REFINEMENTS OF SOME MAJORIZATION TYPE INEQUALITIES

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Abstract. In this paper, we prove refinements of some companion inequalities to the Jensen inequality, namely Slater's inequality and the inequalities obtained by Matić and Pečarić (2000). We also give refinements of majorization type inequalities, generalized weighted Favard and Berwald inequalities.

1. Introduction

Let X be a real normed linear space and let X^* be the algebraic dual space of X, that is, the real space of all linear functionals $x^* : X \to \mathbb{R}$. If $\phi : C \to \mathbb{R}$ is a convex function defined on a convex open subset C of X, then for any fixed point $y \in C$ we can define the abstract subdifferential $\partial \phi(y)$ of ϕ at y as:

$$\partial \phi(y) := \{a^*(y;.) \in X^* : \phi(x) \geqslant \phi(y) + a^*(y;x-y), \text{ for all } x \in C\}$$

The set $\partial \phi(y)$ is non empty ([21, p. 108 Theorem B]). Also, when ϕ is strictly convex, the inequality

$$\phi(x) \geqslant \phi(y) + a^*(y;x-y), \text{ for all } x,y \in C \tag{1}$$

is strict unless $x = y$.

In the simplest case when $\phi : (a,b) \to \mathbb{R}$ is a convex function defined on an open interval (a,b) in $\mathbb{R}$, for any $y \in (a,b)$ we have that $a^*(y;.)$ is given by $a^*(y;x) = mx$, $x \in \mathbb{R}$ where $m \in [\phi'_-(y), \phi'_+(y)]$. For convenience we shall always take $m = \phi'_+(y)$ and in this case (1) becomes

$$\phi(x) \geqslant \phi(y) + \phi'_+(y)(x-y), \text{ for all } x,y \in (a,b). \tag{2}$$

In 1981 Slater has proved an interesting companion inequality to the Jensen's inequality [22].

THEOREM 1.1. *Let* $\phi : (a,b) \to \mathbb{R}$ *be a monotonic convex function,* $x_i \in (a,b)$ *and* $p_i \geqslant 0$ $(i = 1,...,n)$ *with* $P_n = \sum_{i=1}^{n} p_i > 0$. *If* $\sum_{i=1}^{n} p_i \phi'_+(x_i) \neq 0$, *then*

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i \phi(x_i) \leqslant \phi\left(\frac{\sum_{i=1}^{n} p_i \phi'_+(x_i) x_i}{\sum_{i=1}^{n} p_i \phi'_+(x_i)}\right). \tag{3}$$

When ϕ *is strictly convex on* (a,b), *inequality* (3) *becomes equality if and only if* $x_i = c$ *for some* $c \in (a,b)$ *and for all* i *with* $p_i > 0$.

In [18] Pečarić noted that (3) remains true if we replace the condition of monotonicity of ϕ with $\frac{\sum_{i=1}^{n} p_i \phi'_+(x_i) x_i}{\sum_{i=1}^{n} p_i \phi'_+(x_i)} \in (a,b)$, which is more general and can holds for suitable points in (a,b) and not necessarily for monotone functions. In the same paper, one can find the multidimensional case of Slater's inequality. For the recent work on Slater's inequality see [2, 3].

The following theorem has been proved in [15].

THEOREM 1.2. *Let* $\phi : C \to \mathbb{R}$ *be a convex function defined on an open convex subset* C *in the normed real linear space* X. *For the given vectors* $x_i \in C$, $p_i \geqslant 0$ $(i = 1,..,n)$ *such that* $P_n := \sum_{i=1}^{n} p_i > 0$ *and suppose that*

$$\bar{x} := \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i \text{ and } \bar{y} := \frac{1}{P_n} \sum_{i=1}^{n} p_i \phi(x_i). \tag{4}$$

If c,d *are arbitrary vectors in* C, *then we have*

$$\phi(c) + a^*(c; \bar{x} - c) \leqslant \bar{y} \leqslant \phi(d) + \frac{1}{P_n} \sum_{i=1}^{n} a^*(x_i; x_i - d). \tag{5}$$

Also, when ϕ *is strictly convex, we have equality in the left inequality in* (5) *if and only if* $x_i = c$ *holds for all indices* i *with* $p_i > 0$, *while equality holds in the right inequality in* (5) *if and only if* $x_i = d$ *holds for all indices* i *with* $p_i > 0$.

In [8] Favard proved the following result: Let f be a non-negative continuous concave function on $[a,b]$, not identically zero and ϕ be a convex function on $[0,2\tilde{f}]$, where $\tilde{f} = \frac{1}{b-a} \int_a^b f(x) \, dx$, then

$$\int_0^1 \phi\left(2s\tilde{f}\right) ds = \frac{1}{2\tilde{f}} \int_0^{2\tilde{f}} \phi(y) \, dy \geqslant \frac{1}{b-a} \int_a^b \phi[f(x)] \, dx.$$

Favard [8] also proved the following result: Let f be a concave non-negative function on $[a,b] \subset \mathbb{R}$. If $q > 1$, then

$$\frac{1}{b-a} \int_a^b f^q(x) dx \leqslant \frac{2^q}{q+1} \left(\frac{1}{b-a} \int_a^b f(x) dx\right)^q.$$

Some generalizations of the Favard inequality and its reverse are also given in [10, pp. 412–413]. Moreover, Berwald (1947) [5] proved the following generalization of

Favard's inequality [10, p. 413–414]: Let f be a non-negative, continuous concave function, not identically zero on $[a,b]$, and ψ be a continuous and strictly monotonic function on $[0,y_0]$, where y_0 is sufficiently large. If $\bar{z}$ is the unique positive root of the equation

$$\frac{1}{\bar{z}} \int_0^{\bar{z}} \psi(y)\,dy = \frac{1}{b-a} \int_a^b \psi[f(x)]\,dx,$$

then for every function $\phi : [0,y_0] \to \mathbb{R}$ which is convex with respect to ψ, we have

$$\int_0^1 \phi(s\,\bar{z})\,ds = \frac{1}{\bar{z}} \int_0^{\bar{z}} \phi(y)\,dy \geqslant \frac{1}{b-a} \int_a^b \phi[f(x)]\,dx.$$

Berwald [5] also proved the following result: If f is a non-negative concave function on $[a,b]$, then for $0 < r < s$ we have

$$\left[\frac{s+1}{b-a} \int_a^b f^s(x)\,dx \right]^{\frac{1}{s}} \leqslant \left[\frac{r+1}{b-a} \int_a^b f^r(x)\,dx \right]^{\frac{1}{r}}.$$

In [14, 19], some generalizations of Favard and Berwald inequalities to the weighted and multidimensional cases are given. For further extension of these results for integral and discrete case see [11, 12, 20].

In this paper, we give a general inequality in discrete as well as in integral form and from this inequality, we obtain refinements of the inequalities in (5), Jensen and Slater inequalities and also refinement of Slater's inequality for monotone convex function. Finally, we present some refinements of the majorzation type inequalities and the generalized Favard and Berwald inequalities.

2. Main results

The main result states.

THEOREM 2.1. *Let $\phi : C \to \mathbb{R}$ be a convex function defined on an open convex subset C in the normed real linear space X and $x_i, y_i \in C$, $p_i \geqslant 0$, $i = 1,..,n$. Then*

$$\sum_{i=1}^n p_i \phi(x_i) - \sum_{i=1}^n p_i \phi(y_i) - \sum_{i=1}^n p_i a^*(y_i; x_i - y_i)$$

$$\geqslant \left| \left| \sum_{i=1}^n p_i \left| \phi(x_i) - \phi(y_i) \right| - \sum_{i=1}^n p_i \left| a^*(y_i; x_i - y_i) \right| \right| \right| \tag{6}$$

Proof. Take $x = x_i$ and $y = y_i (i = 1, 2, .., n)$ in (1) we have

$$\phi(x_i) \geqslant \phi(y_i) + a^*(y_i; x_i - y_i), \quad i.e,$$
$$\phi(x_i) - \phi(y_i) - a^*(y_i; x_i - y_i) \geqslant 0.$$

Therefore

$$\phi(x_i) - \phi(y_i) - a^*(y_i; x_i - y_i) = |\phi(x_i) - \phi(y_i) - a^*(y_i; x_i - y_i)|$$
$$\geqslant ||\phi(x_i) - \phi(y_i)| - |a^*(y_i; x_i - y_i)||. \quad (7)$$

Multiplying (7) by $p_i \geqslant 0, i = 1, .., n$ and summing over i from 1 to n we get

$$\sum_{i=1}^{n} p_i \phi(x_i) - \sum_{i=1}^{n} p_i \phi(y_i) - \sum_{i=1}^{n} p_i a^*(y_i; x_i - y_i)$$
$$\geqslant \sum_{i=1}^{n} p_i \left| |\phi(x_i) - \phi(y_i)| - |a^*(y_i; x_i - y_i)| \right|$$
$$\geqslant \left| \sum_{i=1}^{n} p_i \left| \phi(x_i) - \phi(y_i) \right| - \sum_{i=1}^{n} p_i \left| a^*(y_i; x_i - y_i) \right| \right| \quad (8)$$

which is equivalent to (6). □

Integral version of Theorem 2.1 for Lebesgue integral can be given:

THEOREM 2.2. *Let* (Ω, A, μ) *be a measure space with* $0 < \mu(\Omega) < \infty$ *and* $\phi :$ $(a, b) \to \mathbb{R}$ *be a convex function. If* $f, g : \Omega \to (a, b)$ *are such that* $\phi(f), \phi'(f), \phi'(f)f,$ $\phi(g), \phi'(g)$ *and* $\phi'(g)g$ *are in* $L^1(\mu)$, *then we have*

$$\int_{\Omega} \phi(f) d\mu - \int_{\Omega} \phi(g) d\mu - \int_{\Omega} \phi'_+(g)(f - g) d\mu$$
$$\geqslant \left| \int_{\Omega} |\phi(f) d\mu - \phi(g)| d\mu - \int_{\Omega} |\phi'_+(g)(f - g)| d\mu \right|. \quad (9)$$

The following theorem is the refinement of the left inequality in (5).

THEOREM 2.3. *Let* $\phi : C \to \mathbb{R}$ *be a convex function defined on an open convex subset* C *in the normed real linear space* X, $x_i \in C$, $p_i \geqslant 0$ $(i = 1, .., n)$ *such that* $P_n = \sum_{i=1}^{n} p_i > 0$ *and* $\bar{x}, \bar{y}$ *be as in* (4). *If* c *is arbitrary vector in* C, *then we have*

$$\bar{y} - \phi(c) - a^*(c; \bar{x} - c) \geqslant \left| \frac{1}{P_n} \sum_{i=1}^{n} p_i \left| \phi(x_i) - \phi(c) \right| - \frac{1}{P_n} \sum_{i=1}^{n} p_i \left| a^*(c; x_i - c) \right| \right|. \quad (10)$$

Proof. By substituting $y_i = c$ in (6) and using the fact that $a^*(c; .)$ is linear functional we get (10). □

The integral version of Theorem 2.3 gives us refinement of the inequality given in [15].

THEOREM 2.4. *Let* (Ω, A, μ) *be a measure space with* $0 < \mu(\Omega) < \infty$ *and* $\phi :$ $(a,b) \to \mathbb{R}$ *be a convex function. If* $f : \Omega \to (a,b)$ *is such that* $\phi(f), \phi'_+(f)$ *and* $\phi'_+(f)f$ *are in* $L^1(\mu)$, *then for any* $c \in (a,b)$ *we have*

$$\bar{g} - \phi(c) - \phi'_+(c)(\bar{f} - c)$$
$$\geqslant \left| \frac{1}{\mu(\Omega)} \int_\Omega |\phi(f)d\mu - \phi(c)|d\mu - \frac{|\phi'_+(c)|}{\mu(\Omega)} \int_\Omega |f - c|d\mu \right|, \tag{11}$$

where $\bar{g} = \frac{1}{\mu(\Omega)} \int_\Omega \phi(f)d\mu$, $\bar{f} = \frac{1}{\mu(\Omega)} \int_\Omega f d\mu$.

The following refinement of Jensen's inequality holds.

COROLLARY 2.5. *Under the assumptions of Theorem* 2.3 *we have*

$$\bar{y} - \phi(\bar{x}) \geqslant \left| \frac{1}{P_n} \sum_{i=1}^n p_i \left| \phi(x_i) - \phi(\bar{x}) \right| - \frac{1}{P_n} \sum_{i=1}^n p_i \left| a^*(\bar{x}; x_i - \bar{x}) \right| \right|. \tag{12}$$

The following refinement of Jenesen's inequality is a simple consequence of Theorem 2.4 (see [1, 9]).

COROLLARY 2.6. *Under the assumptions of Theorem* 2.4 *we have*

$$\bar{g} - \phi(\bar{f}) \geqslant \left| \frac{1}{\mu(\Omega)} \int_\Omega |\phi(f)d\mu - \phi(\bar{f})|d\mu - \frac{|\phi'_+(\bar{f})|}{\mu(\Omega)} \int_\Omega |f - \bar{f}|d\mu \right|. \tag{13}$$

Proof. Set $c = \bar{f}$ in (2.4) we obtain (13). □

The following theorem is the refinement of the right inequality in (5).

THEOREM 2.7. *Let* $\phi : C \to \mathbb{R}$ *be a convex function defined on an open convex subset* C *in the normed real linear space* X, $x_i \in C$, $p_i \geqslant 0$ $(i = 1,..,n)$ *such that* $P_n = \sum_{i=1}^n p_i > 0$ *and* $\bar{y}$ *be as in* (4). *If* d *is arbitrary vector in* C, *then we have*

$$\phi(d) - \bar{y} - \frac{1}{P_n} \sum_{i=1}^n p_i a^*(x_i; d - x_i)$$
$$\geqslant \left| \frac{1}{P_n} \sum_{i=1}^n p_i \left| \phi(d) - \phi(x_i) \right| - \frac{1}{P_n} \sum_{i=1}^n p_i \left| a^*(x_i; x_i - d) \right| \right|. \tag{14}$$

Proof. By substituting $x_i = d$ and $y_i = x_i$ in (6) we get (2.7). □

The following refinement of the inequality given in [15] holds.

COROLLARY 2.8. *Let the assumptions of Theorem 2.7 be satisfied. If there exists a vector $\bar{d} \in C$ such that the corresponding functional $a^*(\bar{d};.) \in \partial \phi(\bar{d})$ satisfies $a^*(\bar{d};.) = \frac{1}{P_n} \sum_{i=1}^{n} p_i a^*(x_i;.)$, then*

$$\phi(\bar{d}) - \bar{y} - \frac{1}{P_n} \sum_{i=1}^{n} p_i a^*(x_i; \bar{d} - x_i)$$

$$\geqslant \left| \frac{1}{P_n} \sum_{i=1}^{n} p_i \left| \phi(\bar{d}) - \phi(x_i) \right| - \frac{1}{P_n} \sum_{i=1}^{n} p_i \left| a^*(x_i; x_i - \bar{d}) \right| \right|. \qquad (15)$$

THEOREM 2.9. *Let (Ω, A, μ) be a measure space with $0 < \mu(\Omega) < \infty$ and $\phi : (a,b) \to \mathbb{R}$ be a convex function. If $f : \Omega \to (a,b)$ is such that $\phi(f), \phi'_+(f)$ and $\phi'_+(f)f$ are in $L^1(\mu)$, then for any $d \in (a,b)$ we have*

$$\phi(d) - \bar{g} - \int_{\Omega} \phi'_+(f)(d - f) d\mu$$

$$\geqslant \left| \frac{1}{\mu(\Omega)} \int_{\Omega} |\phi(d) - \phi(f)| d\mu - \frac{1}{\mu(\Omega)} \int_{\Omega} |\phi'_+(f)(f - d)| d\mu \right|, \qquad (16)$$

where $\bar{g} = \frac{1}{\mu(\Omega)} \int_{\Omega} \phi(f) d\mu$.

COROLLARY 2.10. *Let (Ω, A, μ) be a measure space with $0 < \mu(\Omega) < \infty$ and $\phi : (a,b) \to \mathbb{R}$ be a differentiable function. If $f : \Omega \to (a,b)$ is such that $\phi(f), \phi'(f)$ and $\phi'(f)f$ are in $L^1(\mu)$, then there exists at least one $\bar{d} \in (a,b)$ such that $\phi'(\bar{d}) = \frac{1}{\mu(\Omega)} \int_{\Omega} \phi'(f) d\mu$, and*

$$\phi(\bar{d}) - \bar{g} - \int_{\Omega} \phi'(f)(\bar{d} - f) d\mu$$

$$\geqslant \left| \frac{1}{\mu(\Omega)} \int_{\Omega} |\phi(\bar{d}) - \phi(f)| d\mu - \frac{1}{\mu(\Omega)} \int_{\Omega} |\phi'(f)(f - \bar{d})| d\mu \right|, \qquad (17)$$

where $\bar{g} = \frac{1}{\mu(\Omega)} \int_{\Omega} \phi(f) d\mu$.

COROLLARY 2.11. *Under the assumptions of Theorem 2.7 we have*

$$\phi(\bar{x}) - \bar{y} - \frac{1}{P_n} \sum_{i=1}^{n} p_i a^*(x_i; \bar{x} - x_i)$$

$$\geqslant \left| \frac{1}{P_n} \sum_{i=1}^{n} p_i \left| \phi(\bar{x}) - \phi(x_i) \right| - \frac{1}{P_n} \sum_{i=1}^{n} p_i \left| a^*(x_i; x_i - \bar{x}) \right| \right|. \qquad (18)$$

Proof. By setting $d = \bar{x}$ in (2.7) we obtain (2.11). $\quad \square$

COROLLARY 2.12. *Under the assumptions of Theorem 2.4 we have*

$$\phi(\bar{f}) - \bar{g} - \int_{\Omega} \phi'_+(f)(\bar{f} - f) d\mu$$

$$\geqslant \left| \frac{1}{\mu(\Omega)} \int_{\Omega} |\phi(\bar{f}) - \phi(f)| d\mu - \frac{1}{\mu(\Omega)} \int_{\Omega} |\phi'_+(f)(f - \bar{f})| d\mu. \right. \tag{19}$$

REMARK 2.13. In fact (2.11) is further refinement of the counter part of the Jensen inequality given in [15] and in particular for the counter part of the Jensen inequality given in [6]. Also (2.12) is further refinement of the integral counter part of the jensen inequality given in [15].

Theorem 2.7 for the case when $X = \mathbb{R}$ can be stated which is of interest:

THEOREM 2.14. *Let $\phi : (a,b) \to \mathbb{R}$ be a convex function, $x_i \in (a,b)$, $p_i \geqslant 0$ $(i = 1,..,n)$ such that $P_n = \sum_{i=1}^n p_i > 0$ and $\bar{x}, \bar{y}$ be as in (4), then we have*

$$\phi(d) - \bar{y} - \frac{1}{P_n} \sum_{i=1}^n p_i \phi'_+(x_i)(d - x_i)$$

$$\geqslant \left| \frac{1}{P_n} \sum_{i=1}^n p_i |\phi(d) - \phi(x_i)| - \frac{1}{P_n} \sum_{i=1}^n p_i |\phi'_+(x_i)(d - x_i)| \right|. \tag{20}$$

The following refinement of the Slater's inequality holds:

COROLLARY 2.15. *Let $\phi : (a,b) \to \mathbb{R}$ be a convex function, $x_i \in (a,b)$, $p_i \geqslant 0$ $(i = 1,..,n)$ such that $P_n = \sum_{i=1}^n p_i > 0$ and $\bar{y}$ be as in (4). If $\sum_{i=1}^n p_i \phi'_+(x_i) \neq 0$ such that $\bar{\bar{x}} = \frac{\sum_{i=1}^n p_i \phi'_+(x_i) x_i}{\sum_{i=1}^n p_i \phi'_+(x_i)} \in (a,b)$, then*

$$\phi(\bar{\bar{x}}) - \bar{y} \geqslant \left| \frac{1}{P_n} \sum_{i=1}^n p_i |\phi(\bar{\bar{x}}) - \phi(x_i)| - \frac{1}{P_n} \sum_{i=1}^n p_i |\phi'_+(x_i)(\bar{\bar{x}} - x_i)| \right|. \tag{21}$$

Integral analogue of Corollary 2.15 can be given:

COROLLARY 2.16. *Let (Ω, A, μ) be a measure space with $0 < \mu(\Omega) < \infty$ and $\phi : (a,b) \to \mathbb{R}$ be a convex function. If $f : \Omega \to (a,b)$ is such that $\phi(f), \phi'_+(f)$ and $\phi'_+(f)f$ are all in $L^1(\mu)$, then*

$$\phi(\bar{\bar{f}}) - \frac{1}{\mu(\Omega)} \int_{\Omega} \phi(f) d\mu$$

$$\geqslant \left| \frac{1}{\mu(\Omega)} \int_{\Omega} |\phi(\bar{\bar{f}}) - \phi(f)| d\mu - \frac{1}{\mu(\Omega)} \int_{\Omega} |\phi'_+(f)(f - \bar{\bar{f}})| d\mu \right| \tag{22}$$

holds, whenever $\int_{\Omega} \phi'_+(f) d\mu \neq 0$ and $\bar{\bar{f}} = \frac{\int_{\Omega} \phi'_+(f) f d\mu}{\int_{\Omega} \phi'_+(f) d\mu} \in (a,b)$.

The following theorem is the refinement of Slater's inequality for monotone convex function.

THEOREM 2.17. *Let* $\phi : (a,b) \to \mathbb{R}$ *be a monotonic convex function,* $x_i \in (a,b)$, $p_i \geqslant 0$ $(i = 1,...,n)$ *such that* $P_n = \sum_{i=1}^{n} p_i > 0$ *and* $\bar{y}$ *be as in* (4). *If* $\sum_{i=1}^{n} p_i \phi'_+(x_i) \neq 0$ *and* $I = \left\{ i \in I_n = \{1,2,..,n\} : x_i \geqslant \bar{\bar{x}} = \frac{\sum_{i=1}^{n} p_i \phi'_+(x_i) x_i}{\sum_{i=1}^{n} p_i \phi'_+(x_i)} \right\}$, *then we have*

$$\phi(\bar{\bar{x}}) - \bar{y} \geqslant \left| \frac{1}{P_n} \sum_{i=1}^{n} p_i \mathrm{sgn}(x_i - \bar{\bar{x}}) \left[\phi(x_i) - x_i \phi'_+(x_i) + \bar{\bar{x}} \phi'_+(x_i) \right] + \phi(\bar{\bar{x}}) \left(1 - \frac{2P_I}{P_n} \right) \right|,$$

(23)

where $P_I = \sum_{i \in I} p_i$.

Proof. Consider the case when ϕ is non decreasing

$$\sum_{i=1}^{n} p_i \left| \phi(\bar{\bar{x}}) - \phi(x_i) \right| = \sum_{i \in I} p_i \left(\phi(x_i) - \phi(\bar{\bar{x}}) \right) + \sum_{i \in \bar{I}} p_i \left(\phi(\bar{\bar{x}}) - \phi(x_i) \right)$$

$$= \sum_{i \in I} p_i \phi(x_i) - \sum_{i \in \bar{I}} p_i \phi(x_i) - \sum_{i \in I} p_i \phi(\bar{\bar{x}}) + \sum_{i \in \bar{I}} p_i \phi(\bar{\bar{x}})$$

$$= \sum_{i=1}^{n} p_i \mathrm{sgn}(x_i - \bar{\bar{x}}) p_i \phi(x_i) - \phi(\bar{\bar{x}})(P_I - P_{I_n \setminus I}).$$

(24)

Similarly

$$\sum_{i=1}^{n} p_i \left| \phi'_+(x_i)(\bar{\bar{x}} - x_i) \right| = \sum_{i=1}^{n} p_i \mathrm{sgn}(x_i - \bar{\bar{x}})(x_i - \bar{\bar{x}}) \phi'_+(x_i).$$

(25)

Now by using (24) and (25) in (21) we get (23).

The case when ϕ is non increasing can be treated similarly. □

The integral analogue of Theorem 2.17 can be given:

THEOREM 2.18. *Let* (Ω, A, μ) *be a measure space with* $0 < \mu(\Omega) < \infty$, $\phi : (a,b) \to \mathbb{R}$ *be a monotone convex function and* $f : \Omega \to (a,b)$ *be such that* $\phi(f), \phi'_+(f)$ *and* $\phi'_+(f)f$ *are all in* $L^1(\mu)$. *If* $\int_{\Omega} \phi'_+(f) d\mu \neq 0$ *and* $\Omega' = \left\{ t \in (a,b) : f(t) \geqslant \bar{\bar{f}} = \frac{\int_{\Omega} \phi'_+(f) f d\mu}{\int_{\Omega} \phi'_+(f) d\mu} \right\}$, *then the inequality*

$$\phi(\bar{\bar{f}}) - \bar{g} \geqslant \left| \frac{1}{\mu(\Omega)} \int_{\Omega} \mathrm{sgn}(f - \bar{\bar{f}})[\phi(f) - f\phi'_+(f) + \bar{\bar{f}}\phi'_+(f)] - \phi(\bar{\bar{f}}) \left(1 - \frac{2\mu(\Omega')}{\mu(\Omega)} \right) \right|$$

(26)

holds, where $\bar{g} = \frac{1}{\mu(\Omega)} \int_{\Omega} \phi(f) d\mu$.

3. Refinements of the majorization type inequalities, Favard and Berwald inequalities

The purpose of this section is to give some refinements of the well known results. The following theorem is the refinement of the majorization inequality given in [12, 16, 13].

THEOREM 3.1. *Let* $\phi : (a,b) \to \mathbb{R}$ *be a convex function,* $\mathbf{p} = (p_1, p_2, ..., p_n)$, $\mathbf{x} = (x_1, ..., x_n)$ *and* $\mathbf{y} = (y_1, ..., y_n)$ *be n-tuples such that* $x_i, y_i \in (a,b)$, $p_i \geqslant 0$, $(i = 1, 2, .., n)$ *and satisfying*

$$\sum_{i=1}^{k} p_i y_i \leqslant \sum_{i=1}^{k} p_i x_i \text{ for } k = 1, ..., n-1, \tag{27}$$

and

$$\sum_{i=1}^{n} p_i y_i = \sum_{i=1}^{n} p_i x_i. \tag{28}$$

(i) If $\mathbf{y}$ *is decreasing n-tuple, then the inequality*

$$\sum_{i=1}^{n} p_i \phi(x_i) - \sum_{i=1}^{n} p_i \phi(y_i) \geqslant \left| \sum_{i=1}^{n} p_i \left| \phi(x_i) - \phi(y_i) \right| - \sum_{i=1}^{n} p_i \left| \phi'_+(y_i)(x_i - y_i) \right| \right| \tag{29}$$

holds.

(ii) If $\mathbf{x}$ *is increasing n-tuple, then the inequality*

$$\sum_{i=1}^{n} p_i \phi(y_i) - \sum_{i=1}^{n} p_i \phi(x_i) \geqslant \left| \sum_{i=1}^{n} p_i \left| \phi(y_i) - \phi(x_i) \right| - \sum_{i=1}^{n} p_i \left| \phi'_+(x_i)(y_i - x_i) \right| \right| \tag{30}$$

holds.

Proof. By taking $x = x_i$ and $y = y_i (i = 1, 2, .., n)$ in (2) and following the proof of Theorem 2.1, we have

$$\sum_{i=1}^{n} p_i \phi(x_i) - \sum_{i=1}^{n} p_i \phi(y_i) - \sum_{i=1}^{n} p_i \phi'_+(y_i)(x_i - y_i)$$

$$\geqslant \left| \sum_{i=1}^{n} p_i \left| \phi(x_i) - \phi(y_i) \right| - \sum_{i=1}^{n} p_i \left| \phi'_+(y_i)(x_i - y_i) \right| \right|. \tag{31}$$

If $\mathbf{y}$ is decreasing n-tuple then by using (27), (28) and the convexity of ϕ, we have, $\sum_{i=1}^{n} p_i \phi'_+(y_i)(x_i - y_i) \geqslant 0$ ([16], p. 32) and so

$$\sum_{i=1}^{n} p_i \phi(x_i) - \sum_{i=1}^{n} p_i \phi(y_i) \geqslant \sum_{i=1}^{n} p_i \phi(x_i) - \sum_{i=1}^{n} p_i \phi(y_i) - \sum_{i=1}^{n} p_i \phi'_+(y_i)(x_i - y_i)$$

which together with (3) gives (29). Similarly we can prove (30). □

The following theorem is the refinement of the majorization inequality given by Dragomir in [7].

Theorem 3.2. *Let* $\phi : (a,b) \to \mathbb{R}$ *be a convex function,* $x_i, y_i \in (a,b)$, $p_i \geqslant 0$ $(i = 1, 2, ..., n)$ *with* $P_n = \sum_{i=1}^{n} p_i > 0$. *If* $(x_i - y_i)_{(i=\overline{1,n})}$ *is increasing (decreasing),* $(y_i)_{(i=\overline{1,n})}$ *is increasing (decreasing) and satisfying* (28), *then the inequality*

$$\sum_{i=1}^{n} p_i \phi(x_i) - \sum_{i=1}^{n} p_i \phi(y_i) \geqslant \left| \sum_{i=1}^{n} p_i \left| \phi(x_i) - \phi(y_i) \right| - \sum_{i=1}^{n} p_i \left| \phi'_+(y_i)(x_i - y_i) \right| \right| \quad (32)$$

holds.

Proof. The idea of the proof is similar to the proof of Theorem 3.1. □

Remark 3.3. Let $\phi, y_i, x_i - y_i$ and p_i $(i = 1, .., n)$ be the same as in Theorem 3.2. If in addition, ϕ is increasing and $\sum_{i=1}^{n} p_i x_i \geqslant \sum_{i=1}^{n} p_i y_i$, then (32) holds.

The following theorem is an integral analogue of Theorem 3.1.

Theorem 3.4. *Let* w *be a positive weight function and* $f, g : [a, b] \to [c, d]$ *be integrable functions. Suppose that* $\phi : [c, d] \to \mathbb{R}$ *is a convex function and*

$$\int_a^x g(t)w(t)dt \leqslant \int_a^x f(t)w(t)dt \quad \text{for all } x \in [a, b]$$

and

$$\int_a^b g(t)w(t)dt = \int_a^b f(t)w(t)dt$$

holds.

(i) *If* g *is a decreasing function on* $[a, b]$, *then the following inequality holds*

$$\int_a^b \phi(f) w(t)dt - \int_a^b \phi(g) w(t)dt$$
$$\geqslant \left| \int_a^b w(t) \left| \phi(f) - \phi(g) \right| dt - \int_a^b w(t) \left| \phi'_+(g)(f - g) \right| dt \right|. \quad (33)$$

(ii) *If* f *is an increasing function on* $[a, b]$, *then the following inequality holds*

$$\int_a^b \phi(g) w(t)dt - \int_a^b \phi(f) w(t)dt$$
$$\geqslant \left| \int_a^b w(t) \left| \phi(g) - \phi(f) \right| dt - \int_a^b w(t) \left| \phi'_+(f)(g - f) \right| dt \right|. \quad (34)$$

Remark 3.5. Similarly we can give integral version of Theorem 3.2 which is in fact the refinement of the majorization inequality given in [4].

The following result is needed in the proof of the next theorem.

LEMMA 3.6. ([12]) *Let* $\mathbf{v} = (v_1,..,v_2)$ *be a positive n-tuple. If* $\mathbf{a} = (a_1,..,a_n)$ *is decreasing real n-tuple, then*

$$\sum_{i=1}^{n} a_i v_i \sum_{i=1}^{k} v_i \leqslant \sum_{i=1}^{k} a_i v_i \sum_{i=1}^{n} v_i, \ k = 1,2,..,n. \tag{35}$$

If $\mathbf{a}$ *is increasing real n-tuple, then the reverse inequality holds in* (35).

If $\mathbf{x} = (x_1,..,x_n), \mathbf{y} = (y_1,...,y_n)$ are two n-tuples with $y_i \neq 0$, $i = 1,2,..,n$, then we define the n-tuple $\frac{\mathbf{x}}{\mathbf{y}}$ by $(\frac{x_1}{y_1}, \frac{x_2}{y_2},..,\frac{x_n}{y_n})$.

The following theorem is the refinement of the generalized discrete weighted Favard's inequality.

THEOREM 3.7. *Let* $\phi : (0,1) \to \mathbb{R}$ *be a convex function,* $\mathbf{p} = (p_1,..,p_n)$, $\mathbf{x} = (x_1,...,x_n)$ *and* $\mathbf{y} = (y_1,...,y_n)$ *be positive n-tuples. Assume that* $u_i = \frac{x_i}{\sum_{i=1}^{n} p_i x_i}$, $z_i = \frac{y_i}{\sum_{i=1}^{n} p_i y_i}$, $i = 1,2,..,n$. *Consider the inequalities*

$$\sum_{i=1}^{n} p_i \phi(z_i) - \sum_{i=1}^{n} p_i \phi(u_i) \geqslant \left| \sum_{i=1}^{n} p_i | \phi(z_i) - \phi(u_i) | - \sum_{i=1}^{n} p_i | \phi'_+(u_i)(z_i - u_i) | \right| \tag{36}$$

and

$$\sum_{i=1}^{n} p_i \phi(u_i) - \sum_{i=1}^{n} p_i \phi(z_i) \geqslant \left| \sum_{i=1}^{n} p_i | \phi(u_i) - \phi(z_i) | - \sum_{i=1}^{n} p_i | \phi'_+(z_i)(u_i - z_i) | \right|. \tag{37}$$

(i) *Let* $\frac{\mathbf{x}}{\mathbf{y}}$ *be a decreasing n-tuple. If* $\mathbf{x}$ *is an increasing n-tuple, then* (36) *holds. If* $\mathbf{y}$ *is a decreasing n-tuple, then* (37) *holds.*

(ii) *Let* $\frac{\mathbf{x}}{\mathbf{y}}$ *be an increasing n-tuple. If* $\mathbf{y}$ *is an increasing n-tuple, then* (37) *holds. If* $\mathbf{x}$ *is a decreasing n-tuple, then* (36) *holds.*

Proof. Using Lemma 3.6 for a positive n-tuple $\mathbf{v} = \mathbf{yp}$ and a decreasing n-tuple $\mathbf{a} = \frac{\mathbf{x}}{\mathbf{y}}$, we have,

$$\sum_{i=1}^{k} p_i z_i \leqslant \sum_{i=1}^{k} p_i u_i, \ \text{for } k = 1,2,..,n-1$$

and

$$\sum_{i=1}^{n} p_i z_i = \sum_{i=1}^{n} p_i u_i.$$

Now if $\mathbf{x}$ is increasing, then by using Theorem 3.1(ii), we have (36) and if $\mathbf{y}$ is decreasing, then again by using Theorem 3.1(i), we have (37).

Similarly way can prove the remaining cases. $\square$

COROLLARY 3.8. *Let $\phi : [0,\infty) \to \mathbb{R}$ be convex function, $\mathbf{p} = (p_1,..,p_n)$ be positive n-tuples. Assume that $u_i = \frac{x_i}{\sum_{i=1}^n p_i x_i}, \bar{u}_i = \frac{i-1}{\sum_{i=1}^n p_i(i-1)}, \bar{z}_i = \frac{n-i}{\sum_{i=1}^n p_i(n-i)}, i = 1,2,..,n.$*

(i) *If $\mathbf{x} = (x_1,...,x_n)$ is a positive increasing concave n-tuple, then we have*

$$\sum_{i=1}^n p_i \phi(\bar{u}_i) - \sum_{i=1}^n p_i \phi(u_i) \geqslant \left| \sum_{i=1}^n p_i \big| \phi(\bar{u}_i) - \phi(u_i) \big| - \sum_{i=1}^n p_i \big| \phi'_+(u_i)(\bar{u}_i - u_i) \big| \right|. \quad (38)$$

(ii) *If $\mathbf{x} = (x_1,...,x_n)$ is an increasing convex real n-tuple with $x_1 = 0$, then we have*

$$\sum_{i=1}^n p_i \phi(u_i) - \sum_{i=1}^n p_i \phi(\bar{u}_i) \geqslant \left| \sum_{i=1}^n p_i \big| \phi(u_i) - \phi(\bar{u}_i) \big| - \sum_{i=1}^n p_i \big| \phi'_+(\bar{u}_i)(u_i - \bar{u}_i) \big| \right|. \quad (39)$$

(iii) *If $\mathbf{x} = (x_1,...,x_n)$ is a positive decreasing concave n-tuple, then we have*

$$\sum_{i=1}^n p_i \phi(\bar{z}_i) - \sum_{i=1}^n p_i \phi(u_i) \geqslant \left| \sum_{i=1}^n p_i \big| \phi(\bar{z}_i) - \phi(u_i) \big| - \sum_{i=1}^n p_i \big| \phi'_+(u_i)(\bar{z}_i - u_i) \big| \right|. \quad (40)$$

(iv) *If $\mathbf{x} = (x_1,...,x_n)$ is a decreasing convex real n-tuple with $x_n = 0$, then we have*

$$\sum_{i=1}^n p_i \phi(u_i) - \sum_{i=1}^n p_i \phi(\bar{z}_i) \geqslant \left| \sum_{i=1}^n p_i \big| \phi(u_i) - \phi(\bar{z}_i) \big| - \sum_{i=1}^n p_i \big| \phi'_+(\bar{z}_i)(u_i - \bar{z}_i) \big| \right|. \quad (41)$$

Proof. (i) By taking $y_1 = \varepsilon < \frac{x_1}{x_2}$, $y_i = i - 1$ $(2 \leqslant i \leqslant n)$ and by using the concavity of $\mathbf{x}$ we have $\frac{\mathbf{x}}{\mathbf{y}}$ is a decreasing n-tuple. Now as $\frac{\mathbf{x}}{\mathbf{y}}$ is a decreasing n-tuple and $\mathbf{x}$ is increasing by assumption therefore by using Theorem 3.7 (i) and taking $\varepsilon \to 0$, we have (38).

(ii) If $\mathbf{x}$ is an increasing convex real n-tuple and $x_1 = 0$, then $\frac{x_i}{i-1}$, $(2 \leqslant i \leqslant n)$ is increasing. Now as $\frac{x_i}{i-1}$, $(2 \leqslant i \leqslant n)$ is increasing and also $y_i = i - 1$, $(2 \leqslant i \leqslant n)$ is increasing, therefore by using Theorem 3.7 (ii), we have (39).

Similarly we can prove the remaining cases. □

The following corollary is an application of Theorem 3.7.

COROLLARY 3.9. *Let $\mathbf{p} = (p_1,..,p_n)$, $\mathbf{x} = (x_1,...,x_n)$ and $\mathbf{y} = (y_1,...,y_n)$ be positive n-tuples and $\phi(x) = x^p$, where $p > 1$ or $p < 0$. Consider the inequalities*

$$\left(\frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i y_i} \right)^p - \frac{\sum_{i=1}^n p_i x_i^p}{\sum_{i=1}^n p_i y_i^p} \geqslant \left| \sum_{i=1}^n p_i \right| \left(\frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i y_i} \right)^p \frac{y_i^p}{\sum_{i=1}^n p_i y_i^p} - \frac{x_i^p}{\sum_{i=1}^n p_i y_i^p} \Big|$$

$$- |p| \sum_{i=1}^n p_i \Big| \frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i y_i} \frac{y_i x_i^{p-1}}{\sum_{i=1}^n p_i y_i^p} - \frac{x_i^p}{\sum_{i=1}^n p_i y_i^p} \Big| \right| \quad (42)$$

and

$$\frac{\sum_{i=1}^n p_i x_i^p}{\sum_{i=1}^n p_i y_i^p} - \left(\frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i y_i} \right)^p \geqslant \left| \sum_{i=1}^n p_i \right| \left(\frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i y_i} \right)^p \frac{y_i^p}{\sum_{i=1}^n p_i y_i^p} - \frac{x_i^p}{\sum_{i=1}^n p_i y_i^p} \Big|$$

$$- |p| \sum_{i=1}^n p_i \Big| \left(\frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i y_i} \right)^{p-1} \frac{x_i y_i^{p-1}}{\sum_{i=1}^n p_i y_i^p} - \left(\frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i y_i} \right)^p \frac{y_i^p}{\sum_{i=1}^n p_i y_i^p} \Big| \right|. \quad (43)$$

(i) *Let $\frac{\mathbf{x}}{\mathbf{y}}$ be a decreasing n-tuple. If $\mathbf{x}$ is an increasing n-tuple, then (42) holds. If $\mathbf{y}$ is a decreasing n-tuple, then (43) holds.*

(ii) *Let $\frac{\mathbf{x}}{\mathbf{y}}$ be an increasing n-tuple. If $\mathbf{y}$ is an increasing n-tuple, then (43) holds. If $\mathbf{x}$ is a decreasing n-tuple, then (42) holds.*

The following result is an application of Corollary 3.8.

COROLLARY 3.10. *Let $\mathbf{p} = (p_1, .., p_n)$, $\mathbf{x} = (x_1, .., x_n)$ be a positive n-tuple and $\phi(x) = x^p$, where $p > 1$. Assume that $u_i = \frac{x_i}{\sum_{i=1}^n p_i x_i}$, $\bar{u}_i = \frac{i-1}{\sum_{i=1}^n p_i(i-1)}$, $\bar{z}_i = \frac{n-i}{\sum_{i=1}^n p_i(n-i)}$, $1 \leqslant i \leqslant n$, $w = \frac{\left(\sum_{i=1}^n p_i x_i\right)^p}{\sum_{i=1}^n p_i(i-1)^p}$ and $\bar{w} = \frac{\left(\sum_{i=1}^n p_i x_i\right)^p}{\sum_{i=1}^n p_i(n-i)^p}$.*

(i) *If $\mathbf{x} = (x_1, ..., x_n)$ is an increasing concave n-tuple, then we have*

$$\left(\frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i(i-1)}\right)^p - \frac{\sum_{i=1}^n p_i x_i^p}{\sum_{i=1}^n p_i(i-1)^p} \geqslant w \left| \sum_{i=1}^n p_i \left| \bar{u}_i^p - u_i^p \right| - \sum_{i=1}^n p_i \left| p u_i^{p-1}(\bar{u}_i - u_i) \right| \right|.$$

(ii) *If $\mathbf{x} = (x_1, ..., x_n)$ is an increasing convex real n-tuple with $x_1 = 0$, then we have*

$$\frac{\sum_{i=1}^n p_i x_i^p}{\sum_{i=1}^n p_i(i-1)^p} - \left(\frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i(i-1)}\right)^p \geqslant w \left| \sum_{i=1}^n p_i \left| u_i^p - \bar{u}_i{}^p \right| - \sum_{i=1}^n p_i \left| p \bar{u}_i{}^{p-1}(u_i - \bar{u}_i) \right| \right|.$$

(iii) *If $\mathbf{x} = (x_1, ..., x_n)$ is a decreasing concave n-tuple, then we have*

$$\left(\frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i(n-i)}\right)^p - \frac{\sum_{i=1}^n p_i x_i^p}{\sum_{i=1}^n p_i(n-i)^p} \geqslant \bar{w} \left| \sum_{i=1}^n p_i \left| \bar{z}_i^p - u_i^p \right| - \sum_{i=1}^n p_i \left| p u_i^{p-1}(\bar{z}_i - u_i) \right| \right|.$$

(iv) *If $\mathbf{x} = (x_1, ..., x_n)$ is decreasing convex real n-tuple with $x_n = 0$, then we have*

$$\frac{\sum_{i=1}^n p_i x_i^p}{\sum_{i=1}^n p_i(n-i)^p} - \left(\frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i(n-i)}\right)^p \geqslant \bar{w} \left| \sum_{i=1}^n p_i \left| u_i^p - \bar{z}_i^p \right| - \sum_{i=1}^n p_i \left| p \bar{z}_i^{p-1}(u_i - \bar{z}_i) \right| \right|.$$

The following theorem is the refinement of the generalized weighted Favard's inequality given in [11].

THEOREM 3.11. *Let $w, f, g : [a,b] \to \mathbb{R}^+$ be integrable functions and $\phi : \mathbb{R}^+ \to \mathbb{R}$ be a convex function. Assume that $h(t) = \frac{f(t)}{\int_a^b f(t)w(t)dt}$, $k(t) = \frac{g(t)}{\int_a^b g(t)w(t)dt}$. Consider the inequalities*

$$\int_a^b \phi(k)w(t)dt - \int_a^b \phi(h)w(t)dt$$

$$\geqslant \left| \int_a^b w(t) \left| \phi(k) - \phi(h) \right| dt - \int_a^b w(t) \left| \phi'_+(h)(k-h) \right| dt \right| \qquad (44)$$

and

$$\int_a^b \phi\,(h)\,w(t)dt - \int_a^b \phi\,(k)\,w(t)dt$$

$$\geqslant \left| \int_a^b w(t)\big|\phi\,(h) - \phi\,(k)\,\big|dt - \int_a^b w(t)\big|\phi'_+(k)(h-k)\big|dt \right|. \tag{45}$$

(i) *Let* $\frac{f}{g}$ *be decreasing function on* $[a,b]$. *If* f *is an increasing function on* $[a,b]$, *then* (3.11) *holds. If* g *is decreasing function on* $[a,b]$, *then* (3.11) *holds.*

(ii) *Let* $\frac{f}{g}$ *be increasing function on* $[a,b]$. *If* g *is an increasing function on* $[a,b]$, *then* (3.11) *holds. If* f *is decreasing function on* $[a,b]$, *then* (3.11) *holds.*

REMARK 3.12. If $x \to \phi(x)$ is a convex function, then $x \to \phi(kx)$, $k \in \mathbb{R}$ is also convex function. If f is a positive increasing concave function and $g(t) = t - a$, then (3.11) gives the refinement of the weighted Favard's inequality.

COROLLARY 3.13. *Let* $w,f,g : [a,b] \to \mathbb{R}^+$ *be integrable functions and* $\phi(x) = x^p$, *where* $p > 1$ *or* $p < 0$. *Assume that* $h(t) = \frac{f(t)}{\int_a^b f(t)w(t)dt}$, $k(t) = \frac{g(t)}{\int_a^b g(t)w(t)dt}$. *Consider the inequalities*

$$\left(\frac{\int_a^b f(t)w(t)dt}{\int_a^b g(t)w(t)dt} \right)^p - \frac{\int_a^b f^p(t)w(t)dt}{\int_a^b g^p(t)w(t)dt}$$

$$\geqslant \frac{\left(\int_a^b f(t)w(t)dt\right)^p}{\int_a^b g^p(t)w(t)dt} \left| \int_a^b w(t)\big|k^p - h^p\big|dt - \int_a^b w(t)\big|\phi'_+(h)(k-h)\big|dt \right| \tag{46}$$

and

$$\frac{\int_a^b f^p(t)w(t)dt}{\int_a^b g^p(t)w(t)dt} - \left(\frac{\int_a^b f(t)w(t)dt}{\int_a^b g(t)w(t)dt} \right)^p$$

$$\geqslant \frac{\left(\int_a^b f(t)w(t)dt\right)^p}{\int_a^b g^p(t)w(t)dt} \left| \int_a^b w(t)\big|h^p - k^p\big|dt - \int_a^b w(t)\big|\phi'_+(k)(h-k)\big|dt \right|. \tag{47}$$

(i) *Let* $\frac{f}{g}$ *be decreasing function on* $[a,b]$. *If* f *is an increasing function on* $[a,b]$, *then* (3.13) *holds. If* g *is decreasing function on* $[a,b]$, *then* (3.13) *holds.*

(ii) *Let* $\frac{f}{g}$ *be increasing function on* $[a,b]$. *If* g *is an increasing function on* $[a,b]$, *then* (3.13) *holds. If* f *is decreasing function on* $[a,b]$, *then* (3.13) *holds.*

REMARK 3.14. If f is a positive increasing concave function and if we substitute $g(t) = t - a$, $w(t) \equiv 1$ in (3.13), then we obtain the refinement of the classical Favard's inequality.

The following theorem is the refinement of the majorization inequality given in [12].

THEOREM 3.15. *Let* $\mathbf{p} = (p_1, p_2, ..., p_n)$, $\mathbf{x} = (x_1, ..., x_n)$ *and* $\mathbf{y} = (y_1, ..., y_n)$ *be positive* n*-tuples. Let* $\phi, \psi : [0, \infty) \to \mathbb{R}$ *be such that* ψ *is strictly increasing function and* $\phi \circ \psi^{-1}$ *is convex. Also suppose that*

$$\sum_{i=1}^{k} p_i \psi(y_i) \leqslant \sum_{i=1}^{k} p_i \psi(x_i) \ for \ k = 1, ..., n-1, \tag{48}$$

and

$$\sum_{i=1}^{n} p_i \psi(y_i) = \sum_{i=1}^{n} p_i \psi(x_i) \tag{49}$$

hold.

(i) *If* $\mathbf{y}$ *is decreasing* n*-tuple, then we have*

$$\sum_{i=1}^{n} p_i \phi(x_i) - \sum_{i=1}^{n} p_i \phi(y_i)$$

$$\geqslant \left| \sum_{i=1}^{n} p_i \left| \phi(x_i) - \phi(y_i) \right| - \sum_{i=1}^{n} p_i \left| (\phi \circ \psi^{-1})'_+ (\psi(y_i))(\psi(x_i) - \psi(y_i)) \right| \right|. \tag{50}$$

(ii) *If* $\mathbf{x}$ *is increasing* n*-tuple, then we have*

$$\sum_{i=1}^{n} p_i \phi(y_i) - \sum_{i=1}^{n} p_i \phi(x_i)$$

$$\geqslant \left| \sum_{i=1}^{n} p_i \left| \phi(y_i) - \phi(x_i) \right| - \sum_{i=1}^{n} p_i \left| (\phi \circ \psi^{-1})'_+ (\psi(x_i))(\psi(y_i) - \psi(x_i)) \right| \right|. \tag{51}$$

Proof. By using Theorem 3.1 for the convex function $f(x) = \phi \circ \psi^{-1}(x)$ and the for the n-tuples $(a_1, .., a_n)$ and $(b_1, .., b_n)$, where $a_i = \psi(x_i), b_i = \psi(y_i)$, we obtain the required inequalities. $\square$

Integral version of the above theorem is stated as:

THEOREM 3.16. *Let* w, f, g *be positive integrable functions on* $[a, b]$. *Suppose* $\phi, \psi : [0, \infty) \to \mathbb{R}$ *are such that* ψ *is strictly increasing function and* $\phi \circ \psi^{-1}$ *is convex. Also suppose that*

$$\int_{a}^{x} \psi(g(t)) w(t) dt \leqslant \int_{a}^{x} \psi(f(t)) w(t) dt \ \ for \ all \ x \in [a, b]$$

and

$$\int_{a}^{b} \psi(g(t)) w(t) dt = \int_{a}^{b} \psi(f(t)) w(t) dt$$

hold.

(i) *If g is a decreasing function on* $[a,b]$, *then the following inequality holds*

$$\int_a^b \phi\left(f\right)w(t)dt - \int_a^b \phi\left(g\right)w(t)dt$$

$$\geqslant \left| \int_a^b w(t)\left|\phi\left(f\right) - \phi\left(g\right)\right|dt - \int_a^b w(t)(\phi \circ \psi^{-1})'_+(\psi(g))(\psi(f) - \psi(g))\left|dt\right|.$$

(52)

(ii) *If f is an increasing function on* $[a,b]$, *then the following inequality holds*

$$\int_a^b \phi\left(g\right)w(t)dt - \int_a^b \phi\left(f\right)w(t)dt$$

$$\geqslant \left| \int_a^b w(t)\left|\phi\left(g\right) - \phi\left(f\right)\right|dt - \int_a^b w(t)\left|(\phi \circ \psi^{-1})'_+(\psi(f))(\psi(g) - \psi(f))\right|dt\right|.$$

(53)

The following theorem is the refinement of the generalized discrete weighted Berwald's inequality.

THEOREM 3.17. *Let* $\mathbf{p} = (p_1, p_2, ..., p_n)$, $\mathbf{x} = (x_1, ..., x_n)$ *and* $\mathbf{y} = (y_1, ..., y_n)$ *be positive n-tuples. Suppose* $\phi, \psi : [0, \infty) \to \mathbb{R}$ *are such that* ψ *is continuous and strictly increasing function and* $\phi \circ \psi^{-1}$ *is convex. Let* z_1 *be such that*

$$\sum_{i=1}^n p_i\psi(z_1 y_i) = \sum_{i=1}^n p_i\psi(x_i).$$

(54)

Consider the inequalities

$$\sum_{i=1}^n p_i\phi\left(z_1 y_i\right) - \sum_{i=1}^n p_i\phi\left(x_i\right)$$

$$\geqslant \left| \sum_{i=1}^n p_i\left|\phi\left(z_1 y_i\right) - \phi\left(x_i\right)\right| - \sum_{i=1}^n p_i\left|(\phi \circ \psi^{-1})'_+(\psi(x_i))(\psi(z_1 y_i) - \psi(x_i))\right| \right|$$

(55)

and

$$\sum_{i=1}^n p_i\phi\left(x_i\right) - \sum_{i=1}^n p_i\phi\left(z_1 y_i\right)$$

$$\geqslant \left| \sum_{i=1}^n p_i\left|\phi\left(x_i\right) - \phi\left(z_1 y_i\right)\right| - \sum_{i=1}^n p_i\left|(\phi \circ \psi^{-1})'_+(\psi(z_1 y_i))(\psi(x_i) - \psi(z_1 y_i))\right| \right|.$$

(56)

(i) *Let* $\frac{\mathbf{x}}{\mathbf{y}}$ *be a decreasing n-tuple. If* $\mathbf{x}$ *is an increasing n-tuple, then* (3.17) *holds. If* $\mathbf{y}$ *is a decreasing n-tuple, then* (3.17) *holds.*

(ii) *Let* $\frac{\mathbf{x}}{\mathbf{y}}$ *be an increasing n-tuple. If* $\mathbf{y}$ *is an increasing n-tuple, then* (3.17) *holds. If* $\mathbf{x}$ *is a decreasing n-tuple, then* (3.17) *holds.*

Proof. In [12] authors has shown the existence of z_1 and proved that

$$\sum_{i=1}^{k} p_i \psi(z_1 y_i) \leqslant \sum_{i=1}^{k} p_i \psi(x_i) \text{ for } k = 1, ..., n-1. \tag{57}$$

Now using (54) and (57) in Theorem 3.15, we obtain the required inequalities. □

COROLLARY 3.18. *Let* $\mathbf{p} = (p_1, p_2, ..., p_n)$ *be a positive n-tuple and* $\phi, \psi : [0, \infty)$ $\to \mathbb{R}$ *be such that* ψ *is continuous and strictly increasing function and* $\phi \circ \psi^{-1}$ *is convex. Let* z_1 *and* z_2 *be such that*

$$\sum_{i=1}^{n} p_i \psi(z_1(i-1)) = \sum_{i=1}^{n} p_i \psi(x_i) \tag{58}$$

and

$$\sum_{i=1}^{n} p_i \psi(z_2(n-i)) = \sum_{i=1}^{n} p_i \psi(x_i). \tag{59}$$

(i) If $\mathbf{x}$ *is an increasing concave n-tuple and* $x_1 = 0$, *then we have*

$$\sum_{i=1}^{n} p_i \phi(z_1(i-1)) - \sum_{i=1}^{n} p_i \phi(x_i) \geqslant \left| \sum_{i=1}^{n} p_i \left| \phi(z_1(i-1)) - \phi(x_i) \right| \right.$$

$$\left. - \sum_{i=1}^{n} p_i \left| (\phi \circ \psi^{-1})'_+ (\psi(x_i))(\psi(z_1(i-1)) - \psi(x_i)) \right| \right|.$$

(ii) If $\mathbf{x}$ *is an increasing convex n-tuple and* $x_1 = 0$, *then we have*

$$\sum_{i=1}^{n} p_i \phi(x_i) - \sum_{i=1}^{n} p_i \phi(z_1(i-1)) \geqslant \left| \sum_{i=1}^{n} p_i \left| \phi(x_i) - \phi(z_1(i-1)) \right| \right.$$

$$\left. - \sum_{i=1}^{n} p_i \left| (\phi \circ \psi^{-1})'_+ (\psi(z_1(i-1)))(\psi(x_i) - \psi(z_1(i-1))) \right| \right|.$$

(iii) If $\mathbf{x}$ *is a decreasing concave n-tuple and* $x_n = 0$, *then we have*

$$\sum_{i=1}^{n} p_i \phi(z_2(n-i)) - \sum_{i=1}^{n} p_i \phi(x_i) \geqslant \left| \sum_{i=1}^{n} p_i \left| \phi(z_2(n-i)) - \phi(x_i) \right| \right.$$

$$\left. - \sum_{i=1}^{n} p_i \left| (\phi \circ \psi^{-1})'_+ (\psi(x_i))(\psi(z_2(n-i)) - \psi(x_i)) \right| \right|.$$

(iv) If $\mathbf{x}$ *is a decreasing convex n-tuple and* $x_n = 0$, *then we have*

$$\sum_{i=1}^{n} p_i \phi(x_i) - \sum_{i=1}^{n} p_i \phi(z_2(n-i)) \geqslant \left| \sum_{i=1}^{n} p_i \left| \phi(x_i) - \phi(z_2(n-i)) \right| \right.$$

$$\left. - \sum_{i=1}^{n} p_i \left| (\phi \circ \psi^{-1})'_+ (\psi(z_2(n-i)))(\psi(x_i) - \psi(z_2(n-i))) \right| \right|.$$

The following refinement of the inequalities given in [17] is a simple consequence of the Theorem 3.17.

COROLLARY 3.19. *Let* $\mathbf{p} = (p_1, p_2, ..., p_n)$, $\mathbf{x} = (x_1, ..., x_n)$ *and* $\mathbf{y} = (y_1, ..., y_n)$ *be positive n-tuples and let* $\psi(x) = x^q$, $\phi(x) = x^p$, *where* $0 < q \leqslant p$.
Let $\frac{x}{y}$ *be a decreasing n-tuple. If $\mathbf{x}$ is an increasing n-tuple, then we have*

$$\left(\frac{\sum_{i=1}^{n} p_i x_i^q}{\sum_{i=1}^{n} p_i y_i^q} \right)^{\frac{p}{q}} - \frac{\sum_{i=1}^{n} p_i x_i^p}{\sum_{i=1}^{n} p_i y_i^p} \geqslant \frac{1}{\sum_{i=1}^{n} p_i y_i^p} \left| \sum_{i=1}^{n} p_i \right| \left(\frac{\sum_{i=1}^{n} p_i x_i^q}{\sum_{i=1}^{n} p_i y_i^q} \right)^{\frac{p}{q}} y_i^p - x_i^p \right|$$

$$- \frac{p}{q} \sum_{i=1}^{n} p_i \left| x_i^{p-q} \left(\frac{\sum_{i=1}^{n} p_i x_i^q}{\sum_{i=1}^{n} p_i y_i^q} y_i^q - x_i^q \right) \right| \right|.$$

Similarly we can give other possible results by using Theorem 3.17.

The following corollary is an application of Corollary 3.18.

COROLLARY 3.20. *Let* $\mathbf{p} = (p_1, p_2, ..., p_n)$ *and* $\mathbf{x} = (x_1, ..., x_n)$ *be positive n-tuples and* $\psi(x) = x^q$, $\phi(x) = x^p$, *where* $0 < q \leqslant p$. *Assume that* $u = \sum_{i=1}^{n} p_i (i-1)^p$.
If $\mathbf{x}$ is an increasing n-tuple, then

$$\left(\frac{\sum_{i=1}^{n} p_i x_i^q}{\sum_{i=1}^{n} p_i (i-1)^q} \right)^{\frac{p}{q}} - \frac{\sum_{i=1}^{n} p_i x_i^p}{u} \geqslant \frac{1}{u} \left| \sum_{i=1}^{n} p_i \right| \left(\frac{\sum_{i=1}^{n} p_i x_i^q}{\sum_{i=1}^{n} p_i (i-1)^q} \right)^{\frac{p}{q}} (i-1)^p - x_i^p \right|$$

$$- \frac{p}{q} \sum_{i=1}^{n} p_i \left| x_i^{p-q} \left(\frac{\sum_{i=1}^{n} p_i x_i^q}{\sum_{i=1}^{n} p_i (i-1)^q} (i-1)^q - x_i^q \right) \right| \right|.$$

Similarly we can give other possible results by using Corollary 3.18.

The following theorem is the refinement of the extension of the weighted Berwald inequality given in [11].

THEOREM 3.21. *Let* w, f, g *be positive integrable functions on* $[a, b]$. *Suppose* $\phi, \psi : [0, \infty) \to \mathbb{R}$ *are such that ψ is continuous and strictly increasing function and* $\phi \circ \psi^{-1}$ *is convex. Let z_1 be such that*

$$\int_a^b \psi(z_1 g(t)) w(t) dt = \int_a^b \psi(z_1 f(t)) w(t) dt. \tag{60}$$

Consider the inequalities

$$\int_a^b \phi(z_1 g(t)) w(t) dt - \int_a^b \phi(f(t)) w(t) dt \geqslant \left| \int_a^b w(t) \left| \phi(z_1 g(t)) - \phi(f(t)) \right| dt \right.$$

$$\left. - \int_a^b w(t) \left| (\phi \circ \psi^{-1})'_+ (\psi(f(t))) (\psi(z_1 g(t)) - \psi(f(t))) \right| dt \right| \tag{61}$$

and

$$\int_a^b \phi\left(f(t)\right)w(t)dt - \int_a^b \phi\left(z_1 g(t)\right)w(t)dt \geqslant \left|\int_a^b w(t)\big|\phi\left(f(t)\right) - \phi\left(z_1 g(t)\right)\big|dt\right.$$

$$\left. - \int_a^b w(t)(\phi\circ\psi^{-1})'_+(\psi(z_1 g(t)))(\psi(f(t)) - \psi(z_1 g(t)))\big|dt\right|. \qquad (62)$$

(i) *Let $\frac{f}{g}$ be decreasing function on $[a,b]$. If f is an increasing function on $[a,b]$, then (61) holds. If g is decreasing function on $[a,b]$, then (62) holds.*

(ii) *Let $\frac{f}{g}$ be increasing function on $[a,b]$. If g is an increasing function on $[a,b]$, then (62) holds. If f is decreasing function on $[a,b]$, then (61) holds.*

REMARK 3.22. If $z_1 > 0$, where z_1 is defined in Theorem 3.21, f is a positive increasing concave function and $g(t) = \frac{t-a}{b-a}$, then (61) gives refinement of the weighted Berwald's inequality.

REMARK 3.23. We can obtain integral version of Corollary 3.19 as an application of Theorem 3.21.

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