OPERATOR MONOTONE FUNCTIONS, \( A > B > 0 \) AND \( \log A > \log B \)

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To the memory of my familiar persons passed away by tsunami disaster at Tohoku district on 2011.3.11 with deep sorrow

(Communicated by Y. Seo)

Abstract. Let \( f(t) \) be any non-constant operator monotone function on \( (0, \infty) \) and also let \( A \) and \( B \) be strictly positive operators on a Hilbert space \( H \).

(i) If \( A > B \), then the following inequality holds:

\[
f(A^\alpha) > f(B^\alpha) \quad \text{for all} \quad \alpha \in (0, 1].
\]

(ii) If \( \log A > \log B \), then there exists \( \beta \in (0, 1] \) and following inequality holds:

\[
f(A^\alpha) > f(B^\alpha) \quad \text{for all} \quad \alpha \in (0, \beta].
\]

1. \( A > B > 0, \ \log A \geq \log B \) and \( \log A > \log B \)

A capital letter means a bounded linear operator on a complex Hilbert space \( H \).

An operator \( T \) is said to be \textit{positive} (denoted by \( T \geq 0 \)) if \( (Tx, x) \geq 0 \) for all \( x \in H \) and an operator \( T \) is said to be \textit{strictly positive} (denoted by \( T > 0 \)) if \( T \) is positive and invertible. \textit{Chaotic order} is defined by \( \log A \geq \log B \) for strictly positive operators \( A \) and \( B \), and also \textit{strictly chaotic order} is defined by \( \log A > \log B \) for strictly positive operators \( A \) and \( B \). The well known celebrated Löwner-Heinz inequality asserts that if \( A \geq B \geq 0 \), then \( A^\alpha \geq B^\alpha \) for any \( \alpha \in [0, 1] \). This means that \( t \mapsto t^\alpha \) is operator monotone. Another well known example of operator monotone is \( t \mapsto \log t \) on \( (0, \infty) \), that is, \( \log A \geq \log B \) is weaker than the usual order \( A \geq B \geq 0 \).

We consider the following two operator monotone functions on \( (0, \infty) \) in [8, p. 151] and [10, p. 131]:

\[
\varphi(t) = \frac{t - 1}{\log t} \quad \text{and} \quad \psi(t) = \frac{t \log t - 1}{\log^2 t}.
\]


Keywords and phrases: Löwner-Heinz inequality, chaotic order, strictly chaotic order, operator monotone function.
THEOREM A. Let $A$ and $B$ be strictly positive operators on a Hilbert space $H$. If $\log A > \log B$, then there exists $\beta \in (0, 1]$ and the following inequalities hold:

$$\varphi(A^\alpha) > \varphi(B^\alpha) \quad \text{for all } \alpha \in (0, \beta)$$

and

$$\psi(A^\alpha) > \psi(B^\alpha) \quad \text{for all } \alpha \in (0, \beta)$$

where $\varphi(t)$ and $\psi(t)$ are defined in (1.1).

THEOREM B. There exist strictly positive operators $A$ and $B$ such that $\log A \geq \log B$:

$$\varphi(A^\alpha) \not\geq \varphi(B^\alpha) \quad \text{for any } \alpha > 0$$

and

$$\psi(A^\alpha) \not\geq \psi(B^\alpha) \quad \text{for any } \alpha > 0$$

where $\varphi(t)$ and $\psi(t)$ are defined in (1.1).

(1.2) and (1.4) are shown in [5], and (1.3) and also (1.5) are shown in [6]. In §2, we shall show Theorem 2.2 which is a further extension of Theorem A.

2. Relations among $A > B$, $\log A > \log B$ for $A, B > 0$ and operator monotone functions

We study relations among $A > B$, $\log A > \log B$ for $A, B > 0$ and operator monotone functions for strictly positive operators $A$ and $B$.

THEOREM 2.1. Let $A$ and $B$ be strictly positive operators on a Hilbert space $H$. If $A > B$, then the following inequality holds:

$$f(A) > f(B)$$

for any non-constant operator monotone function $f$ on $[0, \infty)$.

Proof. Let $A > B$. Then there exists some $\varepsilon > 0$ such that

$$A - B \geq \varepsilon I$$

so that $A + s \geq B + s + \varepsilon > B + s$ for $s \geq 0$, so that there exists some $\delta > 0$ such that

$$(B + s)^{-1} - (A + s)^{-1} \geq \delta I.$$ (2.3)

It is well known (for examples, [1], [11]) that a function $f$ on $[0, \infty)$ is an operator monotone if and only if it has the representation

$$f(t) = a + bt + \int_0^\infty \frac{ts}{t+s} dm(s)$$

$$= a + bt + \int_0^\infty \left(s - \frac{s^2}{t+s}\right) dm(s)$$

(2.4)
with \( a \in \mathbb{R} \) and \( b \geq 0 \) and a positive measure \( m \) on \([0, \infty)\). Then (2.2), (2.3) and (2.4) ensure the following (2.5)

\[
f(A) - f(B) = b(A - B) + \int_0^\infty \{(B + s)^{-1} - (A + s)^{-1}\} s^2 dm(s) > 0
\]

(2.5) so that we have (2.1) by (2.5). □

**Theorem 2.2.** Let \( f(t) \) be any non-constant operator monotone function on \([0, \infty)\) and also let \( A \) and \( B \) be strictly positive operators. If \( \log A > \log B \), then there exists \( \beta \in (0, 1] \) and the following inequality holds:

\[
f(A^\alpha) > f(B^\alpha) \quad \text{for all} \quad \alpha \in (0, \beta].
\]

(2.6)

**Proof.** Recall the following obvious relation (2.7):

\[
X > Y > 0 \implies X^\gamma > Y^\gamma \quad \text{for any} \quad \gamma \in (0, 1].
\]

(2.7) In fact \( X > Y > 0 \) ensures \( X \geq Y + \varepsilon I > Y > 0 \) for some \( \varepsilon > 0 \), then \( X^\gamma \geq (Y + \varepsilon I)^\gamma > Y^\gamma \) for any \( \gamma \in (0, 1] \) by Löwner-Heinz inequality and we have (2.7). [3, Corollary 2] asserts that

\[
\log A > \log B \iff \text{there exists} \quad \beta \in (0, 1] \quad \text{such that} \quad A^\beta > B^\beta.
\]

(2.8)

Applying (2.7) for \( \gamma = \frac{\alpha}{\beta} \in (0, 1) \) to (2.8), then we have \( A^\alpha > B^\alpha \) for any \( \alpha \in (0, \beta] \), so that we have (2.6) by Theorem 2.1. □

**Remark 2.1.** We remark that Theorem 2.2 is a further extension of Theorem A since Theorem 2.2 can be available for any non-constant operator monotone functions on \([0, \infty)\).

**Remark 2.2.** For a simple proof of (2.7), we have only to put \( f(t) = t^\gamma \) for \( \gamma \in (0, 1] \) in Theorem 2.1 since \( f(t) \) is a typical well known operator monotone. In fact (2.7) is cited in [2, p. 477, Corollary 8.6.11].

### 3. Concluding remark and a conjecture

It is interesting to point out that an interesting contrast between \( A > B \) and \( \log A > \log B \) for \( A, B > 0 \) as follows.

**Remark 3.1.** Let \( f(t) \) be any non-constant operator monotone function on \([0, \infty)\) and also let \( A \) and \( B \) be strictly positive operators.

(i) If \( A > B \), then the following inequality holds:

\[
f(A^\alpha) > f(B^\alpha) \quad \text{for all} \quad \alpha \in (0, 1].
\]

(3.1)

(ii) If \( \log A > \log B \), then there exists \( \beta \in (0, 1] \) and the following inequality holds:

\[
f(A^\alpha) > f(B^\alpha) \quad \text{for all} \quad \alpha \in (0, \beta].
\]

(3.2)
Proof. If $A > B$, then $A^\alpha > B^\alpha$ for any $\alpha \in (0, 1]$ by (2.7) and we have (3.1) by Theorem 2.1 and (ii) is already shown in Theorem 2.2.

It is reasonable understanding that the condition $\log A > \log B$ in (ii) is weaker than $A > B > 0$ in (i), the corresponding result (3.2) is weaker than (3.1). □

Theorem 2.2, Theorem A and Theorem B suggest the following conjecture.

**Conjecture.** There exist strictly positive operators $A$ and $B$ such that $\log A \geq \log B$, but $f(A^\alpha) \not\geq f(B^\alpha)$ for any non-constant operator monotone function $f(t)$ on $[0, \infty)$ and for any $\alpha > 0$.

We remark that useful and interesting results associated with §2 are discussed in [4], [7] and [9].

**Acknowledgement.** We would like to express our cordial thanks to the referee for useful advice to improve the first version.

**References**