sense:

$$\lim_{q\to 1} H_q(\mathbf{p}) = H_1(\mathbf{p}) \equiv -\sum_{j=1}^{n} p_j \log p_j. \tag{2}$$

Throughout the paper we consider $\mathbf{p} = \{p_1, p_2, \cdots, p_n\}$ and $\mathbf{r} = \{r_1, r_2, \cdots, r_n\}$ with $p_j > 0, r_j > 0$ for all $j = 1, 2, \cdots, n$ to be probability distributions. Tsallis relative entropy (divergence) is given by

$$D_q(\mathbf{p}||\mathbf{r}) \equiv \sum_{j=1}^{n} p_j^q (\ln_q p_j - \ln_q r_j) = -\sum_{j=1}^{n} p_j \ln_q \frac{r_j}{p_j}. \tag{3}$$

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It converges to the classic Kullback-Leibler information,

$$\lim_{q\to 1} D_q(\mathbf{p}||\mathbf{r}) = D_1(\mathbf{p}||\mathbf{r}) \equiv \sum_{j=1}^{n} p_j(\log p_j - \log r_j). \tag{4}$$

For $x > 0$ and $q \geqslant 0$ with $q \neq 1$, the q-exponential function represents the inverse function of the q-logarithm, that is $\exp_q(x) \equiv \{1+(1-q)x\}^{1/(1-q)}$, if $1+(1-q)x > 0$ and vanishes if its argument is nonpositive.

The Jeffreys divergence is defined by

$$J_1(\mathbf{p}||\mathbf{r}) \equiv D_1(\mathbf{p}||\mathbf{r}) + D_1(\mathbf{r}||\mathbf{p}) \tag{5}$$

and the Jensen-Shannon divergence is defined as

$$JS_1(\mathbf{p}||\mathbf{r}) \equiv \frac{1}{2} D_1\left(\mathbf{p}||\frac{\mathbf{p}+\mathbf{r}}{2}\right) + \frac{1}{2} D_1\left(\mathbf{r}||\frac{\mathbf{p}+\mathbf{r}}{2}\right). \tag{6}$$

See [4], [10].

For a continuous and strictly monotonic function ψ on $(0,\infty)$ and $q \geqslant 0$ with $q \neq 1$, Tsallis quasilinear entropy (q-quasilinear entropy; see [8]) is defined by

$$I_q^{\psi}(\mathbf{p}) \equiv \ln_q \psi^{-1}\left(\sum_{j=1}^{n} p_j \psi\left(\frac{1}{p_j}\right)\right). \tag{7}$$

Obviously $I_q^{\ln_q}(\mathbf{p}) = H_q(\mathbf{p})$.

The aim of this paper is to extend the quasilinear entropies from Tsallis statistical viewpoints, providing biparametric generalizations. Additionally, we give such extensions for Jeffreys and Jensen-Shannon divergences. The interested reader is referred to [9] for the study of the uniparametric case.

2. Main results

Before stating the results we establish the notation. The two-parameter extended logarithmic function (see [11]) is given by the formula

$$\ln_{r,q} x \equiv \ln_q \exp \ln_r x = \frac{\exp \frac{1-q}{1-r}\left(x^{1-r}-1\right) - 1}{1-q}.$$

This is a decreasing function with respect to indices. Correspondingly, the inverse function of $\ln_{r,q}$ is denoted by

$$\exp_{r,q} x \equiv \exp_r \log \exp_q x.$$

We start from the Tsallis (r,q)-quasilinear entropies and Tsallis (r,q)-quasilinear divergences as they were defined in [9].

DEFINITION 2.1. For a continuous and strictly monotonic function ψ on $(0,\infty)$ and $r,q \geqslant 0$ with $r,q \neq 1$, the (r,q)-quasilinear entropy is defined by

$$I_{r,q}^{\psi}(\mathbf{p}) \equiv \ln_{r,q} \psi^{-1}\left(\sum_{j=1}^{n} p_j \psi\left(\frac{1}{p_j}\right)\right). \tag{8}$$

The above definition generalizes the Tsallis entropy to the context of quasilinear means. We emphasize here the mathematical significance of our definition. For $\psi(x) = \ln_{r,q}(x)$ we recover the entropic functional used in [11, Section 4]:

$$H_{r,q}(\mathbf{p}) \equiv \sum_{j=1}^{n} p_j \ln_{r,q} \frac{1}{p_j}.$$

This also gives rise to another case of interest

$$I_{\frac{2r-1}{r},q}^{x^{1-r}}(\mathbf{p}) = \ln_q \exp \ln_{\frac{2r-1}{r}} \left(\sum_{j=1}^{n} p_j^r\right)^{\frac{1}{1-r}} = \ln_q \exp\left\{\frac{r}{1-r}\left[\left(\sum_{j=1}^{n} p_j^r\right)^{\frac{1}{r}} - 1\right]\right\}, \tag{9}$$

which coincides with Arimoto's entropy for $q=1$ and $r = 1/\beta$, cf. [2], and with R-norm information measure, for $q=1$ and $R=r$, cf. [3].

DEFINITION 2.2. For a continuous and strictly monotonic function ψ on $(0,\infty)$ and $r,q \geqslant 0$ with $r,q \neq 1$, the (r,q)-quasilinear divergence is defined by

$$D_{r,q}^{\psi}(\mathbf{p}||\mathbf{r}) \equiv -\ln_{r,q} \psi^{-1}\left(\sum_{j=1}^{n} p_j \psi\left(\frac{r_j}{p_j}\right)\right). \tag{10}$$

For $\psi(x) = \ln_{r,q}(x)$ we obtain the following entropy that extends the usual Tsallis divergence:

$$D_{r,q}(\mathbf{p}||\mathbf{r}) \equiv -\sum_{j=1}^{n} p_j \ln_{r,q} \frac{r_j}{p_j}.$$

By analogy to the entropy computation, we find the following Arimoto type divergence:

$$D_{\frac{2r-1}{r},q}^{x^{1-r}}(\mathbf{p}||\mathbf{r}) = -\ln_q \exp\left\{-\frac{r}{1-r}\left[1 - \left(\sum_{j=1}^{n} p_j^r r_j^{1-r}\right)^{\frac{1}{r}}\right]\right\}. \tag{11}$$

LEMMA 2.3. (Young's inequality) *Let m, $n \geqslant 0$ and $p,q \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1$. If $p < 0$ (then $0 < q < 1$) or $0 < p < 1$ (then $q < 0$), then one has $\frac{m^p}{p} + \frac{n^q}{q} \leqslant mn$.*

Firstly we obtain an inequality involving the deformation of the logarithm which is of interest in itself.

LEMMA 2.4. *Let* $r>0$, $r\neq 1$. *Assume* $p,q\in\mathbb{R}$ *satisfy* $\frac{1}{1-p}+\frac{1}{1-q}=1$. *If* $1<p<2$ *or if* $1<q<2$, *then*

$$\ln_{r,p}x+\ln_{r,q}y\leqslant \exp\left(\ln_r x+\ln_r y\right)-1. \tag{12}$$

Proof. Using Young's inequality we obtain

$$\begin{aligned}\ln_p \exp\ln_r x+\ln_q \exp\ln_r y &= \frac{\exp\left((1-p)\ln_r x\right)-1}{1-p}+\frac{\exp\left((1-q)\ln_r y\right)-1}{1-q}\\ &\leqslant \exp\left(\ln_r x+\ln_r y\right)-1. \quad \square\end{aligned}$$

We can see this inequality illustrated in the figure below.

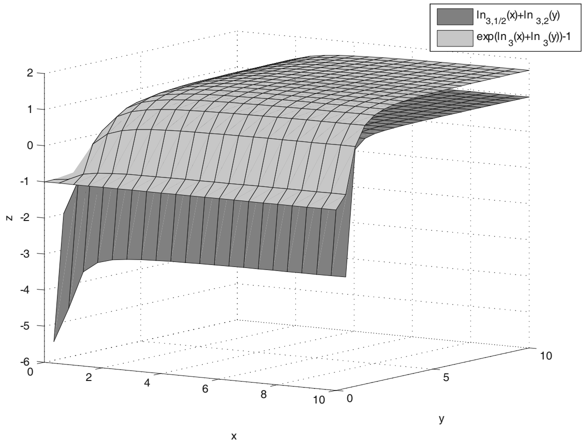

Figure 1: $\ln_{3,1/2}x+\ln_{3,2}y\leqslant \exp\left(\ln_3(x)+\ln_3(y)\right)-1$

When the parameters approach the value 1, the lemma reduces to the known inequality $\log xy\leqslant xy-1$.

PROPOSITION 2.5. *Let* r *be a real number. Assume* $p,q\in\mathbb{R}_+$ *satisfy* $p=\frac{1}{q}$. *If* $1<p<2$ *or if* $1<q<2$, *then*

$$D_{r,p}(\mathbf{p}||\mathbf{r})+H_{2-r,2-q}(\mathbf{p})\geqslant 1-\sum_{j=1}^{n}p_j\exp\left(\ln_r\frac{r_j}{p_j}+\ln_r p_j\right). \tag{13}$$

Proof. We apply Lemma 2.4. Since we have

$$\ln_{r,q}y=-\ln_{2-r,2-q}\frac{1}{y} \tag{14}$$

for all $y>0$, we get

$$\ln_{r,p}x+\ln_{r,q}y=\ln_{r,p}x-\ln_{2-r,2-q}\frac{1}{y}\leqslant \exp\left(\ln_r x+\ln_r y\right)-1.$$

Putting $x = \frac{r_j}{p_j}$ and $y = p_j$ and multiplying $-p_j$ and then taking the sum on both sides, it follows

$$-\sum_{j=1}^{n} p_j \ln_{r,p} \frac{r_j}{p_j} + \sum_{j=1}^{n} p_j \ln_{2-r,2-q} \frac{1}{p_j} \geqslant 1 - \sum_{j=1}^{n} p_j \exp\left(\ln_r \frac{r_j}{p_j} + \ln_r p_j\right)$$

which implies the inequality (13). The proof is completed. □

The case $r = 1$ reduces to a nicer form and was already proved in [9, Proposition 5.3] using the same technique.

Our next extensions incorporate known entropies and divergences: Jeffreys and Jensen-Shannon divergences are extended in the context of Tsallis theory to their biparametric forms.

DEFINITION 2.6. Let the Jeffreys (r,q)-divergence be

$$J_{r,q}(\mathbf{p}||\mathbf{r}) \equiv D_{r,q}(\mathbf{p}||\mathbf{r}) + D_{r,q}(\mathbf{r}||\mathbf{p}) \tag{15}$$

and the Jensen-Shannon (r,q)-divergence be

$$JS_{r,q}(\mathbf{p}||\mathbf{r}) \equiv \frac{1}{2} D_{r,q}\left(\mathbf{p}||\frac{\mathbf{p}+\mathbf{r}}{2}\right) + \frac{1}{2} D_{r,q}\left(\mathbf{r}||\frac{\mathbf{p}+\mathbf{r}}{2}\right). \tag{16}$$

We find that $J_{r,q}(\mathbf{p}||\mathbf{r}) = J_{r,q}(\mathbf{r}||\mathbf{p})$ and $JS_{r,q}(\mathbf{p}||\mathbf{r}) = JS_{r,q}(\mathbf{r}||\mathbf{p})$. That is, these divergences are symmetric.

S. Furuichi [7, Theorem 3.5] obtained the following refinement of Young's inequality:

LEMMA 2.7. (Refinement of Young's inequality) *Let a, $b \geqslant 0$ and $\lambda < 0$ or $\lambda > 1$. Then one has*

$$\lambda a + (1-\lambda) b \leqslant a^{\lambda} b^{1-\lambda} + \alpha \left(\sqrt{a} - \sqrt{b}\right)^2,$$

where $\alpha = \min\{\lambda, 1-\lambda\}$.

We can reformulate this in a more convenient way.

LEMMA 2.8. *Let m, $n \geqslant 0$ and $p,q \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1$. If $p < 0$ (then $0 < q < 1$) or $0 < p < 1$ (then $q < 0$), then one has*

$$\frac{m^p}{p} + \frac{n^q}{q} - mn \leqslant \alpha \left(m^{p/2} - n^{q/2}\right)^2, \tag{17}$$

where $\alpha = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

REMARK 2.9. Under the same assumptions, J. M. Aldaz (see [1, Lemma 2.1]) established for $p > 1$ (then $q > 1$) that

$$\alpha\left(m^{p/2}-n^{q/2}\right)^2 \leqslant \frac{m^p}{p}+\frac{n^q}{q}-mn \leqslant (1-\alpha)\left(m^{p/2}-n^{q/2}\right)^2. \tag{18}$$

The double inequality (18) is a direct application of a more general result concerning a refinement and a reverse of the Jensen's inequality provided in [5]. See also [6] for other helpful details.

Let the following conditions be fulfilled: $r > 0$, $r \neq 1$. In what follows $\alpha = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$. Hence we derive the following result.

LEMMA 2.10. *Assume $p,q \in \mathbb{R}$ satisfy $\frac{1}{1-p}+\frac{1}{1-q}=1$. If $1 < p < 2$ or if $1 < q < 2$, then*

$$\ln_{r,p} x+\ln_{r,q} y-\alpha\left[\exp\left(\frac{1-p}{2}\ln_r x\right)-\exp\left(\frac{1-q}{2}\ln_r y\right)\right]^2 \leqslant \exp\left(\ln_r x+\ln_r y\right)-1. \tag{19}$$

Proof. Using the above refinement of Young's inequality we obtain

$$\begin{aligned}\ln_p \exp\ln_r x+\ln_q \exp\ln_r y &= \frac{\exp\left((1-p)\ln_r x\right)-1}{1-p}+\frac{\exp\left((1-q)\ln_r y\right)-1}{1-q}\\ &\leqslant \exp\left(\ln_r x+\ln_r y\right)\\ &\quad +\alpha\left[\exp\left(\frac{1-p}{2}\ln_r x\right)-\exp\left(\frac{1-q}{2}\ln_r y\right)\right]^2-1. \quad \square\end{aligned}$$

We are now in a position to state and prove the following:

THEOREM 2.11. *Assume $p,q \in \mathbb{R}$ satisfy $\frac{1}{1-p}+\frac{1}{1-q}=1$. If $1 < p < 2$ or if $1 < q < 2$, then*

$$\begin{aligned}J_{r,p}(\mathbf{p}||\mathbf{r})+J_{r,q}(\mathbf{p}||\mathbf{r}) \geqslant 2-&\sum_{j=1}^{n}\left[p_j\exp\left(2\ln_r\frac{r_j}{p_j}\right)+r_j\exp\left(2\ln_r\frac{p_j}{r_j}\right)\right]\\ &-\alpha\sum_{j=1}^{n}\left[p_jE\left(\frac{r_j}{p_j}\right)+r_jE\left(\frac{p_j}{r_j}\right)\right],\end{aligned} \tag{20}$$

where $E(x)=\left[\exp\left(\frac{1-p}{2}\ln_r x\right)-\exp\left(\frac{1-q}{2}\ln_r x\right)\right]^2$.

Proof. In Lemma 2.10, we get for $x = y$

$$\ln_{r,p} x+\ln_{r,q} x \leqslant \exp\left(2\ln_r x\right)-1+\alpha\left[\exp\left(\frac{1-p}{2}\ln_r x\right)-\exp\left(\frac{1-q}{2}\ln_r x\right)\right]^2. \tag{21}$$

Putting $x=\frac{r_j}{p_j}$, multiplying $-p_j$ and then taking the sum on both sides, it follows

$$\begin{aligned} D_{r,p}(\mathbf{p}||\mathbf{r})+D_{r,q}(\mathbf{p}||\mathbf{r}) &= -\sum_{j=1}^{n} p_j \ln_{r,p}\frac{r_j}{p_j} - \sum_{j=1}^{n} p_j \ln_{r,q}\frac{r_j}{p_j} \\ &\geqslant 1-\sum_{j=1}^{n} p_j \exp\left(2\ln_r \frac{r_j}{p_j}\right) - \alpha\sum_{j=1}^{n} p_j E\left(\frac{r_j}{p_j}\right). \end{aligned} \tag{22}$$

Putting $x=\frac{p_j}{r_j}$, multiplying $-r_j$ and then taking the sum on both sides, it follows

$$\begin{aligned} D_{r,p}(\mathbf{r}||\mathbf{p})+D_{r,q}(\mathbf{r}||\mathbf{p}) &= -\sum_{j=1}^{n} r_j \ln_{r,p}\frac{p_j}{r_j} - \sum_{j=1}^{n} r_j \ln_{r,q}\frac{p_j}{r_j} \\ &\geqslant 1-\sum_{j=1}^{n} r_j \exp\left(2\ln_r \frac{p_j}{r_j}\right) - \alpha\sum_{j=1}^{n} r_j E\left(\frac{p_j}{r_j}\right). \end{aligned} \tag{23}$$

Summing the inequalities (22) and (23) the proof is completed. □

REMARK 2.12. Using (14), under the same assumptions as in Theorem 2.11, we get analogously

$$\begin{aligned} J_{2-r,2-p}(\mathbf{p}||\mathbf{r})+J_{2-r,2-q}(\mathbf{p}||\mathbf{r}) \leqslant{}& \sum_{j=1}^{n}\left[p_j \exp\left(2\ln_r \frac{p_j}{r_j}\right)+r_j \exp\left(2\ln_r \frac{r_j}{p_j}\right)\right]-2 \\ &+\alpha\sum_{j=1}^{n}\left[p_j E\left(\frac{p_j}{r_j}\right)+r_j E\left(\frac{r_j}{p_j}\right)\right]. \end{aligned} \tag{24}$$

We omit the computation. From (20) and (24) we get

$$\begin{aligned} & J_{r,p}(\mathbf{p}||\mathbf{r})+J_{r,q}(\mathbf{p}||\mathbf{r})-J_{2-r,2-p}(\mathbf{p}||\mathbf{r})-J_{2-r,2-q}(\mathbf{p}||\mathbf{r}) \\ \geqslant{}& 4-\sum_{j=1}^{n}(p_j+r_j)\left[\exp\left(2\ln_r \frac{p_j}{r_j}\right)+\exp\left(2\ln_r \frac{r_j}{p_j}\right)\right] \\ & -\alpha\sum_{j=1}^{n}(p_j+r_j)\left[E\left(\frac{p_j}{r_j}\right)+E\left(\frac{r_j}{p_j}\right)\right]. \end{aligned} \tag{25}$$

Our last result reads as follows.

THEOREM 2.13. *Assume* $p,q\in\mathbb{R}$ *satisfy* $\frac{1}{1-p}+\frac{1}{1-q}=1$. *If* $1<p<2$ *or if* $1<q<2$, *then*

$$\begin{aligned} JS_{r,p}(\mathbf{p}||\mathbf{r})+JS_{r,q}(\mathbf{p}||\mathbf{r}) \geqslant{}& 1-\frac{1}{2}\sum_{j=1}^{n}\left[p_j \exp\left(2\ln_r \frac{p_j+r_j}{2p_j}\right)+r_j \exp\left(2\ln_r \frac{p_j+r_j}{2r_j}\right)\right] \\ &-\frac{\alpha}{2}\sum_{j=1}^{n}\left[p_j E\left(\frac{p_j+r_j}{2p_j}\right)+r_j E\left(\frac{p_j+r_j}{2r_j}\right)\right], \end{aligned} \tag{26}$$

where $E(x)=\left[\exp\left(\frac{1-p}{2}\ln_r x\right)-\exp\left(\frac{1-q}{2}\ln_r x\right)\right]^2$.

Proof. From (22), for $\mathbf{r} \to \frac{\mathbf{p}+\mathbf{r}}{2}$ we get

$$D_{r,p}\left(\mathbf{p}||\frac{\mathbf{p}+\mathbf{r}}{2}\right)+D_{r,q}\left(\mathbf{p}||\frac{\mathbf{p}+\mathbf{r}}{2}\right) \geqslant 1-\sum_{j=1}^{n} p_j \exp\left(2\ln_r \frac{p_j+r_j}{2p_j}\right)-\alpha\sum_{j=1}^{n} p_j E\left(\frac{p_j+r_j}{2p_j}\right). \tag{27}$$

Analogously from (23) for $\mathbf{p} \to \frac{\mathbf{p}+\mathbf{r}}{2}$ we get

$$D_{r,p}\left(\mathbf{r}||\frac{\mathbf{p}+\mathbf{r}}{2}\right)+D_{r,q}\left(\mathbf{r}||\frac{\mathbf{p}+\mathbf{r}}{2}\right) \geqslant 1-\sum_{j=1}^{n} r_j \exp\left(2\ln_r \frac{p_j+r_j}{2r_j}\right)-\alpha\sum_{j=1}^{n} r_j E\left(\frac{p_j+r_j}{2r_j}\right). \tag{28}$$

This completes the proof. □

REMARK 2.14. Similarly the following inequality holds:

$$\begin{aligned} &JS_{2-r,2-p}(\mathbf{p}||\mathbf{r})+JS_{2-r,2-q}(\mathbf{p}||\mathbf{r}) \\ &\leqslant -1+\frac{1}{2}\sum_{j=1}^{n}\left[p_j \exp\left(2\ln_r \frac{2p_j}{p_j+r_j}\right)+r_j \exp\left(2\ln_r \frac{2r_j}{p_j+r_j}\right)\right] \\ &\quad +\frac{\alpha}{2}\sum_{j=1}^{n}\left[p_j E\left(\frac{2p_j}{p_j+r_j}\right)+r_j E\left(\frac{2r_j}{p_j+r_j}\right)\right]. \end{aligned} \tag{29}$$

As we have seen in all these examples, in many cases the use of the (r,q) - generalized logarithmic function nicely completes the picture obtained with the q-logarithm and can be useful in applied areas (signal and image processing, information theory).

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