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OPERATOR MONOTONE FUNCTIONS, $A > B > 0$ AND $\log A > \log B$

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To the memory of my familiar persons
passed away by tsunami disaster
at Tohoku district on 2011.3.11
with deep sorrow

(*Communicated by Y. Seo*)

Abstract. Let $f(t)$ be any non-constant operator monotone function on $[0,\infty)$ and also let A and B be strictly positive operators on a Hilbert space H.

(i) *If $A > B$, then the following inequality holds*:

$$f(A^{\alpha}) > f(B^{\alpha}) \quad \text{for all} \quad \alpha \in (0,1].$$

(ii) *If* $\log A > \log B$, *then there exists* $\beta \in (0,1]$ *and following inequality holds*:

$$f(A^{\alpha}) > f(B^{\alpha}) \quad \text{for all} \quad \alpha \in (0,\beta].$$

1. $A > B > 0$, $\log A \geqslant \log B$ and $\log A > \log B$

A capital letter means a bounded linear operator on a complex Hilbert space H.

An operator T is said to be *positive* (denoted by $T \geqslant 0$) if $(Tx,x) \geqslant 0$ for all $x \in H$ and an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. *Chaotic order* is defined by $\log A \geqslant \log B$ for strictly positive operators A and B, and also *strictly chaotic order* is defined by $\log A > \log B$ for strictly positive operators A and B. The well known celebrated Löwner-Heinz inequality asserts that if $A \geqslant B \geqslant 0$, then $A^{\alpha} \geqslant B^{\alpha}$ for any $\alpha \in [0,1]$. This means that $t \mapsto t^{\alpha}$ is operator monotone. Another well known example of operator monotone is $t \mapsto \log t$ on $(0,\infty)$, that is, $\log A \geqslant \log B$ is weaker than the usual order $A \geqslant B \geqslant 0$.

We consider the following two operator monotone functions on $(0,\infty)$ in [8, p. 151] and [10, p. 131]:

$$\varphi(t) = \frac{t-1}{\log t} \quad \text{and} \quad \psi(t) = \frac{t\log -t+1}{\log^2 t}. \tag{1.1}$$

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THEOREM A. *Let A and B be strictly positive operators on a Hilbert space H. If* $\log > \log B$, *then there exists* $\beta \in (0,1]$ *and the following inequalities hold:*

$$\varphi(A^{\alpha}) > \varphi(B^{\alpha}) \quad \text{for all} \quad \alpha \in (0,\beta) \tag{1.2}$$

and

$$\psi(A^{\alpha}) > \psi(B^{\alpha}) \quad \text{for all} \quad \alpha \in (0,\beta) \tag{1.3}$$

where $\varphi(t)$ and $\psi(t)$ are defined in (1.1).

THEOREM B. *There exist strictly positive operators A and B such that* $\log A \geqslant \log B$*:*

$$\varphi(A^{\alpha}) \ngeqslant \varphi(B^{\alpha}) \quad \text{for any} \quad \alpha > 0 \tag{1.4}$$

and

$$\psi(A^{\alpha}) \ngeqslant \psi(B^{\alpha}) \quad \text{for any} \quad \alpha > 0 \tag{1.5}$$

where $\varphi(t)$ and $\psi(t)$ are defined in (1.1).

(1.2) and (1.4) are shown in [5], and (1.3) and also (1.5) are shown in [6].

In §2, we shall show Theorem 2.2 which is a further extension of Theorem A.

2. Relations among $A > B$, $\log A > \log B$ for $A, B > 0$ and operator monotone functions

We study relations among $A > B$, $\log A > \log B$ for $A, B > 0$ and operator monotone functions for strictly positive operators A and B.

THEOREM 2.1. *Let A and B be strictly positive operators on a Hilbert space H. If $A > B$, then the following inequality holds:*

$$f(A) > f(B) \tag{2.1}$$

for any non-constant operator monotone function f on $[0,\infty)$.

Proof. Let $A > B$. Then there exists some $\varepsilon > 0$ such that

$$A - B \geqslant \varepsilon I \tag{2.2}$$

so that $A + s \geqslant B + s + \varepsilon > B + s$ for $s \geqslant 0$, so that there exists some $\delta > 0$ such that

$$(B+s)^{-1} - (A+s)^{-1} \geqslant \delta I. \tag{2.3}$$

It is well known (for examples, [1], [11]) that a function f on $[0,\infty)$ is an operator monotone if and only if it has the representation

$$\begin{aligned} f(t) &= a + bt + \int_0^{\infty} \frac{ts}{t+s} dm(s) \\ &= a + bt + \int_0^{\infty} \left(s - \frac{s^2}{t+s}\right) dm(s) \end{aligned} \tag{2.4}$$

with $a \in \mathbb{R}$ and $b \geqslant 0$ and a positive measure m on $[0,\infty)$.

Then (2.2), (2.3) and (2.4) ensure the following (2.5)

$$f(A) - f(B) = b(A-B) + \int_0^\infty \{(B+s)^{-1} - (A+s)^{-1}\} s^2 dm(s) > 0 \tag{2.5}$$

so that we have (2.1) by (2.5). □

THEOREM 2.2. *Let $f(t)$ be any non-constant operator monotone function on $[0,\infty)$ and also let A and B be strictly positive operators.*

If $\log A > \log B$, then there exists $\beta \in (0,1]$ and the following inequality holds:

$$f(A^\alpha) > f(B^\alpha) \quad \textit{for all} \quad \alpha \in (0,\beta]. \tag{2.6}$$

Proof. Recall the following obvious relation (2.7):

$$X > Y > 0 \Longrightarrow X^\gamma > Y^\gamma \quad \text{for any} \quad \gamma \in (0,1]. \tag{2.7}$$

In fact $X > Y > 0$ ensures $X \geqslant Y + \varepsilon I > Y > 0$ for some $\varepsilon > 0$, then $X^\gamma \geqslant (Y+\varepsilon I)^\gamma > Y^\gamma$ for any $\gamma \in (0,1]$ by Löwner-Heinz inequality and we have (2.7).

[3, Corollary 2] asserts that

$$\log A > \log B \Longleftrightarrow \quad \text{there exists} \quad \beta \in (0,1] \quad \text{such that} \quad A^\beta > B^\beta. \tag{2.8}$$

Applying (2.7) for $\gamma = \dfrac{\alpha}{\beta} \in (0,1]$ to (2.8), then we have $A^\alpha > B^\alpha$ for any $\alpha \in (0,\beta]$, so that we have (2.6) by Theorem 2.1. □

REMARK 2.1. We remark that Theorem 2.2 is a further extension of Theorem A since Theorem 2.2 can be available for *any non-constant operator monotone functions* on $[0,\infty)$.

REMARK 2.2. For a simple proof of (2.7), we have only to put $f(t) = t^\gamma$ for $\gamma \in (0,1]$ in Theorem 2.1 since $f(t)$ is a typical well known operator monotone. In fact (2.7) is cited in [2, p. 477, Corollary 8.6.11].

3. Concluding remark and a conjecture

It is interesting to point out that an interesting contrast between $A > B$ and $\log A > \log B$ for $A, B > 0$ as follows.

REMARK 3.1. Let $f(t)$ be any non-constant operator monotone function on $[0,\infty)$ and also let A and B be strictly positive operators.

(i) If $A > B$, then the following inequality holds:

$$f(A^\alpha) > f(B^\alpha) \quad \text{for all} \quad \alpha \in (0,1]. \tag{3.1}$$

(ii) If $\log A > \log B$, then there exists $\beta \in (0,1]$ and the following inequality holds:

$$f(A^\alpha) > f(B^\alpha) \quad \text{for all} \quad \alpha \in (0,\beta]. \tag{3.2}$$

Proof. If $A > B$, then $A^\alpha > B^\alpha$ for any $\alpha \in (0,1]$ by (2.7) and we have (3.1) by Theorem 2.1 and (ii) is already shown in Theorem 2.2.

It is reasonable understanding that the condition $\log A > \log B$ in (ii) is weaker then $A > B > 0$ in (i), the corresponding result (3.2) is weaker than (3.1). □

Theorem 2.2, Theorem A and Theorem B suggest the following conjecture.

CONJECTURE. *There exist strictly positive operators A and B such that $\log A \geqslant \log B$, but $f(A^\alpha) \not\geqslant f(B^\alpha)$ for any non-constant operator monotone function $f(t)$ on $[0,\infty)$ and for any $\alpha > 0$.*

We remark that useful and interesting results associated with §2 are discussed in [4], [7] and [9].

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